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THE
INDUCTION MOTOR

THE THEORY, DESIGN, AND APPLICATION
OF ALTERNATING-CURRENT MACHINES
INCLUDING FRACTIONAL H.P. MOTORS

BY

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Preface to the Second Edition

In the new edition of this book opportunity has been taken to enlarge the scope of its contents and to bring it up to date. Most of the book has been entirely rewritten. An attempt has been made to cover the theory, design, and applications of the motor. It may appear to some that emphasis has been laid too heavily on design and design principles, but it is my belief that the machine is best understood, in all its forms, when design principles are thoroughly mastered. Much additional material has been introduced in connection with single-phase machines, selsyns, the three-phase series and shunt commutator motors, a new theory of the single-phase motor, and also the application of symmetrical components to conditions of unbalance. Fractional-horse-power motors have received fairly full treatment and typical designs are worked out for single- and three-phase types.

In preparing and writing this second edition, I am greatly indebted to the works of the late B. A. Behrend, of Boston—the great pioneer in this field; to the work of Prof. Waldo Lyon; to Messrs. Wagner and Evans; to Dr. Liwschitz-Garik; and to the American Institute of Electrical Engineers. I also wish to express grateful thanks to The Westinghouse Electric Corporation for photos of fractional h.p. machines; to The B.T.H. Co., of Rugby, for photos, curves, and data on control and selsyns; to The English Electric Co., of Stafford, for excellent photos of machines, windings, and regulators; and to Messrs. Clarke Chapman for photos of machines and windings, and also for drawings. I hope the book will have a wide appeal to students and engineers.

H. VICKERS

February, 1949

Preface to the First Edition

THIS book is intended to introduce the student to the theory and design of induction motors.

An attempt has been made to deal with the latest developments in speed and power-factor control, and to incorporate most of the theory connected with the motor and its applications.

While the scope of the book is large, it is hoped that no section of the work is lacking in thoroughness. An endeavour has been made to place the design of induction motors on a firmer scientific basis than it has rested upon heretofore.

In this work I have largely drawn on my own experience, but I am deeply conscious of my great debt to various writers.

In the course of the work many technical journals, and especially the *Proceedings of the American Institute of Electrical Engineers*, have been consulted and, in addition, use has been made of the works of Steinmetz.

Due acknowledgment has been made, throughout the book, of the sources from which the information has been drawn.

It is hoped that the book will make a large appeal to engineers and students of the various technical colleges and universities.

H. VICKERS

VANCOUVER
CANADA

Contents

PAGE
Preface to Second Edition . . . v
Preface to First Edition . . . vi
List of Plates and Insets . . . xi

INTRODUCTORY . . . 1

Historical account of the development of the induction motor

CHAPTER I

THE POLYPHASE INDUCTION MOTOR . . . 7

Manner of working, vector diagrams, equivalent circuit; deduction of approximate circle diagram and characteristics of the motor deduced therefrom; equivalent resistances and reactances; dispersion coefficient, maximum power factor

Relation of leakages to performance, complete summary of all the properties of the motor and relation of the motor to the circle diagram

CHAPTER II

THE SQUIRREL-CAGE ROTOR . . . 55

Calculation of all quantities referred to the stator. Equivalent resistance and leakage per bar. The double squirrel cage

Analysis of performance and curves

CHAPTER III

THE CIRCLE DIAGRAM . . . 68

The more exact calculation of the circle diagram from the equivalent circuit. Demonstration of the principle of inversion and determination of the radius and co-ordinates of the centre

CHAPTER IV

SINGLE-PHASE INDUCTION MOTOR . . . 80

The cross-field theory applied to the motor and its operating characteristic determined. A completely new theory developed for the motor

CHAPTER V

THE TWO REVOLVING FIELD THEORY . . . 105

Development of the two revolving field theory. Deduction of performance and the equivalent circuit. Influence of reverse field on the torque and heating computed

CHAPTER VI PAGE

ELECTRIC BRAKING OF INDUCTION MOTORS 134

Braking by direct current

CHAPTER VII

SPEED CONTROL OF INDUCTION MOTORS 146

Speed control by resistance in the rotor circuit; rolling-mill motor with permanent slip regulator, rolling-mill motor with automatic slip regulator

Pole-changing. The cascade control and its theory; shunt cascade motor and its windings

Speed control by varying the supply frequency. The Kramer system. The Scherbius system

Speed and power-factor control by concatenation with three-phase series and shunt commutator motors

Circle diagram and circuit diagrams for the foregoing

CHAPTER VIII

THE THREE-PHASE SERIES AND SHUNT COMMUTATOR MOTORS 211

The theory of three-phase series and shunt commutator motors. Circle diagrams relating to them

CHAPTER IX

PHASE ADVANCING 240

The Leblanc system, the Scherbius system, the Miles-Walker advancer and the Kapp vibrator

CHAPTER X

INDUCTION MOTOR WITH D.C. SECONDARY EXCITATION . . 250

The theory of the synchronous induction motor. Calculation of slip for pulling into step. Considerations relating to the design of synchronous induction motors

CHAPTER XI

THE REVOLVING FIELD 266

Analysis of m.m.f. diagrams. Harmonics and their estimation for both stator and rotor, influence on performance

CHAPTER XII

NOISE—CAUSES AND TREATMENT 280

CHAPTER XIII

TESTING OF INDUCTION MOTORS. 286

CHAPTER XIV

INDUCTION MOTOR WINDINGS 293

Stator windings—mush, concentric, and double-layer. Winding rules. Parallel circuits

Rotor windings—mush and barrel windings, non-academic wave windings. Location of abnormal steps, starts, and ends of phases. D.C. wave windings opened out to form three-phase windings

Discussion of windings

CHAPTER XV

PAGE

SELSYNS AND AUTOSYNS 320

Power selsyns. Instrument selsyns. Circuit diagrams. Theory of operation and vector diagrams

CHAPTER XVI

UNBALANCED CONDITIONS IN INDUCTION MOTORS . . . 330

Unbalanced three-phase voltages—positive and negative sequence equivalent circuits, torque developed

Unsymmetrical rotor windings

Operation of three-phase machine with single-phase rotor

CHAPTER XVII

THE INDUCTION REGULATOR, PHASE CONVERTER, AND GENERATOR 349

Theory of single- and three-phase regulators. Use of twin regulators. Automatic control and its circuits. Phase conversion. Induction motor as generator

CHAPTER XVIII

REGULATION AND STABILITY OF THE INDUCTION MOTOR . 363

CHAPTER XIX

DESIGN PRINCIPLES AND DESIGN 367

The D^2L derivation of single-, two-, and three-phase machines, the output coefficient and factors on which it depends. Discussion of values of specific electric and magnetic loadings and their determinations. Relation of D to L for best power factor

Leakage permeance for various types of slots. Zigzag permeance and its calculation. Calculation of overhang leakage. Magnetizing and short-circuit current

Calculation of power-factor curves, dispersion coefficient, etc.

CHAPTER XX

SHORT-PITCH WINDINGS 433

Calculation of the effect of pitch on copper saving and also the effect on leakage inductance

CHAPTER XXI

DESIGN FOR MEDIUM-POWER SLIP-RING MOTOR . . . 441

The design of a 250 b.h.p., three-phase, 50 c/s, 440 V, 1500/1480 slip-ring induction motor is dealt with fully

CHAPTER XXII

DESIGN FOR LARGE SLIP-RING MOTOR 461

The design of a 3750 b.h.p., three-phase, 40 c/s, 2750 V, 400 r.p.m. slip-ring motor

CHAPTER XXIII

DESIGN FOR SQUIRREL-CAGE INDUCTION MOTOR . . . 468

The design of a 15 b.h.p., three-phase, 50 c/s, 400 V, 1000 r.p.m. motor

CHAPTER XXIV PAGE

DESIGN FOR DOUBLE-SQUIRREL-CAGE MOTOR 474

The design of a 12 b.h.p., 220 V, 50 c/s, 1500/1455 r.p.m. motor

CHAPTER XXV

DESIGN OF FRACTIONAL HORSE-POWER MOTORS 487

The different types of fractional h.p. motors are discussed and typical designs given for three-phase and single-phase fractional h.p. motors

CHAPTER XXVI

HOW TO ADAPT WINDINGS TO A GIVEN FRAME . . . 510

CHAPTER XXVII

RATINGS OF MOTORS 513

CHAPTER XXVIII

CONTROL DEVICES 517

Manual starters and on-the-line starters. Magnetic contactors. Push-button control circuits. Auto-starters. Star-delta starters. Overload relays. thermal and solenoid contactors

Index 527

List of Plates and Insets

FACING PAGE

I. SQUIRREL-CAGE ROTOR: STATOR FOR COMMON TYPE OF INDUCTION MOTOR 4

II. THREE-PHASE TYPE OF ROTOR . . . 5

III. BRUSH-LIFTING AND SHORT-CIRCUITING GEAR: CONTINUOUSLY-RATED EXTERNAL SLIP-RINGS WITH SHORT-CIRCUITING DEVICE ONLY 20

IV. CONTINUOUSLY-RATED INTERNAL SLIP-RINGS FOR LARGE OUTPUT MOTOR: BRUSH-LIFTING AND SHORT-CIRCUITING GEAR ON EXTERNAL SLIP-RINGS OF MEDIUM-OUTPUT INDUCTION MOTOR 21

V. DOUBLE SQUIRREL-CAGE ROTOR 88

VI. SQUIRREL-CAGE ROTOR: SINGLE-PHASE CAPACITOR MOTOR STATOR 89

VII. SINGLE-PHASE MOTOR TYPE CAP: SPLASH-PROOF TYPE LIFE-LINE MOTOR 96

VIII. ILGNER SET FOR ROLLING-MILL DRIVE 97

IX. ILGNER SET, WITH EXCITER SET FOR REVERSING RAIL-MILL MOTOR 212

X. MUSH-WOUND ROTOR IN PROCESS OF WINDING: MUSH WINDING FOR THE STATOR OF A THREE-PHASE MOTOR, IN PROCESS OF WINDING 213

XI. THREE-PHASE, THREE-PLANE, SPLIT-PHASE, HAIR-PIN WINDING 228

XII. A TWO-BAR-PER-SLOT, TWO-LAYER, HAIR-COIL, WAVE-WOUND STATOR WINDING 229

XIII. THE COMPONENT PARTS OF A SELSYN RECEIVER OR TRANSMITTER 340

XIV. THREE-PHASE AUTOMATIC VOLTAGE REGULATOR FOR 10 PER CENT VARIATION ON A 250 kVA, 400 V FEEDER 341

XV. TWIN INDUCTION REGULATOR FOR A 20,000 kVA INTERCONNECTOR 356

PLATE FACING PAGE

XVI. TOTALLY-ENCLOSED FAN-COOLED LIFE-LINE MOTOR (CUT-AWAY VIEW): DRIP-PROOF MOTOR . . . 357

XVII. CAPACITOR-START MOTOR TYPE FJ: END BRACKET OF TYPE CAP MOTOR (SINGLE-PHASE) SHOWING STARTING CAPACITOR SWITCH 472

XVIII. CONTROL SWITCHES FOR F.H.P. MOTORS . . 473

XIX. THREE-HORSE-POWER REVERSING CONTACTOR STARTER 480

XX. CONTACTOR STARTER WITH MECHANICALLY-INTERLOCKED TRIPLE-POLE CONTACTOR 481

XXI. THE KMC72 AUTO-TRANSFORMER STARTER FOR 50 B.H.P. 400 V SQUIRREL-CAGE MOTOR . . . 488

XXII. REAR VIEW OF UNIT SEEN IN PLATE XXI . . 489

XXIII. STAR-DELTA STARTER 496

XXIV. CONTACTOR STARTER FOR SQUIRREL-CAGE MOTORS UP TO 50 H.P. 497

INSETS

FIG. FACING PAGE

14.11. THREE-PHASE, LAP-TYPE, STAR-CONNECTED, ROTOR WINDING, SIX-POLE, 63 SLOTS 318

14.12. THREE-PHASE, LAP-TYPE, STAR-CONNECTED, ROTOR WINDING, SIX-POLE, 72 SLOTS 318

14.13. THREE-PHASE, LAP-TYPE, STAR-CONNECTED, ROTOR WINDING, EIGHT-POLE, 96 SLOTS 318

21.11. A.C. SLIP-RING MOTOR, DETAILED DRAWING . 458

Introductory

It will be interesting to the student of electrical engineering to have some account of the fascinating history of the induction motor. The discovery by Gambey, the instrument-maker of Paris, that a compass needle, when disturbed and set oscillating, comes to rest more quickly when it is in the vicinity of copper, than when wood is near it, was made in 1824. At that time also Barlow and Marsh, at Woolwich, had observed the effect on a magnetic needle of rotating it near a sphere of iron. Arago published, in 1824, an account of an experiment with a compass needle within rings of different materials. In this experiment he pushed the needle aside to about 45° and counted the number of oscillations made by the needle before the swing decreased to 10°. With a ring of wood the number of oscillations was 145; with a copper ring 66; and with a stout copper ring only 33. In 1825 he suspended a compass needle over a rotating copper disc and found that, by turning the disc slowly, the needle is deviated out of the magnetic meridian. By rotating the disc fast enough he found that continuous rotation of the needle could be produced. The brilliant discovery by Faraday, in 1831, of electromagnetic induction provided the solution to the question of the origin of the forces present in the above experiments of Gambey, Barlow and Marsh, and Arago. Faraday showed that the rotation of the Arago disc was due to induced currents, set up in the disc by relative motion of disc and compass needle. From 1831 to 1879 this valuable discovery produced no further results. In June, 1879, Mr. Walter Baily read a paper, before the Physical Society of London, on "A Mode of Producing Arago's Rotations." Baily used a fixed electromagnet with four magnet cores joined to a yoke.

The four magnet cores were about 4 in. long and each was wound with about 150 turns of insulated copper wire of 2·5 mm diameter. The coils were connected two and two in series, similar to two independent horse-shoe magnets and were set diagonally across one to another.

The two circuits were connected separately to a revolving commutator, built up of a simple arrangement of springs and contact strips mounted on a piece of wood, with a wire handle by which it was turned. By rotation, the currents from two batteries were caused to be reversed alternately in the two circuits, and this gave rise to the following changes in polarity of the four poles.

N O   N N   O N   S N   S O
↘   ↓   ↙   ←   ↖
O S   S S   S O   S N   O N

In this rotating magnetic field a copper disc was suspended. He stated: "The rotation of the disc is due to that of the magnetic field in which it is suspended, and we should expect that, if a similar motion of the field could be produced by any other means the result would be a similar motion of the disc." He also suggested that if a whole circle of poles were arranged under the disc, successively excited in opposite pairs, the series of impulses all tend to make the disc revolve in one direction around the axis, and added: "In one extreme case, when the number of electromagnets is infinite, we have the case of a uniform rotation of the magnetic field, such as we obtain by rotating permanent magnets." It is clear that Mr. Baily had grasped the fundamental principle of action of the induction motor, and the motor he exhibited before the Physical Society, in 1879, was the first induction motor, but it needed later important discoveries of methods for producing the revolving field by means of alternating currents to make it the useful machine that it is to-day. The next discovery was made by Marcel Deprez in 1883.

Deprez fed alternating current to a coil, which produced an alternating or oscillating field along the *OX* axis. He supplied another coil, whose magnetic axis made an angle of 90° with the *OX* axis, with alternating current, whose phase difference was 90° in time from the current in the first coil, and showed that a revolving field of constant amplitude could be produced. The frequency of the two currents was the same. He also showed that if the two currents were of equal period, but not of equal amplitude, an elliptically rotating field was produced. The number of turns in each coil was the same.

Professor Ferraris arrived at the same conclusions as Baily and Deprez in 1885, and apparently without knowing of the work of either. His paper on "Electrodynamic Rotations Produced by Means of Alternating Currents" was published in 1888. He suggested the method of obtaining currents, differing in phase by nearly 90°, by inserting a resistance in one winding and inductance in the other, thus making the ratio of $\frac{\text{reactance}}{\text{resistance}}$ small in one winding and large in the other. This method, it may be noted, is largely used for starting up single-phase motors.

Then followed the great work of Nikola Tesla between 1887 and 1891. His researches placed the induction motor on a sound foundation. His patents were sold to the Westinghouse Co. of America, whose pioneer efforts in this field must be recognized. In that period, however, the only a.c. supply circuits were single phase, and the frequencies were 133 and 125 c/s. These supply circuits were obviously unsuitable for the development of the motor.

In 1891 the Electrotechnical Exhibition at Frankfort was held, and three-phase transmission of power was demonstrated. Two turbine-driven three-phase generators were installed at Lauffen, generating 1400 A at 55 V. The frequency was 40 c/s. Three-phase transformers were installed, at each end of the line, to raise the voltage to 8000 V at Lauffen, and to reduce it to 65 V at Frankfort. The distance of transmission was 110 miles. This bold experiment demonstrated the feasibility of three-phase transmission of energy. The load consisted of a 100 h.p. three-phase motor and also lamps. Several German firms exhibited different types of three-phase induction motors at the exhibition. One three-phase, 3 b.h.p. motor had the three-phase supply brought into the rotor by three slip-rings, the secondary circuit, being the stator winding, consisted of a closed-circuit winding. Several motors, built by the Oerlikon Co., and designed by the late C. E. L. Brown, were shown. One such motor was a three-phase, 20 b.h.p. motor. It had a distributed stator winding and a squirrel-cage rotor, and a small air-gap. The squirrel-cage rotor was the invention of Mr. Dolivo-Dobrowolsky, who co-operated with Mr. Brown in the design of these motors. This motor of Brown's closely resembled in construction the motor of to-day. From the short account given, it will be realized that much progress was made purely as the result of experiment, and that much theoretical investigation was needed to explain the reactions taking place in the motor, and also to show how it could be designed to give the characteristics desired. To that end it was necessary to give a lucid theory of alternating currents. Thomas H. Blakesley gave a series of ten brilliant papers in the *Electrician* in 1885. He discussed, for the first time, alternating current phenomena by means of polar diagrams. Then followed the work of the late Professor Gisbert Kapp in his papers contributed to the Institute of Civil and Electrical Engineers in 1890.

In 1892, F. Bedell and A. C. Crehore published their book, *Alternating Currents*, in which polar diagrams were used and applied to the theory of the transformer and the locus of the primary e.m.f. of the current transformer was shown to be a circle. In 1894, Kapp gave a very lucid elementary account of the phenomena in induction motors. The polar diagram was developed and included the primary resistance and leakage. His diagram was given for each point of the load, and gave no general solution showing how the different characteristics varied with the load.

In 1895, Blondel gave his papers on "Some General Properties of Revolving Magnetic Fields." In these papers, published in *Eclairage Electrique*, he unfolded the theory of the composition of magnetic fluxes, including leakage fluxes. In 1895 also, the late Mr. B. A. Behrend proved that the locus of the primary current of the alternating-current transformer is a circle in the polar diagrams, provided the primary resultant magnetic field is constant. The circle locus of the induction motor was also shown by A. Heyland in 1894. There has been much controversy about priority in this discovery. Part of the credit must be given to Dr. Bedell, who stated: "In any circuit or apparatus with constant reactance and variable power consumption, the current will have a circle locus if the supply voltage is constant." This was first shown by Bedell and Crehore in 1892.

Bedell also states: "That the induction motor nearly fulfils those conditions and that its current locus is practically the arc of a circle, was first shown by Heyland in 1894." It is also stated that Kapp and Behn-Eschenberg first pointed out the identity of the theory of the a.c. transformer and the induction motor in 1893 and 1894.

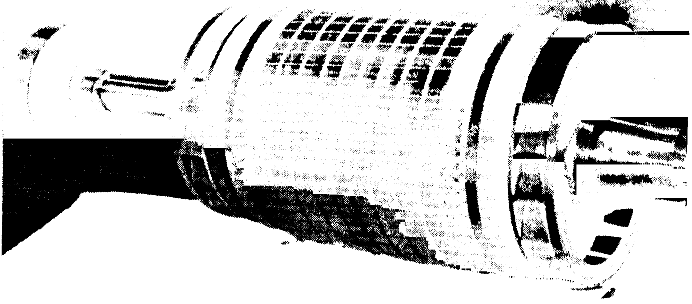
The circle diagram in use to-day is undoubtedly due to Behrend, and one is impressed with the beautiful simplicity of the diagram, and the ease with which the characteristics of the motor are determined from it, commends it to the designer. My first knowledge of this motor came from his stimulating book, the first edition of which was published in 1901. The second edition was published in 1921. Behrend was greatly interested in this motor and, when the first edition of this book appeared in 1925, he invited me to Boston and expressed great pleasure at my entry into this field, which he had made so much his own and to which he had contributed so much. I visited him in Boston. To me he was most gracious. I was impressed with his personality and especially with his modesty. He was a great friend of Heaviside, and I was surprised to see on the walls of his home photographs of Heaviside at various ages.

Even at that time the general outline was clearly seen, but there remained many important questions to be answered. The circle diagram clearly demonstrated the need for a small air-gap length, for high power factor, and a large ideal short-circuit current. All the main characteristics, such as torque, power, current, slip, efficiency are readily determined from the Behrend circle diagram. Behrend gave an empirical formula for the dispersion coefficient in the form $\sigma = C\frac{\delta}{\tau}$, where δ = gap length, τ = pole pitch, and C = a factor which varied with slot dimensions and other things.

This led some to assume that the best power factor was obtainable by using larger and larger ratios of diameter to core length for a given D^2L.

Closer analysis has shown that this is not true, and the exact relation of D to L for best power factor was given by me in my book in 1925.

Although the relation of dimensions to characteristics was known about 1900 or so, it remained to determine the effect of harmonics, due to the distribution of the winding, and also due to slots, on the performance of the motor. Analysis of the m.m.f. diagrams showed that several rotating fields were produced, the fundamental and various harmonics. These fields rotate at different speeds with respect to the rotor, and some rotate in the same direction, some in the opposite direction, and produce both driving and retarding torques. Such harmonics may, and do, produce noise and vibration,



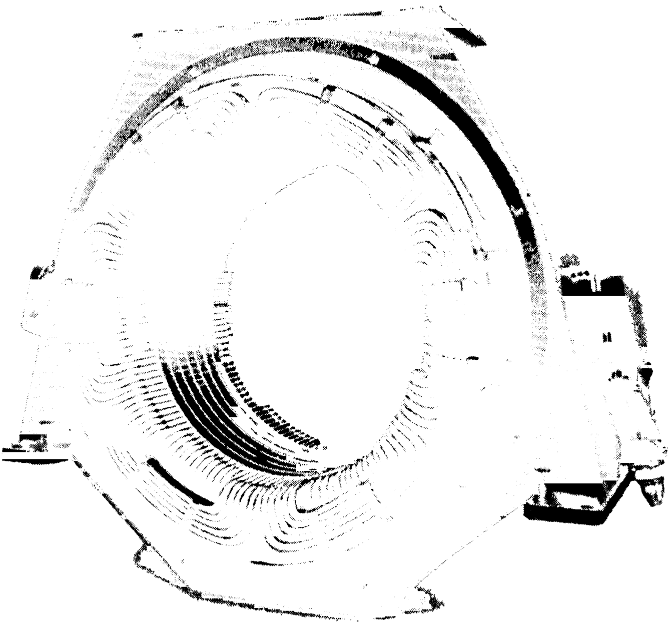


PLATE I

(*Upper*) SQUIRREL-CAGE ROTOR

(*Courtesy English Electric Co., Ltd.*)

(*Lower*) STATOR FOR COMMON TYPE OF INDUCTION MOTOR

(*Courtesy English Electric Co., Ltd.*)

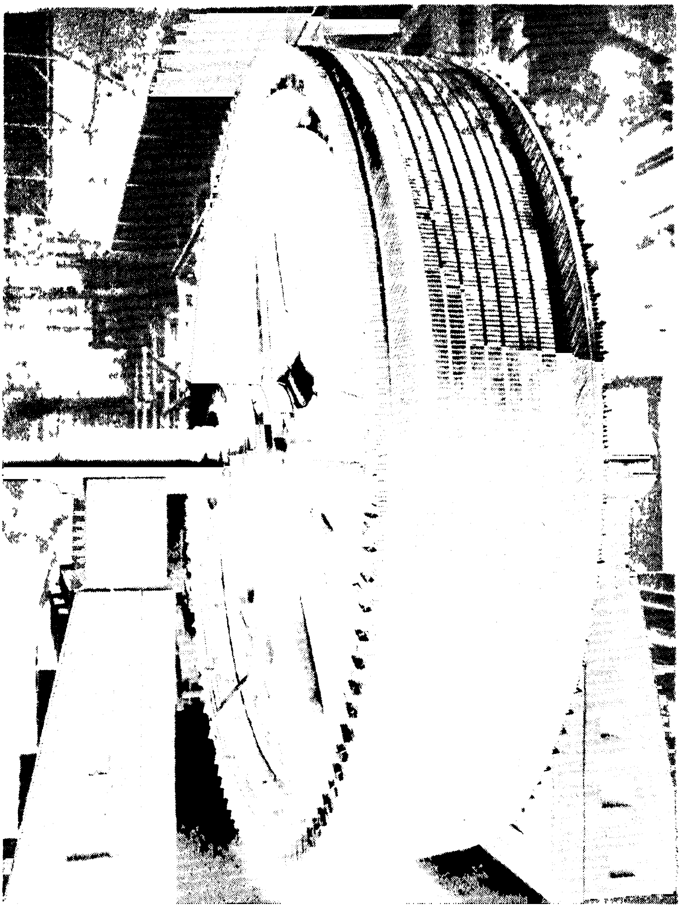


PLATE II THREE-PHASE TYPE OF ROTOR

(Courtesy English Electric Co., Ltd.)

and may, by producing saddle backs in the torque curve, prevent the machine from accelerating to full speed. Especially is this true in pole-changing motors. This question of noise has been investigated by H. Fritze and Kron and Chapman. A large number of papers has been written on this very important subject.

The question of improved efficiency has resulted in improvements in the manufacture of steel laminations, by the introduction of silicon in various percentages. Brands known as Stalloy and Super-Stalloy have been introduced, and are used in those cases where it is necessary to keep down the iron losses, in totally enclosed machines, and especially in machines for 400 c/s, such as are used in connection with the automatic pilot for planes.

Then the question of eddy-current losses in conductors has been thoroughly investigated by A. B. Field and others.

The trend is towards greater and greater output from a given mass of materials, and this can only be effected by scientific design and proper proportioning of the machine. The introduction of Silicone varnishes for insulation has removed the conservative temperature rises formerly allowed, and resulted in smaller machines for a given output and speed.

Ventilation is another problem, which has received much attention and is one of the most important factors in increasing output from a given frame. Much more research is still needed on this important question.

The question of speed control in induction motors has received much attention. The induction motor is essentially a constant-speed machine, like the d.c. shunt motor, and this is, in some cases, rather a serious drawback. There are many industrial applications where speed control is necessary and the induction motor is the ideal motor for many such applications, being simple, rugged, and reliable, but various methods must be adopted to secure efficient speed control which spoil its simplicity. The methods adopted are: (1) pole-changing; (2) cascade connection; (3) cascading with a commutator motor; (4) change of frequency of supply. All these methods will be dealt with later, but the work of Mr. Louis J. Hunt deserves special mention for his genius and originality in producing the cascade motor bearing his name. I should like to pay him the greatest tribute, for I was associated with him in the early stages of his invention. His name should never be forgotten, for his work bears the stamp of genius.

The induction motor is now the most widely used of all machines. It is doubtful whether the large power systems, now in such extensive use, would have been developed if this motor had not been developed. What it means to the economy of the world is appreciated by few people outside the engineering world. In the fractional horse-power field, its development is phenomenal and it is applied in every form of industrial and domestic work. In this field it takes the form of single-phase and three-phase types. We shall be dealing with the problems of single-phase capacitator motors in the text, and with their characteristics.

In analysing the performance of both three-phase and single-phase motors, we are frequently faced with unbalanced conditions, such as unbalanced voltages or unbalanced windings. These interesting problems will be solved by the use of the theory of symmetrical components, due largely to the late Dr. C. L. Fortescue. An explanatory account of the theory will be given in the text, and applications made to many interesting problems.

CHAPTER I

The Polyphase Induction Motor

THE induction motor is the most extensively used of all alternating-current motors. In its simplest form it admits of robust mechanical construction, and its ruggedness and ability to stand rough usage make it a most desirable type of industrial motor. It consists essentially of a stationary member, called the stator, and a rotating member called the rotor. The active part of the stator consists of a core of laminations of sheet steel of about 0·5 mm in thickness. These laminations are slotted on their inner periphery, and assembled in a steel yoke.

In these slots is placed a winding of the required number of phases, which may be of the concentric type, of the mush type, or of the barrel type with diamond-shaped coils. Plate I (lower) shows the stator for a common type of induction motor.

The rotor may be of the squirrel-cage type, consisting of bars of copper or aluminium placed in the rotor slots and connected at each end by a solid ring of copper, aluminium or brass. Plate I (upper) shows a normal type of squirrel-cage rotor.

If the starting requirements are such as to demand large starting torque with low starting current, then a rotor of the wound type with slip-rings will be used, and starting resistances will be used. This rotor is usually of the three-phase type with either mush or diamond coils. Plate II shows this type of rotor. While the stator may be of single-phase or of polyphase type, the rotor is always of the polyphase type, i.e. either two-phase or three-phase.

In the analysis of the magnetomotive force diagrams of the usual types of windings, which will be given in Chapter XI, it is shown that with symmetrical polyphase windings, a rotating field is produced when these windings are supplied with symmetrical polyphase currents. A symmetrical m phase winding is defined as one in which the starts of the phases are displaced around the periphery by $\frac{2\pi}{m}$ or $\frac{360}{m}$ electrical degrees. Each pole pitch corresponds to π electrical radians or 180 electrical degrees, and, of course, the double pole-pitch is 360 electrical degrees or 2π radians. Thus, the starts of the phases in a three-phase machine must be displaced around the periphery by two-thirds of the pole pitch or 120 electrical degrees, i.e. $\frac{360}{3}$°. The ends of the phases will be displaced by the same amount from each other. Now if we supply such a polyphase winding with alternating currents which vary sinusoidally with respect to time, and in which the time phase difference is $\frac{2\pi}{m}$, it will be shown that the magnetomotive forces of such polyphase windings can be represented by a series of rotating waves, each of constant amplitude, rotating at speeds which vary inversely as their order. The principal wave, the fundamental, rotates at synchronous speed, i.e. at $\frac{120f}{p}$ r.p.m. The fundamental is of constant amplitude and rotates at synchronous speed around the machine. The smaller waves are called "harmonics." In the non-chorded windings and also the chorded windings which can be replaced by non-chorded windings, no even harmonics are present. In these three-phase windings we have odd harmonics as follows: $1, -5, +7, -11, +13, -17, +19$, etc.

The minus sign indicates that these harmonics travel in the opposite direction to the fundamental wave. Thus, the 5th, 11th, 17th harmonics, etc., travel in the negative direction, while the 1st, 7th, 13th, 19th travel in the same direction as the fundamental. The orders of the various harmonics are given by the following equation—

$$\beta = \alpha_1 m_1 + 1$$

where α is a positive or negative integer and m_1 is the number of stator phases.

Thus, $\alpha_1 = 0$, $\alpha_1 = -2$; $\alpha_1 = +2$, $\alpha_1 = -4$, etc.; $\beta = +1$, $\beta = -5$; $\beta = +7$, $\beta = -11$.

The above values for β, viz. $1, -5, +7, -11, +13, -17, +19$, show the various harmonics for a non-chorded three-phase winding.

It will be noted that there is no third harmonic in the three-phase winding. The pole pitch for each harmonic is, of course, equal to the pole pitch for the fundamental divided by the order of the harmonic.

The fundamental fact is that, under the conditions stated, several revolving fields are produced in the machine, each of constant amplitude, and if the speed of the fundamental wave is R.P.M.$_s$, then the speeds of the harmonics are $\frac{\text{R.P.M.}_s}{\beta}$, where β = order of the harmonic.

Thus, the fifth harmonic rotates at one-fifth of the speed of the fundamental, the seventh at one-seventh of the speed of the fundamental, etc.

These rotating fields generate e.m.f.s in both stator and rotor windings. The e.m.f.s of the various harmonic fields generated in the stator are all of fundamental frequency, given by the usual equation—

$$\frac{\text{poles}}{2} \times \text{r.p.s.} = \text{frequency}$$

It will be clear that, since the number of poles for the various harmonics vary directly as their order, and the speeds vary inversely as their order, the generated frequency of the e.m.f.s in the stator are all equal.

There are other harmonics present, in addition to those mentioned above, which are introduced by the rotor and stator slots. All these rotating fields are responsible for e.m.f.s generated in both stator and rotor windings and their resulting currents. By the interaction of fluxes and currents, driving and retarding torques will be produced, which may affect the characteristics of the motor. It will be our purpose to study the effects of these harmonics in a later section, and the steps which may be taken to reduce or eliminate their effects. At present it is our object to get a clear understanding of the reactions taking place in the machine and its manner of working.

We will suppose there is a symmetrical three-phase winding supplied with three-phase alternating current, and we will suppose that the fundamental wave of rotating flux only is present. The direction of rotation of the fundamental wave will depend on the sequence of the phases. It will be shown that the revolving field amplitude always lies above the winding group in which the current is a maximum, i.e. the sequence in which the current reaches its maximum in the various phases determines the direction of rotation. Thus, by reversing the connection of two of the terminals, the direction of the rotation may be reversed. Assume that a rotor with a three-phase symmetrical winding exists. The revolving field will generate e.m.f.s in stator and in rotor. At standstill these e.m.f.s are of the same frequency. If the rotor circuits are closed, currents will be set up in the rotor circuits of the same frequency as the rotor e.m.f.s. Since, at standstill, the frequency of the rotor currents is equal to the supply frequency, the leakage reactance of the rotor per phase will be relatively high compared to the resistance, and hence the angle of lag of the rotor current behind the e.m.f. and flux will be large, and this will be true for each phase.

Now we know that conductors carrying current in a magnetic field are subjected to a force or torque, and this force can be calculated for each conductor of the rotor, once we know the current in the conductor and the value of the field density in which it lies. Let us assume the field rotates in the clockwise direction as shown below in Fig. 1.1, and let us assume the flux is distributed sinusoidally.

The direction of the e.m.f.s and currents in the rotor conductors is shown by the dots and crosses. Now, if one places the *left* hand along the conductors in such a manner that the flux passes from the palm to the back, and the current flows from the wrist to the finger tips, then the direction of the force on the conductors is towards the right as shown by the direction in which the thumb points. The rotor, therefore, moves in the direction of movement of the field. It should be clear that the torque at starting is reduced by the angle of lag of the rotor current behind its e.m.f. Now, if this lag could be reduced, the starting torque would be increased. This is usually effected by adding resistance in series with the rotor.

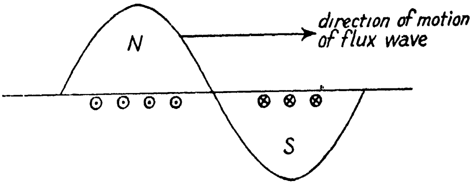


FIG. 1.1

Fig. 1.2 shows a sketch of the three-phase wound rotor motor with starting resistance.

Since $\tan\phi_2 = \dfrac{\text{rotor reactance per phase}}{\text{rotor resistance per phase}}$

it follows that the lag angle can be reduced by the addition of resistance in series with each phase of the rotor. The rotor therefore accelerates, and as its speed increases, the rate of cutting of the flux by the rotor conductors decreases and is dependent on the relative velocity of the flux and rotor.

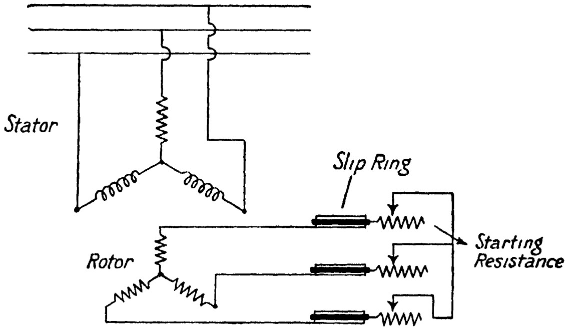


FIG. 1.2

If ω_0 = speed of the field in radians per second

ω = speed of the rotor in radians per second.

Then the relative speed $= \omega_0 - \omega$; and this expressed as a fraction of ω_0, the synchronous speed, is called the slip. If s = the slip, $s = \dfrac{\omega_0 - \omega}{\omega_0}$

Since the rotor frequency = pairs of poles × slip revolutions per second, it is clear that the frequency of the rotor currents $= s \times f$, where $f =$ supply frequency.

At standstill $s = 1$, since $\omega = 0$ at standstill.

At synchronous speed $\omega = \omega_0$ and $s = 0$. Thus the frequency of the rotor currents varies from f at standstill, to sf at slip s.

The speed will continue to increase until the driving torque equals the resisting torque. With no load on the machine, the speed rises to a value not far from synchronous speed. The slip at no-load is a fraction of 1 per cent.

Vector Diagram at No-load

The current in the rotor at no-load being relatively small, the current in the stator is that necessary to produce the flux. This flux is in time phase with the current in the stator winding, neglecting magnetic hysteresis. The e.m.f. generated in one phase lags 90° behind the flux linking the winding, and is in phase with the rotating flux cutting the conductors. Thus when the flux linking a coil is a maximum, the e.m.f. is zero.

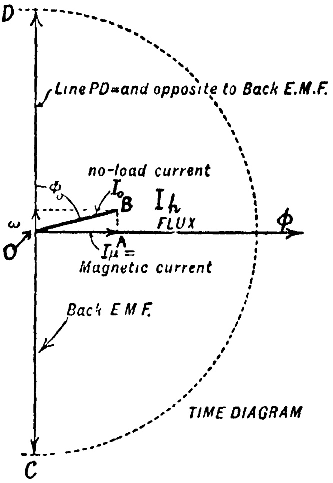


FIG. 1.3. VECTOR DIAGRAM AT NO-LOAD

The vector diagram at no-load is shown in Fig. 1.3.

OA = vector of flux

$OA = I_\mu$ = magnetizing current per phase

$AB = I_h$ = watt component of no-load current per phase

$OB = I_0$ = no-load current per phase

OC = back e.m.f. in the stator

OD = applied p.d., sensibly equal to OC, per phase

$\cos \widehat{DOB} = \Phi_0$ = power factor at no-load.

The no-load current per phase consists of a magnetizing component OA, and a watt component AB, per phase, to supply the no-load losses.

On-load Vector Diagram

Under full load the motor will have a slip such that sufficient e.m.f. will be generated, per phase, in the rotor to produce sufficient torque to equal the retarding torque due to the load and the friction and windage losses. Fig. 1.4 shows the on-load vector diagram.

It will be observed from the diagram, that the current flows in the rotor in such a direction as to tend to demagnetize the stator. In other words the amp-turns of the rotor winding tend to oppose the amp-turns of the stator. At no-load the applied p.d. per phase is sensibly equal to the back e.m.f. generated in the stator by the revolving field.

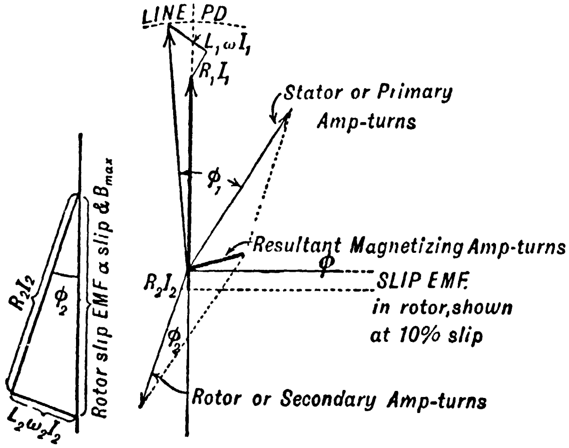

LINE PD
$L_1 \omega I_1$
$R_1 I_1$
Stator or Primary Amp-turns
ϕ_1
Resultant Magnetizing Amp-turns
ϕ
SLIP EMF in rotor, shown at 10% slip
$R_2 I_2$
ϕ_2
Rotor or Secondary Amp-turns
$R_2 I_2$
ϕ_2
Rotor slip EMF α slip & B_{max}
$L_2 \omega_2 I_2$

FIG. 1.4. VECTOR DIAGRAM ON-LOAD

Actually we have

$$\mathbf{V} = \mathbf{E} + \mathbf{Z}_s \mathbf{I}_{os} = \mathbf{E} + (r + jx_s)\mathbf{I}_{os} \qquad . \qquad . \ (1.1)$$

where $\mathbf{V}$ = applied volts per phase
$-\mathbf{E}$ = back e.m.f. per phase
$\mathbf{Z}_s$ = leakage impedance per phase
$= r + jx_s$
r = stator resistance per phase
x_s = leakage reactance per phase of the stator

On-load also—

$$\mathbf{V} = \mathbf{E}_1 + (r + jx_s)\mathbf{I}$$

where— $\mathbf{E}_1$ = back e.m.f. on-load
$\mathbf{I}$ = current per phase in stator on-load

The addition in these equations is vectorial, vector quantities being indicated in bold-face type.

Virtually there is only a 3 per cent difference in the flux from no-load to full load, so we can assume the flux remains sensibly constant. For this to be so, it follows there must flow in the stator a component of current which offsets, by its magnetizing action, the demagnetizing effect of the rotor current.

The flux in the machine is produced by the resultant of the stator and rotor magnetomotive forces. Since the flux exists almost at the same value at no-load, it follows that the stator current on-load consists of two components, namely, the no-load current I_0, and the load component of the current.

The effective ampere-turns of the rotor = effective ampere-turns in the stator, due to the load component of the current.

The e.m.f.s in rotor and stator will have the same relation to the flux as at no-load. Now, however, the rotor carries current lagging behind the e.m.f., and this lag is produced by leakage fluxes. This lag will depend also on frequency of the rotor currents which is proportional to the slip. At standstill this lag is important, for the slip is high, and the angle large. At full-load speed the slip may be anything from 2 to, say, 5 per cent, and the lag is small. As in a transformer this lag is reflected into the stator circuit, so at standstill the power factor is low, increasing as the slip decreases from 1 to the full-load value. Plates III and IV (facing pages 20 and 21) show examples of brush-lifting and short-circuiting gear and slip-rings.

Circle Diagram of the Motor

Let us consider a three-phase motor, and let—

T_1 = turns per stator phase in series
T_2 = turns per rotor phase in series
$\hat{\phi}$ = maximum value of flux per pole
f = supply frequency
s = slip
$\bar{E}_1$ = back e.m.f. in stator per phase
$\bar{E}_2$ = e.m.f. generated in the rotor per phase at standstill
K_1 = breadth factor of stator winding for fundamental
K_2 = breadth factor of rotor winding for the fundamental
K_3 = coil span factor of the stator winding for the fundamental
K_4 = coil span factor of the rotor winding for the fundamental

Then
$$\bar{E}_1 = 4{\cdot}44 \times K_1 \times K_3 \times \hat{\phi} \times T_1 \times f \times 10^{-8} \quad . \ (1.2)$$
$$\bar{E}_2 = 4{\cdot}44 \times K_2 \times K_4 \times \hat{\phi} \times T_2 \times f \times 10^{-8} \quad . \ (1.3)$$

The e.m.f. per phase in the rotor, at slip s, $= s\bar{E}_2$—

$$\therefore \quad s\bar{E}_2 = 4{\cdot}44 \times K_2 \times K_4 \times \hat{\phi} \times T_2 \times sf \times 10^{-8} \quad . \ (1.4)$$

If L_2 = coefficient of self inductance of the rotor winding per phase in henrys, due to *leakage* flux

and $\omega_0 = 2\pi \times f$

and $\bar{I}_2$ = rotor current per phase in amperes

then
$$sE_2 = I_2\sqrt{R_2^2 + s^2L_2^2\omega_0^2} \quad . \quad . \quad . \quad (1.5)$$

and
$$I_2 = \frac{sE_2}{\sqrt{R_2^2 + s^2L_2^2\omega_0^2}} = \frac{E_2}{\sqrt{\dfrac{R_2^2}{s^2} + L_2^2\omega_0^2}} \quad . \quad . \quad (1.6)$$

where R_2 = rotor resistance per phase.

Equation (1.6) shows that the rotor current is equal to the e.m.f. generated per phase at standstill, divided by $\sqrt{\left(\frac{R_2}{s}\right)^2 + L_2^2\omega_0^2}$.

Now $L_2\omega_0$ is the reactance per phase of the rotor at full supply frequency, i.e. at standstill.

We see, therefore, that the rotor current at standstill, provided the total rotor resistance $= \frac{R_2}{s}$ = the actual rotor resistance per phase, divided by the slip, is equal to the rotor current when running under load with slip s, and it is obvious that the phase relation of the rotor current I_2 to the rotor e.m.f. is the same in each case, for $\tan\phi = \frac{sL_2\omega_0}{R_2}$ in each case.

Therefore, as far as current and power-factor relations are concerned, the action of the machine, when running normally under load, is exactly the same as at rest, provided we make the new resistance in the rotor circuit per phase $= \frac{R_2}{s}$.

Now
$$\frac{R_2}{s} = R_2 + R_2\frac{(1-s)}{s} \quad . \quad . \quad . \quad (1.7)$$

Now if we multiply each side of equation (1.7) by I_2^2, we have—
$$I_2^2\frac{R_2}{s} = I_2^2R_2 + I_2^2R_2\frac{(1-s)}{s} \quad . \quad . \quad (1.8)$$

Substituting for I_2 its value from equation (1.6)—
$$\frac{sE_2 \times I_2}{\sqrt{R_2^2 + s^2L_2^2\omega_0^2}} \times \frac{R_2}{s} = I_2^2R_2 + I_2^2R_2\frac{(1-s)}{s} \qquad (1.9)$$

i.e.
$$E_2 \times I_2 \times \cos\phi_2 = I_2^2R_2 + I_2^2R_2\frac{(1-s)}{s} \quad . \quad (1.10)$$

Now considering the two terms on the right of equation (1.10) we have $I_2^2R_2$, which is the rotor copper loss per phase and the second term, namely, $I_2^2R_2\frac{(1-s)}{s}$ represents the additional loss per phase which is equivalent to the gross output of the machine when running under load with a rotor current per phase I_2 at slip s.

With a three-phase rotor—

$$\text{Gross mechanical output} = 3I_2^2R_2\frac{(1-s)}{s} \text{ in watts.} \quad (1.11)$$

With an m-phase rotor, where m is an integer greater than 1, the gross mechanical output $= m \times I_2^2 R_2 \dfrac{(1-s)}{s}$ in watts.

We see, therefore, by making the resistance in the rotor circuit per phase $= \dfrac{R_2}{s}$, the machine is brought to rest and the current, power, and power-factor relations in the rotor are the same as when running under load with a slip s.

Thus, at rest, the motor becomes a transformer; a rather leaky transformer, and all the theory of the constant potential transformer can be directly applied to it.

Now looking at equations (1.2) and (1.3), we see that—

$$E_2 = \frac{E_1 \times T_2 \times K_2 \times K_4}{T_1 \times K_1 \times K_3} \quad . \quad . \quad . \ (1.12)$$

We have also shown that I consists of two components, the no-load current and a component which is called the load component. The effective component of the stator current which neutralizes the magnetizing action of the rotor current is—

$$\frac{I_2}{\alpha} = I_2 \times \frac{T_2 \times K_2 \times K_4}{T_1 \times K_1 \times K_3} = \frac{I_2}{\alpha} = I_1' \quad . \quad . \ (1.13)$$

where $\quad \alpha =$ ratio of transformation $= \dfrac{T_1 \times K_1 \times K_3}{T_2 \times K_2 \times K_4}$. (1.14)

Now—

$$I_1' = \frac{I_2}{\alpha} = \frac{E_2}{\alpha\sqrt{\left(\frac{R_2}{s}\right)^2 + L_2^2\omega_0^2}} = \frac{E_1}{\alpha^2\sqrt{\left(\frac{R_2}{s}\right)^2 + L_2^2\omega_0^2}} \quad (1.15)$$

$$= \frac{E_1}{\sqrt{\left(\frac{R_2\alpha^2}{s}\right)^2 + (L_2\omega_0\alpha^2)^2}} \quad . \quad . \quad . \quad . \quad . \ (1.16)$$

Thus, if we multiply R_2 and L_2, each by α^2, i.e. by $\left(\dfrac{T_1 \times K_1 \times K_3}{T_2 \times K_2 \times K_4}\right)^2$, we obtain the component of the stator current corresponding to the load, namely, I_1'.

It is clear also that—

$$(I_1')^2 R_2' = \frac{I_2^2}{\alpha^2} \times R_2 \times \alpha^2$$

$$= I_2^2 \times R_2 \quad . \quad . \quad . \quad . \ (1.17)$$

where $R_2' = R_2 \times \alpha^2$, the rotor resistance referred to the stator per phase

and $\quad L_2' = L_2 \times \alpha^2$, the rotor leakage self-inductance referred to the stator per phase.

We can now build up our equivalent circuit of the motor, in which all quantities are referred to the stator (Fig. 1.5).

Across AB we have the applied voltage per phase $\bar{V}_1$.

Clearly $\bar{V}_1 = (R_1 + jL_1\omega_0)\bar{I}_1 + \bar{E}_1$ . . . (1.18)

Note $-\bar{E}_1 =$ back e.m.f.

$+\bar{E}_1 =$ component of applied P.D. to overcome E_1.

Clearly also the voltage across $CD = +\bar{E}_1$. This is also the voltage, which is applied to rotor circuit per phase, when all quantities of rotor resistance and reactance are referred to the stator.

Across CD we have connected one phase of the rotor, in which all quantities are referred to the stator by multiplying them by the square of the ratio of transformation. We see that we have the rotor resistance referred, namely, $R_2 \times \alpha^2$, and the rotor inductance referred, namely, $L_2 \times \alpha^2$, and the resistance representing the gross mechanical load also referred, namely, $R_2 \times \alpha^2 \frac{(1 - s)}{s}$. These are shown in series across the points C and D, between which we have the voltage $+\bar{E}_1$, i.e. the voltage which overcomes the back e.m.f. in the stator. It should be remarked that in all polyphase circuits, which are symmetrical, the quantities we require to estimate are the same for *each* phase, and hence it is sufficient to consider *one* phase only. Our diagrams refer to *one* phase only.

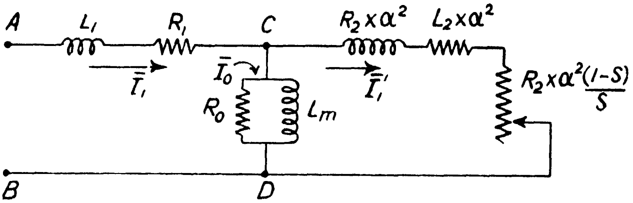


FIG. 1.5

The back e.m.f. $-\bar{E}_1$ is produced by the flux ϕ per pole, and this requires a current for its production. It is the magnetizing current $\bar{I}_\mu$, shown in our vector diagram. It leads $-\bar{E}_1$ by 90°. Hysteresis and eddy-current losses are produced by the revolving field in the core of the stator and also in the rotor and also eddy-current losses in the conductors of the machine. A watt component of the current is needed to supply these losses. This component, namely $\bar{I}_h$, is in phase with $+\bar{E}_1$, i.e. with the component of applied volts across CD. Across CD is connected a purely inductive coil, whose reactance at supply frequency $= X_m$.

Then $$I_\mu = \frac{+\bar{E}_1}{X_m}$$

In parallel with it is connected a non-inductive resistance R_0.

Its magnitude $= \frac{\bar{E}_1}{\bar{I}_h}$, i.e. it is of such a value as to give the watt component of current $\bar{I}_h$. The product of $\bar{E}_1$ and $\bar{I}_h$ gives the F and W losses, no-load copper loss + iron losses. The vector sum of $\bar{I}_\mu$ and $\bar{I}_h$ gives $\bar{I}_0$, the no-load current.

The current flowing through the resistance and leakage reactance of the stator is the vector sum of $\bar{I}_0$ and $\bar{I}_1'$ and equals $\bar{I}_1$. The resistance of the stator per phase $= R_1$ and its *leakage* coefficient of self inductance is L_1. It will simplify the results somewhat if we assume that the circuit connected across C and D, namely, L_m and R_0, is removed to the terminals A and B. This is equivalent to assuming the voltage across them is constant and the flux, therefore, constant. Actually the voltage across them varies slightly, due to the leakage impedance drop in R_1 and L_1. The circuit in Fig. 1.5 can, however, be reduced to a series circuit, with an effective resistance and effective reactance in series across the mains. Assuming, therefore, L_m and R_0 to be transferred across A and B, we have—

$$\bar{I}_1' = \frac{\bar{V}}{\sqrt{\left(R_1 + \frac{R_2'}{s}\right)^2 + (L_1 + L_2')^2\omega_0^2}} \quad . \quad . \quad . \quad (1.19)$$

where $R_2' = R_2 \times \alpha^2$ and $L_2' = L_2 \times \alpha^2$.

The current $\bar{I}_1'$ = rotor current referred to the stator—

$$\bar{I}_1' = \frac{\bar{V}}{(L_1 + L_2')\omega_0} \times \frac{(L_1 + L_2')\omega_0}{\sqrt{\left(R_1 + \frac{R_2'}{s}\right)^2 + (L_1 + L_2')^2\omega_0^2}} \quad (1.20)$$

but

$$\frac{(L_1 + L_2')\omega_0}{\sqrt{\left(R_1 + \frac{R_2'}{s}\right)^2 + (L_1 + L_2')^2\omega_0^2}} = \frac{\text{reactance}}{\text{impedance}} \quad . \quad (1.21)$$

$$= \sin\phi$$

$$\therefore \qquad \bar{I}_1' = \frac{\bar{V}}{(L_1 + L_2')\omega_0}\sin\phi \quad . \quad . \quad . \quad (1.22)$$

$$= I_{sci}\sin\phi \quad . \quad . \quad . \quad . \quad (1.23)$$

It is clear that $\bar{I}_1'$ is represented by the chord in the semicircle OB, for $OB = OA \sin\phi$ (Fig. 1.6).

$$OA = \frac{\bar{V}}{(L_1 + L_2')\omega_0}$$

Equation (1.23) obviously represents the polar equation of a circle.

It will be noted that the diameter of the circle

$$= \frac{\text{applied volts per phase}}{\text{effective reactance per phase}}$$

In any purely reactive coil, the current will lag by 90° behind the volts across the coil. The current represented by OA in Fig. 1.6 is drawn lagging 90° behind V, for its value $= \dfrac{\bar{V}}{(L_1 + L_2')\omega_0}$, and is determined by the effective inductance of the machine referred to the stator, i.e. $(L_1 + L_2')$.

The current OA is called the ideal short-circuit current, for it is the current per phase in the stator which would flow at standstill if the machine possessed reactance only.

As the load changes on the motor, the *load* component of the stator current $I_1' = \dfrac{I_2}{\alpha}$ moves over the circle and is represented by OB in Fig. 1.6. Since the motor possesses resistance also, the actual load component of the stator current at standstill will be represented by a vector OP, where—

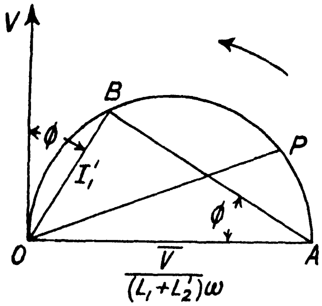


FIG. 1.6

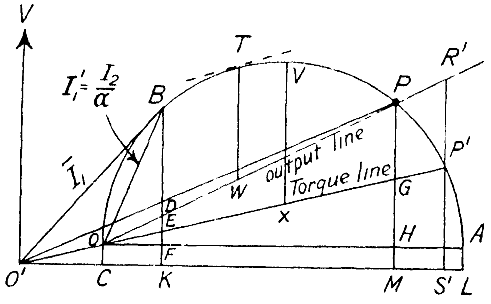


FIG. 1.7

$$OP = \frac{\text{applied volts per phase}}{\text{effective impedance of the machine at standstill}}$$

i.e. OP = actual load component of the stator current at standstill

$$= \frac{\bar{V}}{\sqrt{(R_1 + R_2')^2 + (L_1 + L_2')^2\omega_0^2}} \quad . \quad . \quad . \quad (1.24)$$

And the power factor $(s = 1)$ at *standstill*

$$= \cos \phi_s$$

$$= \frac{R_1 + R_2'}{\sqrt{(R_1 + R_2')^2 + (L_1 + L_2')^2\omega_0^2}} \quad . \quad . \quad (1.25)$$

The total primary current on load is obtained by adding vectorially the no-load current I_0 to I_1'.

Now, draw OC downwards (Fig. 1.7), parallel to $O'V$ and make OC = watt component of the stator current per phase necessary to supply the no-load losses. The diagram refers to one phase only and all quantities on the diagram are phase quantities.

$$OC = \frac{\text{total iron losses} + \text{friction and windage loss} + \text{no-load copper losses}}{\text{number of stator phases} \times \text{volts per phase on stator}}$$

The no-load losses are assumed to remain constant. This, of course, is not true. The flux per pole decreases on-load, due to the impedance drop in the stator, and actually the back e.m.f. E_1 decreases for $\mathbf{V} = + \mathbf{E}_1 + Z_1 \mathbf{I}_1$ (vector addition). Therefore, the iron losses in the stator decrease slightly, but as the slip increases with the load, the iron losses in the rotor increase, for they depend on the slip frequency. The friction and windage loss fall as the speed falls, i.e. when the slip increases. No important error is made by assuming the no-load losses to remain constant, i.e. *OC* is assumed constant. Now set off *CO′* parallel to *OA* and make it equal to the magnetizing current per phase. Then *O′O* is the no-load current. We thus transfer our origin of vectors from *O* to *O′*.

The point *O* on the circle represents the no-load point and the point *P* the short-circuit point. Draw from any point on the circle lines perpendicular to *O′L*. The stator current per phase is represented by lines drawn from the origin *O′* to the point in question.

At *B*, which represents some point on the circle corresponding to a certain load, *O′B* is the stator current per phase. *O′B* can be resolved into a component *O′K*, which is a wattless current, and *BK*, which is a watt current. It will be noted that, as the load increases, and the point *B* moves over the circle to the right, the wattless component of the stator current increases from *O′C*, at no-load, to *O′M* at standstill. This is due to the increase of stator and rotor leakage flux, which increases as the currents in stator and rotor increase. The watt component of *O′B*, i.e. *BK* represents the power input to the stator. The total power input at the point *B* $= m_1 \times V \times BK$, where m_1 = number of stator phases. The stator power factor at point *B* = cos *BO′V*.

It is, perhaps, desirable at this point to notice that high power factor, at any load, depends on making *O′K* as small as possible; that is, the magnetizing current per phase, *O′C*, must be as small as possible, and also the leakage fluxes of stator and rotor must be kept small. This means a small air-gap length, for the magnetizing current depends on the length of the air-gap, and there must be no saturation in the iron part of the circuit. The leakage fluxes will be discussed in a later chapter. At no-load the leakage fluxes are small, for the current is small, and so virtually all the flux per pole crosses the air-gap and enters the rotor. At standstill the leakage fluxes are large, and most of the flux per pole exists as leakage flux, i.e. flux which follows local paths and does not follow the useful path.

It will also be clear that maximum power factor is obtained when

$O'B$ is tangent to the semicircle, the maximum power factor being determined by the cosine of the angle the tangent to the circle from O' makes with $O'V$.

It is also clear that the maximum input to the motor is determined by the radius of the semicircle. Thus, it is important to make the effective leakage reactance of the machine low for high short-circuit current. Now at P, $O'P$ is the primary current, and its watt component is PM, which is the power component at standstill. The power input, at standstill, which is equal to $\bar{V} \times PM$ per phase is absorbed in stator and rotor copper losses. PH represents the copper losses due to the component of current OP. When the load component of the stator current is OB, it will be shown that the copper losses in stator and rotor are represented by DF, for $\frac{DF}{PH} = \frac{OB^2}{OP^2}$

For
$$\frac{DF}{PH} = \frac{OF}{OH} = \frac{OB \cos BOF}{OP \cos POH} \quad . \quad . \quad . \quad (1.26)$$

$$\cos BOF = \frac{OB}{OA} \text{ and } \cos POH = \frac{OP}{OA} \quad . \quad . \quad . \quad (1.27)$$

$$\therefore \quad \frac{DF}{PH} = \frac{OB^2}{OP^2} \quad . \quad . \quad . \quad . \quad . \quad (1.28)$$

Now let GH represent the watt component of the *stator* current per phase corresponding to the stator copper losses with primary current OP at short-circuit.

The PG must represent the watt component corresponding to the *rotor* copper loss per phase at standstill.

Join OG, then EF represents the watt component of current corresponding to the stator copper loss per phase with the load component OB, and DE represents the rotor copper loss per phase under the same conditions.

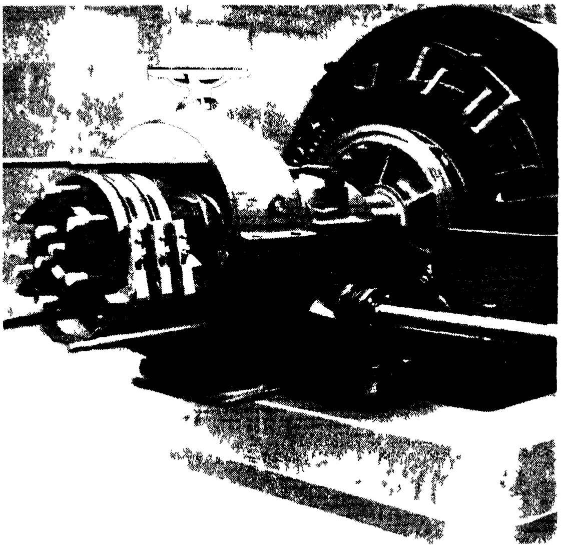
Since the watt input current per phase with current $O'B = BK$, FK corresponds to the no-load losses per phase, EF corresponds to the stator copper loss per phase with the load component OB, DE corresponds to the rotor copper loss per phase, and BD must represent the output.

Summing up, we have, with current $O'B$ in the stator—

$m_1\bar{V} \times BK$ = input to motor in watts (total)
$m_1\bar{V} \times BD$ = output in watts (total)
$m_1\bar{V} \times DE$ = rotor copper losses (total)
$m_1\bar{V} \times EF$ = stator copper losses due to load component of current OB

The power factor, when the stator current is $O'B = \cos BO'V$.

For any other value of the load, all we need to do is to draw a line from the point of the circle, corresponding to the load, perpendicular to OA, then the output is represented by the vertical intercept



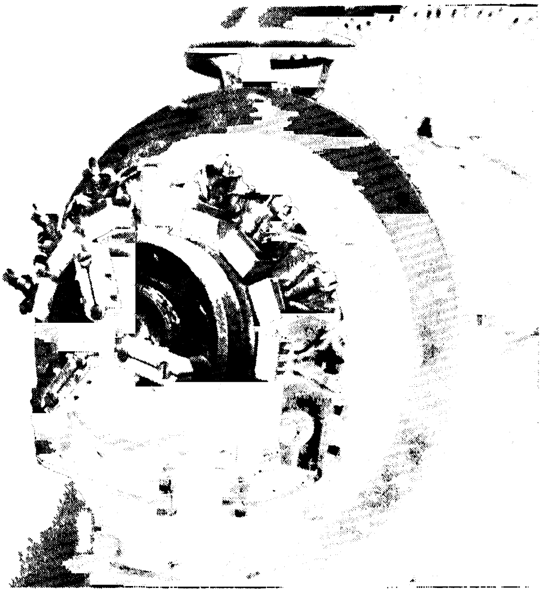


PLATE III

(*Upper*) BRUSH-LIFTING AND SHORT-CIRCUITING GEAR

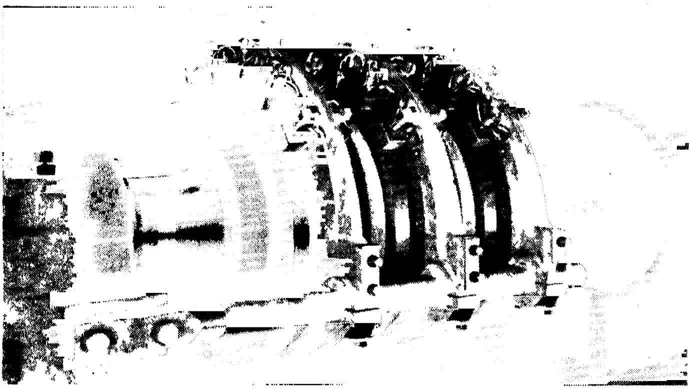
This is a 2430 h p 1500 r p m motor The end plate and enclosing cover have been removed

(*Courtesy English Electric Co Ltd*)

(*Lower*) CONTINUOUSLY-RATED EXTERNAL SLIP-RINGS WITH SHORT-CIRCUITING DEVICE ONLY

The enclosing cover has been removed

(*Courtesy English Electric Co , Ltd*)



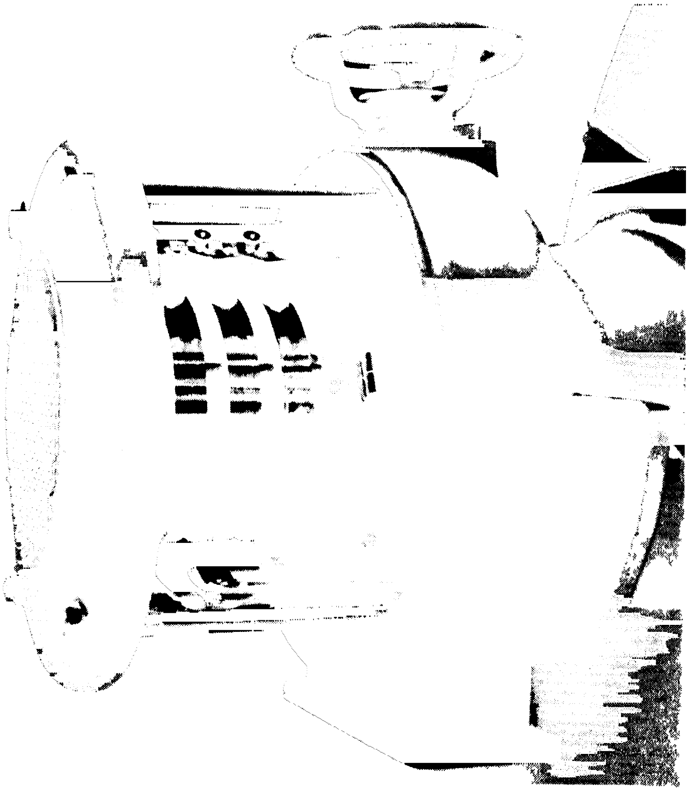


PLATE IV

(*Upper*) CONTINUOUSLY-RATED INTERNAL SLIP-RINGS FOR LARGE-OUTPUT MOTOR

(*Courtesy English Electric Co., Ltd.*)

(*Lower*) BRUSH-LIFTING AND SHORT-CIRCUITING GEAR ON EXTERNAL SLIP-RINGS OF MEDIUM-OUTPUT INDUCTION MOTOR

(*Courtesy English Electric Co., Ltd.*)

(F 591)

between the point on the circle and the line OP; the intercept between the lines OP and OG, made by this perpendicular, represents the rotor copper loss; and, similarly, the intercept, on the same line, made by the lines OG and OH represents the stator copper loss.

Since efficiency $= \frac{\text{output}}{\text{input}}$, it is equal to $\frac{BD}{BK}$

The slip is equal to the—

$$\frac{\text{rotor copper loss}}{\text{rotor input}} = \frac{DE}{BE} \quad . \quad . \quad . \quad (1.29)$$

for rotor input = output + rotor copper loss.

BE, therefore, represents the rotor input.

The line OP is the *output* line and OG represents the *torque* line, for it will be shown that the torque × synchronous speed, expressed in watts, equals the rotor input.

It should be stated that the locus of the primary current is a circle only provided the inductances do not vary with the current. Actually with large values of the currents, i.e. when the currents approach standstill values, the leakage flux paths may become saturated, and if this occurs L_1 and L_2' decrease, with the result that the current increases beyond the value obtained by assuming no saturation. The result is that the relation between applied volts per phase and short-circuit current is not linear, but bends up as shown in Fig. 1.8.

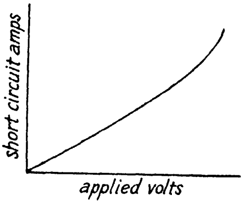


FIG. 1.8

This will obviously distort the circle diagram and give larger values for the starting torque than are calculated by assuming no saturation. Increase of eddy-current losses in conductors and iron due to large currents will increase the effective resistance, and this will tend to make the current curve decrease or bend down.

Torque and Slip Relations

We have shown that the input to the rotor per phase $= I_2^2 \frac{R_2}{s}$

The copper loss per phase $= I_2^2 R_2$.

The difference is the output, namely, per phase

$$= I_2^2 R_2 \frac{(1 - s)}{s} \quad . \quad . \quad . \quad . \quad (1.30)$$

If ω = angular velocity in radians per second

$= \omega_0(1 - s)$

and T = torque.

Then the output $= T\omega_0(1 - s) = m_2 I_2^2 R_2 \left(\frac{1 - s}{s}\right)$

$$T\omega_0 = \frac{I_2^2 R_2}{s} \times m_2 \quad . \quad . \quad . \quad (1.31)$$

m_2 = number of rotor phases.

Therefore, the torque, multiplied by the synchronous speed = the input to the rotor

$$= m_2 I_2^2 \frac{R_2}{s} \text{ watts.} \quad . \quad . \quad . \quad (1.32)$$

The torque, multiplied by the synchronous speed, expressed in watts, is usually referred to as "the torque in synchronous watts."

At standstill—

$$s = 1 \text{ and } T\omega_0 = m_2 I_2^2 R_2 \quad . \quad . \quad . \quad (1.33)$$

Therefore, at standstill, the torque in synchronous watts = loss in the rotor due to copper losses.

Also, from equation (1.31), we see that the slip—

$$s = \frac{\text{rotor copper loss}}{\text{rotor input}} \quad . \quad . \quad . \quad (1.34)$$

From equation (1.31) $T\omega_0 = \frac{I_2^2 R_2}{s} \times m_2$

$$= m_2 \times \frac{s E_2^2 \times R_2}{R_2^2 + s^2 L_2^2 \omega_0^2} \quad . \quad (1.35)$$

where $\omega_0 = 2\pi f$.

For small values of the slip s—

$$T\omega_0 \simeq \frac{m_2 s E_2^2}{R_2} \quad . \quad . \quad . \quad . \quad (1.36)$$

i.e. for small values of the slip, the torque in synchronous watts is proportional to the slip, and for light loads and small s, the relation between slip and torque is linear.

For large loads and large slips—

$$T\omega_0 \simeq \frac{m_2 E_2^2 R_2}{s L_2^2 \omega_0^2} \quad . \quad . \quad . \quad . \quad (1.37)$$

i.e. the relation between torque and slip is a rectangular hyperbola.

It will be noted that the torque is proportional to the square of E_2, i.e. proportional approximately to the square of the phase voltage applied, for E_2 is proportional to V.

Thus, it will be seen how important it is to see that the applied voltage is not low, when carrying out starting torque tests in which guarantees have been given.

It will be appreciated that the starting current with squirrel-cage rotors may vary from four to six times full-load current, and the power factor is low and there may be a large drop of voltage, especially when the motor is supplied from an alternator of small or moderate capacity.

Maximum Torque and Slip at which it Occurs

The torque in synchronous watts $= \dfrac{m_2 s\bar{E}_2^2 R_2}{R_2^2 + s^2 X_2^2} = \tau$ . . . (1.38)

where $X_2 = L_2\omega_0$.

$$\frac{d\tau}{ds} = \frac{m_2\bar{E}_2^2 R_2\{R_2^2 + s^2X_2^2\} - 2sX_2^2(m_2 s\bar{E}_2^2 R_2)}{(R_2^2 + s^2x_2^2)^2} \quad . \quad (1.39)$$

For a maximum $\dfrac{d\tau}{ds} = 0$.

$$\therefore \qquad R_2^2 + s^2X_2^2 = 2s^2X_2^2 \quad . \quad . \quad . \quad (1.40)$$

$$\therefore \qquad R_2^2 = s^2X_2^2 \quad . \quad . \quad . \quad . \quad (1.41)$$

$$\therefore \qquad s^2 = \frac{R_2^2}{X_2^2} \quad . \quad . \quad . \quad . \quad (1.42)$$

$$\text{i.e.} \qquad s_m = \pm\frac{R_2}{X_2} = \pm\frac{R_2}{L_2\omega_0} \quad . \quad . \quad . \quad (1.43)$$

That maximum torque occurs when the slip—

$$s_m = \pm\frac{R_2}{L_2\omega_0} \quad . \quad . \quad . \quad . \quad (1.44)$$

When $s_m = 1$, i.e. at standstill, maximum torque occurs when *rotor resistance per phase = rotor reactance per phase at standstill*

Thus, it is possible, by using a wound rotor machine and external resistance, to satisfy this relation for maximum torque at starting.

The maximum torque is obtained by substituting for s the value $\dfrac{R_2}{X_2}$

Substitution for s in equation (1.38) gives—

$$\text{max. torque} = \frac{m\bar{E}_2^2}{2X_2} \quad . \quad . \quad . \quad (1.45)$$

The maximum torque is independent of the *rotor* resistance, and is inversely proportional to the rotor reactance at standstill.

It will be seen, from Fig. 1.9, that the torque is plotted as function of the slip for various ratios of R_2 to X_2.

To obtain the necessary starting torque it is usual to use slip-ring type motors, when the large currents taken by squirrel-cage machines are prohibitive, and starting resistances are used to reduce the starting current. These resistances are cut-out as the speed rises.

Speed variation is also obtained by adding resistance to the rotor circuit as in rolling-mill motors, but it is obtained at greatly reduced efficiency, for the efficiency is always less than the speed as a percentage of synchronous speed.

The torque in synchronous watts = rotor input. With full-load current in the rotor at starting, we have full-load rotor loss. The slip $= \dfrac{\text{rotor copper loss}}{\text{rotor input}}$. The starting torque with full-load current, expressed as a percentage of full-load torque, is numerically equal to the slip at full load. The starting torque, expressed as a percentage of full-load torque, with a given current $\bar{I}$ in the rotor

$$= \left(\frac{\bar{I}}{\bar{I}_f}\right)^2 \times \text{per cent slip at full load} \qquad (1.46)$$

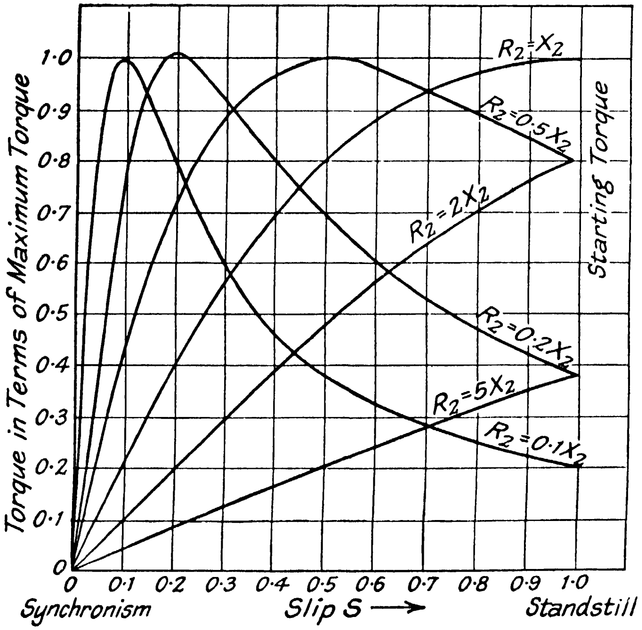


FIG. 1.9

where $\bar{I}_f$ = full-load rotor current, and the slip at full load

$$= \frac{\text{rotor copper loss at full load}}{\text{full-load torque in synchronous watts}}$$

The starting torque with rotor current $\bar{I}$ in synchronous watts = rotor copper loss with current $\bar{I}$.

$$\text{Therefore, } \frac{\text{starting torque with rotor current } I}{\text{full-load torque with rotor current } I_f}$$

$$= \frac{\text{rotor copper loss with current } I}{\text{rotor copper loss with current } I_f} \times \text{slip at full load}$$

$$= \left(\frac{\bar{I}}{\bar{I}_f}\right)^2 \times \text{slip at full load} \qquad (1.47)$$

The starting torque can be increased by increasing the current or by increasing the slip. A limit is set to the slip by rotor heating and the reduction in efficiency produced.

A large starting current of low power factor has a bad effect on the voltage regulation of the power system, and so for high starting torque a compromise must be made.

It is due to this effect on voltage regulation of the supply system that squirrel-cage motors are limited in size in Britain, and for outputs of over 5 b.h.p. it is usual to use rotors of the wound type, i.e. slip-ring type motors. In Canada and America this is not so and one finds the squirrel-cage type used for outputs of 200 h.p. and more. With the slip-ring type, full-load torque is obtained with approximately full-load current, and the cost of the energy used in starting is much less than with the squirrel-cage type. This may be an important matter where frequent starting of large numbers of motors is made.

The torque of a squirrel-cage machine is much more uniform than that of a wound rotor machine. The squirrel-cage conductors act powerfully to reduce flux variation and so tend to preserve a state of uniform torque.

With a wound rotor machine, the torque varies with varying position of the rotor with respect to the stator, due to varying impedance and varying value of zigzag leakage. It is important to keep the air-gap reluctance as constant as possible by choosing suitable slot ratios, and to keep the flux distribution as constant as possible both in value and in spacial distribution.

Construction for Slip

The slip $= \dfrac{\text{rotor copper loss}}{\text{rotor input}} = \dfrac{DE}{BE}$, in Fig. 1.7 for point B on the circle.

It is not easy to measure this ratio accurately from the diagram, so another construction is given in Fig. 1.10.

Join LP in Fig. 1.10 and produce LP to Q, and from Q draw QE at right angles to NG, the torque line. P is the standstill point.

Note: Q is any point on LP produced.

The triangles $B'NX$ and LCE are similar. So also are the triangles LQE and MNX.

$$\therefore \qquad \frac{B'X}{NX} = \frac{LE}{CE} \qquad (1.48)$$

and

$$\frac{LE}{EQ} = \frac{MX}{NX} \qquad (1.49)$$

$$\therefore \qquad CE = \frac{MX}{B'X} \times EQ \qquad (1.50)$$

The slip $= \dfrac{MX}{B'X}$; but this is proportional to CE, for QE is constant.

Therefore, CE is proportional to the slip for the primary current OB' in Fig. 1.10. Let QE represent 100 per cent slip, then the ratio $\frac{CE}{QE}$ gives directly to the same scale the percentage slip for the stator current OB'. C is the point of intersection of $B'L$ and QE. It is thus clear that, to obtain the slip directly, we must join the points on the circle representing the different primary currents to the point L. Then the intersection of these lines with QE gives the percentage slip directly, the percentage slip being given by the length of the line from the point of intersection to E.

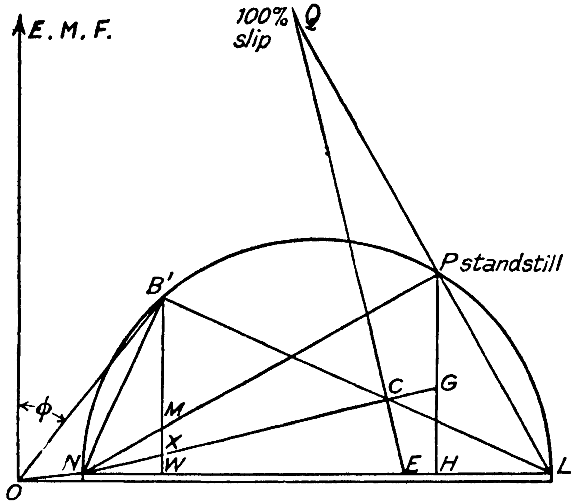


FIG. 1.10

It is interesting to note also that, at the point P', in Fig. 1.7, which is the point of intersection of the torque line OG with the circle, corresponds to infinite slip. $P'R'$ represents the mechanical power, as for any other point on the circle, but it is now a power *input*. This is used to supply the rotor copper loss. The primary power input, corresponding to $P'S'$, is used in primary copper losses, since the iron loss is small at this point.

Output and Slip for Maximum Output

$$\text{The torque, in synchronous watts} = \frac{m_2 s\bar{E}_2{}^2 R_2}{R_2{}^2 + s^2 X_2{}^2}$$

$$\text{The output} = \frac{m_2 s\bar{E}_2{}^2 R_2}{\omega_0(R_2{}^2 + s^2 X_2{}^2)} \times \omega_0(1 - s)$$

$$= \frac{m_2 \bar{E}_2{}^2 R_2 \times s(1 - s)}{R_2{}^2 + s^2 X_2{}^2} \text{ watts} \quad . \quad . \quad (1.51)$$

$$= \frac{C \times s(1 - s)}{R_2{}^2 + s^2 X_2{}^2}$$

where $C = m_2E_2^2R_2$.

$$\frac{d\,(\text{output})}{ds} = \frac{C(1-2s)(R_2^2+s^2X_2^2) - 2sX_2^2Cs(1-s)}{R_2^2+s^2X_2^2} \qquad (1.52)$$

for a maximum—

$$(1-2s)(R_2^2+s^2X_2^2) = 2s^2X_2^2(1-s)$$

i.e. $$R_2^2 + s^2X_2^2 - 2sR_2^2 - 2s^3X_2^2 = 2s^2X_2^2 - 2s^3X_2^2$$

i.e. $$R_2^2 - 2sR_2^2 = s^2X_2^2$$

$$\therefore \qquad s = \frac{-2R_2^2 \pm \sqrt{4R_2^4 + 4R_2^2X_2^2}}{2X_2^2} \quad . \quad . \quad (1.53)$$

$$= \frac{-R_2^2}{X_2^2} \pm \frac{R_2}{X_2}\sqrt{\frac{R_2^2}{X_2^2}+1}$$

$$= -a^2 \pm a\sqrt{a^2+1} \quad . \quad . \quad . \quad . \quad (1.54)$$

where $a = \dfrac{R_2}{X_2}$

The plus sign before the radical refers to motor action, the minus sign to generator action.

Equation (1.54) gives the slip at which maximum output occurs. To find the maximum output of the motor, we must substitute the value for s, obtained in equation (1.54), in equation (1.51).

We obtain for the maximum output—

$$\frac{m_2E_2^2}{2X_2}(\sqrt{a^2+1}-a) \quad . \quad . \quad . \quad (1.55)$$

$$= \frac{m_2E_2^2}{2X_2^2}(Z_2 - R_2) \text{ watts} \quad . \quad . \quad . \quad (1.56)$$

where Z_2 = impedance of the rotor per phase at standstill.

$$= \sqrt{R_2^2 + X_2^2} \quad . \quad . \quad . \quad . \quad (1.57)$$

The following table shows the effect of rotor resistance on maximum output and also the slip at which maximum output and maximum torque occurs—

$a = \dfrac{R_2}{X_2}$	Maximum Output		Maximum Torque	
	Relative Magnitude of Maximum Output $\sqrt{a^2+1}-a$	Slip at Maximum Output	Relative Magnitude at Maximum Torque	Slip at $T_{\max}$
5	0·099	0·495	1	5
2	0·236	0·472	1	2
1	0·414	0·414	1	1
0·5	0·618	0·309	1	0·5
0·2	0·820	0·164	1	0·2
0·1	0·905	0·090	1	0·1

Fig. 1.11 shows the mechanical output at different speeds as a function of rotor resistance R_2.

In deducing the equations for maximum torque and maximum output, $\bar{E}_2$ was assumed constant. The equations can be calculated in terms of $\bar{V}$ the supply volts.

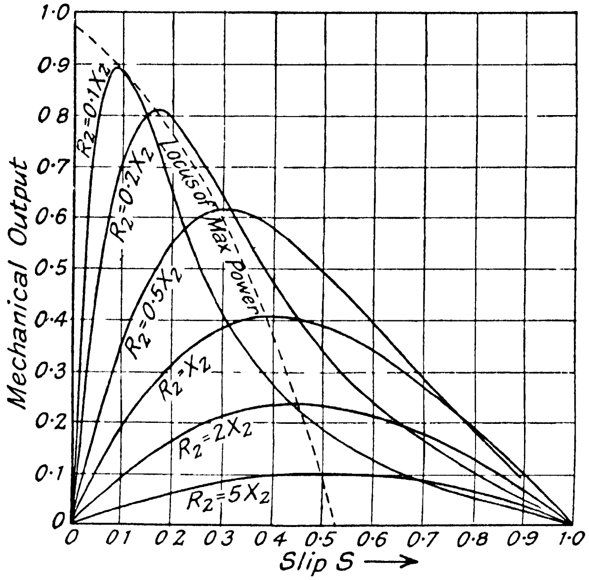


Fig. 1.11

Approximately

$$V = I_1'\left\{(r_1 + jx_1) + \left(\frac{r_2'}{s} + jx_2'\right)\right\} \quad . \quad . \quad (1.58)$$

V = supply volts per phase

r_1 and x_1 = stator resistance and reactance respectively per phase

r_2' = rotor resistance referred to the stator per phase

x_2' = rotor reactance referred to the stator per phase

I_1' = load component of stator current per phase

$$V = I_1'\left\{\left(r_1 + \frac{r_2'}{s}\right) + j(x_1 + x_2')\right\}$$

$$s\bar{V} = \bar{I}_1\sqrt{(r_1 s + r_2')^2 + s^2(x_1 + x_2')^2} \quad . \quad . \quad . \quad (1.59)$$

Also

$$s\bar{E}_2 = \bar{I}_2\sqrt{r_2^2 + s^2 x_2^2}$$

$$= \frac{\bar{I}_1' \times \alpha\sqrt{(r_2')^2 + s^2(x_2')^2}}{\alpha^2}$$

and

$$\alpha s\bar{E}_2 = \bar{I}_1'\sqrt{(r_2')^2 + s^2(x_2')^2} \quad . \quad . \quad . \quad (1.60)$$

where α = ratio of transformation.

$$\frac{I_2}{\alpha} = I_1', \; r_2' = r_2 \times \alpha^2, \text{ and } x_2' = x_2 \times \alpha^2$$

$$\bar{E}_2^2 = \frac{\bar{V}^2}{\alpha^2} \times \frac{((r_2')^2 + s^2(x_2')^2)}{(r_1 s + r_2')^2 + s^2(x_1 + x_2')^2} \quad . \quad . \quad . \tag{1.61}$$

The torque in synchronous watts—

$$= \frac{m_2 \times s\bar{E}_2^2 \times r_2}{r_2^2 + s^2 x_2^2}$$

$$= \frac{m_2 \times s\bar{E}_2^2 \times \dfrac{r_2'}{\alpha^2}}{\dfrac{((r_2')^2 + s^2(x_2')^2)}{\alpha^4}}$$

(m_2 assumed = m_1)

$$= \frac{m_2 \times \alpha^2 s\bar{E}_2^2 \times r_2'}{(r_2')^2 + s^2(x_2')^2} \quad . \quad . \quad . \tag{1.62}$$

Substituting from equation (1.61) for $\bar{E}_2$ in terms of $\bar{V}$, we have torque in synchronous watts

$$= \frac{m_2 \bar{V}^2 \times s r_2'}{(r_1 s + r_2')^2 + s^2(x_1 + x_2')^2} \cdot \quad . \quad . \tag{1.63}$$

and since the synchronous speed is 2π × synchronous revolutions per second—

$$\omega_0 = \frac{4\pi f}{p}$$

where p = number of poles.

The torque—

$$= \frac{m_2 \bar{V}^2 \times p \times s r_2'}{4\pi f (r_1 s + r_2')^2 + s^2(x_1 + x_2')^2} \quad . \quad . \tag{1.64}$$

If we differentiate equation (1.64) with regard to the slip and equate to zero, we get the slip at maximum torque

$$= \frac{r_2'}{\sqrt{r_1^2 + (x_1 + x_2')^2}} \quad . \quad . \quad . \tag{1.65}$$

and the maximum torque

$$= \frac{p\bar{V}^2}{4\pi f} \times \frac{m_2}{2\{r_1 + \sqrt{r_1^2 + (x_1 + x_2')^2}\}} \quad . \quad . \tag{1.66}$$

In this determination of the maximum torque we developed the relation between $\bar{E}_2$ and $\bar{V}'$, and neglected the effect of the no-load current on the impedance drop. Since $\bar{V}$ appears as a square, any error produced by neglecting I_0 will be exaggerated. In all equations for torque and power where V occurs, the vector difference of V and the primary impedance drop due to the no-load current should appear.

Results, sufficiently approximate, will be obtained by assuming the $\boldsymbol{V} - \boldsymbol{I}_0 x_1$ for $\bar{V}$ in the equations.

We have shown that the torque in synchronous watts = input to the rotor

$$= m_2 E_2 I_2 \cos \phi_2$$

Now $$E_2 = 4{\cdot}44 K_2 K_4 \times T_2 \times \hat{\phi} \times f \times 10^{-8}$$

$$f = \frac{p \times \text{revs per min. (syn.)}}{120}$$

and $$T_2 = \frac{Z_2}{2m_2}$$

where Z_2 = total number of conductors on the rotor

Therefore, torque in synchronous watts

$$= m_2 \times 4{\cdot}44 \times K_2 K_4 \times \frac{Z_2}{2m_2} \times \frac{\hat{\phi} \times p \times \text{r.p.m.}_{\text{syn}} \times I_2 \cos \phi_2}{120 \times 10^8} \quad (1.67)$$

The torque in lb-ft

$$= \frac{4{\cdot}44 K_2 K_4 \times Z_2 \times \hat{\phi} \times p \times I_2 \cos \phi_2 \times 7{\cdot}04}{240 \times 10^8} \quad . \ (1.68)$$

$$= 0{\cdot}1303 \times 10^{-8} \times p \times \hat{\phi} \times Z_2 I_2 \cos \phi_2 \times K_2 K_4 \quad (1.69)$$

This is a very useful relation.

p = number of poles

T_2 = turns per rotor phase in series

$\hat{\phi}$ = maximum flux per pole

Z_2 = total conductors on the rotor

I_2 = r.m.s. value of rotor current and

$\cos \phi_2$ = cosine of the angle between E_2 and I_2.

The torque of a polyphase motor in ft-lb = a constant × total flux × total rotor ampere-conductors × $\cos \phi_2$.

Note that $$\frac{\text{torque in lb-ft} \times \text{syn. r.p.m.}}{7{\cdot}04} = \text{syn. watts.}$$

Therefore, torque in lb-ft × $\text{r.p.m.}_{\text{syn}}$

$$= 7{\cdot}04 \times \text{power represented by the syn. watts} \quad . \ (1.70)$$

Equivalent Stator and Rotor Quantities

Let T_1 = turns in series on the stator per phase and

T_2 = turns in series on the rotor per phase.

Then $$\hat{E}_1 = 4{\cdot}44 \times T_1 \times K_1K_3 \times \hat{\phi} \times f \times 10^{-8} \qquad (1.71)$$

$$\hat{E}_2 = 4{\cdot}44 \times T_2 \times K_2 \times K_4 \times \hat{\phi} \times f \times 10^{-8} \qquad (1.72)$$

$$\frac{\hat{E}_1}{\hat{E}_2} = \frac{T_1 \times K_1 \times K_3}{T_2 \times K_2 \times K_4} = \frac{T_1 f_1}{T_2 \times f_2} \qquad (1.73)$$

$f_1 = K_1 \times K_3$ = winding factor for the fundamental wave

= product of distribution factor and coil span factor

f_2 = the same for the rotor.

Now $$K_1 = \frac{\sin q_1 \frac{\lambda}{2}}{q_1 \sin \frac{\lambda}{2}} \text{ and } K_3 = \cos\frac{\varepsilon}{2} \qquad (1.74)$$

q_1 = number of slots *per pole per phase* in the stator

q_2 = number of slots per pole per phase for the rotor

λ = electrical slot-pitch angle

$$= \frac{180^\circ}{\text{number of slots per pole}} \qquad (1.75)$$

λ' = electrical slot pitch angle in rotor

$$f_2 = \frac{\sin q_2 \frac{\lambda'}{2}}{q_2 \sin \frac{\lambda'}{2}} \times \cos\frac{\varepsilon'}{2} \qquad (1.76)$$

and ε and ε' = *deficiency* from full pitch in electrical degrees, of stator and rotor coils respectively.

Instead of the factor $\cos\frac{\varepsilon}{2}$ we may use $\sin\frac{\beta}{2}$, where β = actual span of the coil in electrical degrees.

For the *n*th *harmonic*, we have—

$$f_n = \frac{\sin qn \frac{\lambda}{2}}{q \sin n \frac{\lambda}{2}} \times \cos n\frac{\varepsilon}{2} \qquad (1.77)$$

where f_n = winding factor for the *n*th harmonic.

Rotor Current Referred to Stator

Since the ampere-turns (effective), due to the load component in the stator must equal the ampere-turns effective of the rotor, we have—

$$m_1 T_1 I_1' K_1 K_3 = m_2 T_2 I_2 K_2 K_4 \qquad . \quad . \quad (1.78)$$

$$\therefore \qquad I_1' = \frac{m_2}{m_1} \times \frac{T_2}{T_1} \times \frac{K_2 K_4 \bar{I}_2}{K_1 K_3} = \frac{\bar{I}_2}{\alpha} \qquad . \quad . \quad (1.79)$$

where
$$\alpha = \text{ratio } \frac{m_1 T_1 K_1 K_3}{m_2 T_2 K_2 K_4} \qquad . \quad . \quad . \quad (1.80)$$

The load component of current in the stator is given by equation (1.79) $= \frac{\bar{I}_2}{\alpha}$.

Rotor Impedance Referred to Stator

Let Z_2' = equivalent rotor impedance referred to the stator.

$$Z_2' = \frac{\bar{E}_2'}{\bar{I}_1'} = \bar{E}_2 \times \frac{T_1 \times f_1 \times m_1 T_1 \times f_1}{T_2 \times f_2 \times m_2 T_2 \times f_2 \times \bar{I}_2} \qquad . \quad (1.81)$$

$$= \frac{\bar{E}_2}{\bar{I}_2} \times \frac{m_1 T_1^2 \times f_1^2}{m_2 T_2^2 \times f_2^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.82)$$

$$= Z_2 \times \frac{m_1 T_1^2 \times f_1^2}{m_2 T_2^2 \times f_2^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.83)$$

If $m_1 - m_2$, then—

$$Z_2' = Z_2 \times \left(\frac{T_1 \times f_1}{T_2 \times f_2}\right)^2 \quad . \quad . \quad . \quad (1.84)$$

$T_1 \times f_1$ is called the *effective* number of stator turns per phase and $T_2 \times f_2$ the number of *effective* rotor turns per phase.

The resistances and reactances of the rotor are referred to the stator in the same way as the impedances.

Geometrical Relations from the Circle Diagram

Most of the quantities we have dealt with can be read directly from the circle diagram. In Fig. 1.12, the maximum power is proportional to *GH*, where *GH* is the largest vertical intercept between the circle and the output line. Draw a line tangential to the circle, and parallel to *JF*. This line will touch the circle at *G*. Then the—

$$\text{max. h.p.} = \frac{m_1 \times \bar{V} \times GH}{746} \quad . \quad . \quad . \quad (1.85)$$

$$GH \sin \phi = GM$$

$$GH = \frac{GM}{\sin \phi} = \frac{GP - MP}{\sin \phi} \quad . \quad . \quad (1.86)$$

$$= \frac{GP - JP \cos \phi}{\sin \phi} \quad . \quad . \quad . \quad (1.87)$$

Since $$GP = JP$$

and $$JF = 2GP \sin \phi \quad . \quad . \quad . \quad . \quad (1.88)$$

$$\therefore \quad GP = \frac{JF}{2 \sin \phi} \quad . \quad . \quad . \quad . \quad (1.89)$$

$$\therefore \quad GH = \frac{JF(1 - \cos \phi)}{2 \sin^2 \phi} \quad . \quad . \quad . \quad . \quad (1.90)$$

$$= \frac{JF(1 - \cos \phi)}{2(1 - \cos^2 \phi)} = \frac{JF}{2(1 + \cos \phi)} \quad . \quad (1.91)$$

and $$\text{max. h.p.} = \frac{m \times \bar{V} \times JF}{2(1 + \cos \phi) \times 746} \quad . \quad . \quad . \quad (1.92)$$

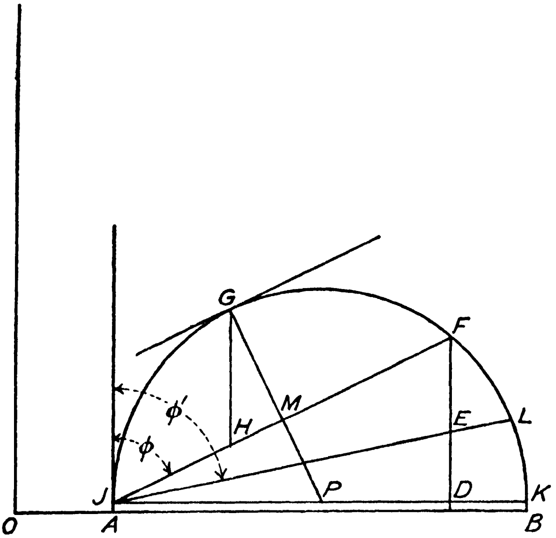


FIG. 1.12

where $\cos \phi$ = the power factor at short-circuit of the current JF and $JF = JK \sin \phi$.

In dealing with the motor, JK, the ideal short-circuit current is usually known, or JF is known from test.

In a similar manner the maximum torque is proportional to the maximum vertical intercept between the circle and the torque line JL.

$$\text{max. torque} = \frac{m_1 V \times JL}{2(1 + \cos \phi')} \quad . \quad . \quad (1.93)$$

and $$JL = JK \sin \phi' \quad . \quad . \quad . \quad (1.94)$$

where ϕ' is the angle JL makes with the vertical.

Torque and Slip Relations

If one neglects the stator copper loss, the diagram takes the simple form shown in Fig. 1.13.

Join DL, where L = standstill point, and produce DL to meet the vertical through M in E.

$\triangle$'s DQM and MOA are similar and so are also the $\triangle$'s DEM and MRA.

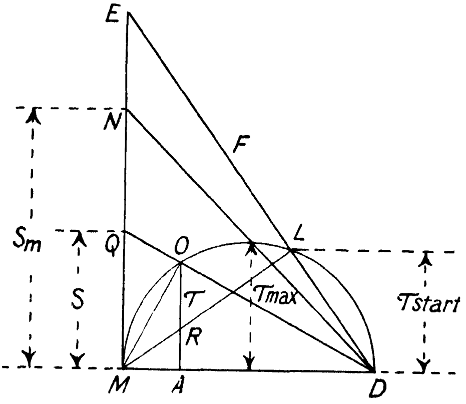


FIG. 1.13

$$\therefore \qquad \frac{RA}{MA} = \frac{MD}{ME} \text{ and } \frac{MQ}{MD} = \frac{MA}{OA} \qquad \text{(1.95)}$$

$$\therefore \qquad MQ \times OA = RA \times ME \qquad \text{(1.96)}$$

$$\therefore \qquad MQ = \frac{RA}{OA} \times ME \qquad \text{(1.97)}$$

but $RA \propto$ loss in the rotor (neglecting stator loss), $OA \propto$ torque.

$$\frac{RA}{OA} = \text{slip} = \frac{MQ}{ME} \qquad \text{(1.98)}$$

where ME represents the slip at standstill.

Let $\tau_{\max}$ = maximum torque

s_m = slip at maximum torque

and τ = torque at any other slip $s = OA$.

Then
$$\frac{\tau}{AD} = \frac{MQ}{MD} = \frac{MQ}{MN} = \frac{s}{s_m} \qquad \text{(1.99)}$$

$$\frac{MA}{\tau} = \frac{s}{s_m} \text{ or } MA = \tau \times \frac{s}{s_m}$$

$$MD = MA + AD = \tau \left\{ \frac{s}{s_m} + \frac{s_m}{s} \right\} \qquad \text{(1.100)}$$

$$\tau_{\max} = \tfrac{1}{2}MD$$

$$\therefore \qquad \frac{\tau}{\tau_{\max}} = \frac{2}{\dfrac{s}{s_m} + \dfrac{s_m}{s}} \qquad . \quad . \quad . \quad (1.101)$$

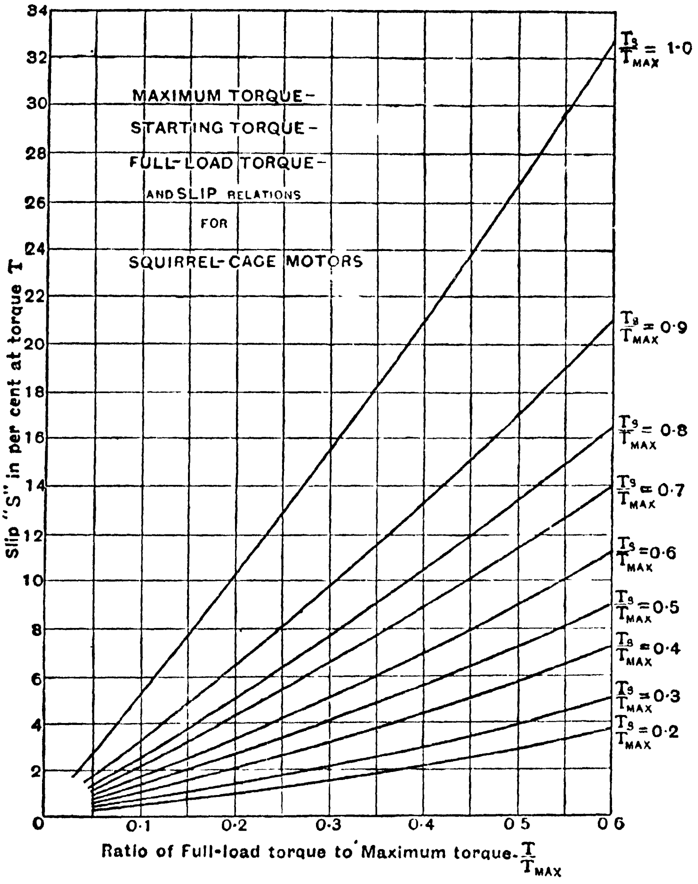


Fig. 1.14

At the starting point $s = 1$

$$\therefore \qquad \frac{\tau_{\text{start}}}{\tau_{\max}} = \frac{2}{\dfrac{1}{s_m} + s_m} \qquad . \quad . \quad . \quad (1.102)$$

These relations are very useful and to show their utility we shall consider an example. A squirrel-cage motor is to be designed for an overload torque capacity of 150 per cent and 20 per cent margin, with a starting torque of 125 per cent of full-load torque. It is required to find the slip at full load.

We have—

$$\frac{\tau}{\tau_{\max}} = 0{\cdot}4 \times 0{\cdot}8 = 0{\cdot}32$$

$$\frac{\tau_{\text{start}}}{\tau_{\max}} = 1{\cdot}25 \times 0{\cdot}32 = 0{\cdot}4$$

From the above equations $s_m = 21$ per cent and $s = 3{\cdot}68$ per cent. Again it is required to find what starting torque can be obtained from a squirrel-cage motor at 100 per cent overload torque and a slip of $2\frac{1}{2}$ per cent at full load.

$$\frac{\tau}{\tau_{\max}} = 0{\cdot}5 \times 0{\cdot}9, \text{ allowing 10 per cent margin}$$

From the equation above, $s_m = 10{\cdot}5$ per cent and

$$\frac{\tau_{\text{start}}}{\tau_{\max}} = 0{\cdot}208$$

i.e.

$$\tau_{\text{start}} = \frac{0{\cdot}208}{0{\cdot}45} \text{ full-load torque}$$

$$= 0{\cdot}46 \text{ of full-load torque}$$

The relationship between slip and torque is shown in Fig. 1.14.

It is advisable in using these curves to allow a margin of 10 to 15 per cent on the guaranteed overload torque capacity, on account of the neglect of the stator copper losses and ripples on torque-slip.

The Efficiency

If we ignore all losses, except the rotor copper loss, and since efficiency $= \dfrac{\text{output}}{\text{input}}$ then the output per phase

$$= \bar{E}_2 \bar{I}_2 \cos \phi_2 - \bar{I}_2{}^2 R_2$$

The rotor input per phase

$$= \bar{E}_2 \bar{I}_2 \cos \phi_2$$

Therefore, the efficiency, neglecting all losses but rotor copper losses

$$= 1 - \frac{\bar{I}_2{}^2 R_2}{\bar{E}_2 \bar{I}_2 \cos \phi_2}$$

$$= 1 - s \quad . \quad . \quad . \quad . \quad . \quad (1.103)$$

Therefore, the efficiency is always less than the speed as a percentage of synchronous speed, since there are other losses in addition to the rotor copper loss, for

$$1 - s = 1 - \frac{(\omega_0 - \omega)}{\omega_0} = \frac{\omega}{\omega_0} \quad . \quad . \quad (1.104)$$

The Dispersion Co-efficient

If we neglect the no-load watt component of current, our circle diagram assumes the simple form of Fig. 1.15.

So long as the voltage is maintained constant at the stator terminals, the flux linked with the stator winding is constant, neglecting resistance drop. With locked rotor the flux is almost entirely leakage flux. At synchronous speed, there is no rotor current and the stator flux is free to enter the rotor. Hence *OB* represents the current necessary to drive the flux through the leakage paths, while *OA* represents the current required to send the same flux through the useful path and the leakage path in parallel.

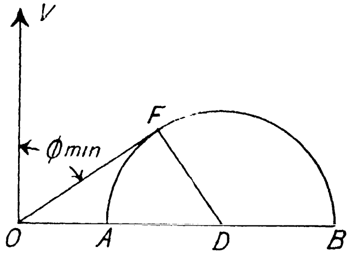


Fig. 1.15

The leakage factor of any circuit

$$= \frac{\text{total flux}}{\text{useful flux}}$$

$$= \frac{\text{useful flux} + \text{leakage flux}}{\text{useful flux}}$$

$$= 1 + \frac{\text{leakage flux}}{\text{useful flux}} \quad . \quad . \quad . \quad . \quad . \quad (1.105)$$

Now flux $= \dfrac{\text{m.m.f.}}{\text{reluctance}} = \text{m.m.f.} \times \text{permeance}$; and the currents are proportional to the m.m.f.s.

$$\therefore \frac{OB}{OA} = \frac{\text{reluctance of leakage paths}}{\text{joint reluctance of useful paths + leakage paths}}$$

$$= \frac{\text{permeance of leakage path + permeance of useful path}}{\text{permeance of leakage path}}$$

$$= 1 + \frac{\text{permeance of useful path}}{\text{permeance of leakage path}}$$

$$= 1 + \frac{\text{useful flux}}{\text{leakage flux}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.106)$$

Also $$\frac{OB}{OA} = \frac{OA + AB}{OA} = 1 + \frac{AB}{OA} \quad . \quad . \quad (1.107)$$

$$\therefore \quad \frac{AB}{OA} = \frac{\text{useful flux}}{\text{leakage flux}}$$

The leakage factor $= 1 + \dfrac{OA}{AB}$ . . . . . (1.108)

The symbol "σ" has been used to represent another leakage ratio, called the "dispersion coefficient."

Behn-Eschenberg uses for the dispersion coefficient the ratio $\dfrac{OA}{OB} = \dfrac{\text{leakage flux}}{\text{total flux}}$. We shall adopt this definition of the dispersion coefficient, i.e. $\sigma = \dfrac{\text{magnetizing current per phase}}{\text{ideal short-circuit current per phase}}$.

Hobart and others use another coefficient,

i.e. $$\frac{OA}{AB} = \frac{\text{leakage flux}}{\text{useful flux}} = v$$

The leakage factor $= 1 + v$

$$= 1 + \frac{OA}{OB - OA} = 1 + \frac{\sigma}{1 - \sigma}$$

$$= \frac{1}{1 - \sigma} \quad . \quad . \quad . \quad . \quad . \quad (1.109)$$

The value of the dispersion coefficient has a most important influence on the power factor of the motor. If high power factor is needed, then a small value of σ is indicated. That is we must have a small magnetizing current OA, and a large ideal short-circuit current. Now OA is determined principally by the length of air-gap used, for the magnetizing current is proportional to the gap length.

The air-gap length must be kept small, indeed as small as is mechanically permissible, and there must be no saturation in the iron. Thus, one finds in these motors air-gaps of the order of 0·3 mm to 1 mm, depending on the size of motor, and, of course, in very small motors, whose output is but a few watts, the gap length may be as small as 0·006 in.

Large ideal short-circuit current means, of course, small leakage flux reactance, since the ideal short-circuit current $= \dfrac{\text{flux per pole}}{\text{leakage flux per pole per ampere}}$ it is clear that the dispersion coefficient depends chiefly on the ratio of $\dfrac{\text{length of air-gap}}{\text{pole pitch}}$.

On a given diameter, and with a given core length, the pole area, and therefore the flux per pole, is dependent on the pole pitch; the flux per pole will be dependent on the pole pitch

As the pole pitch decreases, as the number of poles increases, the number of slots per pole per phase is decreased, and the winding is more concentrated, the number of conductors per pole per phase is increased for a given supply voltage, and hence the leakage reactance is increased. Thus, one must expect both the magnetizing current *OA* to increase and the ideal short-circuit current to decrease as the number of poles in the machine is increased.

Thus, it is natural for high-speed induction motors, i.e. machines with a small number of poles to have small values for the dispersion coefficient. There is thus no difficulty in obtaining high power factor in high-speed motors; but as the number of poles increases, and the speed is low, σ progressively increases, and it becomes difficult to obtain high power factor. Now looking at our diagram, Fig. 1.15, we get maximum power factor when *OF* is tangent to the circle.

Maximum power factor

$$= \cos FOV = \frac{FD}{OD} = \frac{AB}{2 \times OD}$$

$$= \frac{AB}{2OA + AB}$$

$$= \frac{1}{2v + 1}$$

$$= \frac{1}{\frac{2\sigma}{1 - \sigma} + 1} = \frac{1 - \sigma}{1 + \sigma} \quad . \quad . \quad . \quad (1.110)$$

where $v = \frac{OA}{AB}$

Thus, the maximum power factor—

$$= \frac{1 - \sigma}{1 + \sigma} \quad . \quad . \quad . \quad . \quad (1.111)$$

The influence of σ on the power factor is thus seen to be all-important. In machines of two-, four-, and six-poles, it is easy to obtain power factors of 90 per cent and over, but when σ becomes of the order of 0·1 or more, it is clear that the power factor becomes lower than 90 per cent, indeed, for $\sigma = 0{\cdot}1$, $\cos \phi_{\max} = \frac{0{\cdot}9}{1{\cdot}1} = 81{\cdot}75$ per cent and the power factor at full load is much lower.

Now air-gaps which are too small, give trouble due to zigzag leakage fluxes and other causes, so the gap should be as large as is compatible with good power factor. These points will receive further consideration later, but it is desirable to point out that unduly short air-gaps are not necessary in high-speed motors, and in some cases, in the author's experience, machines have failed mechanically due to this desire to obtain unduly large power factors by using very small gap lengths.

The following test results, taken at the Electrical Engineering Laboratory of the University of British Columbia, illustrate the characteristic curves of the three-phase induction motor, and may be helpful to the student. They were taken by one of my former students, Mr. Walter Lind.

THREE-PHASE INDUCTION MOTOR: DIRECT DETERMINATION OF CHARACTERISTICS

Apparatus. Machine No. 3 and No. 17. (C.G.E. induction motor and d.c. generator set.) Ratings—

Induction motor	D.C. generator
60 c/s	10 kW
38·5 A	80 A
220 V	125 V (at no-load and full load)
1800 r.p.m. at no-load	1800 r.p.m.
Exciter voltage 125 V	

Procedure. The d.c. generator was used to load the induction motor. The slip was determined with a stroboscopic disc for light loads.

Observations—

Induction Motor										D.C. Generator			
Volts		Amperes		Watts									
ϕ_1	ϕ_3	ϕ_1	ϕ_3	W_1	W_2	$W_1 + W_2$ (Input)	W_2/W_1	P F	Slip (r.p.m.)	Term Volts	Arm Amperes I_a	Field Amperes I_{sh}	Watts Output
231	231	11 3	12 0	1590	− 680	910	− 0 427	0 23	3 5	—	0 0	—	—
230	232	11 5	12 2	1850	450	1 400	− 0 243	0 33	6 2	120 5	0 0	1 69	203
229	229	14 6	15 6	3120	775	3 895	0 248	0 69	22 0	120 5	20 0	1 73	2 644
224	226	20 2	22 0	4450	2200	6 650	0 495	0 87	40 0	121 0	41 1	1 74	5 295
226	227	26 8	29 3	5980	3450	9 430	0 577	0 91	60 0	121 5	59 9	1 75	7 714
226	226	35 7	38 0	7900	4950	12 850	0 626	0 93	80 0	121 0	81 6	1 75	10 505
225	227	43 7	46 2	9580	6350	15 930	0 663	0 94	110 0	121 5	98 5	1 82	12 780

Resistance Measurements (after the load test)—

Armature			Series Field			Interpoles			Total $R_a + R_s + R_i$ ohms
Volts	Amperes	R_a ohms	V	I_s	R_s ohms	V	I_i	R_i ohms	
0 55	16 0	0 0344	0 24	17 5	0 0137	0 41	28 3	0 0145	Mean
1 24	37 0	0 0336	0 46	35 0	0 0131	0 50	33 4	0 0149	value
1 11	33 0	0 0336	0 45	33 0	0 0136	—	—	—	0 0618

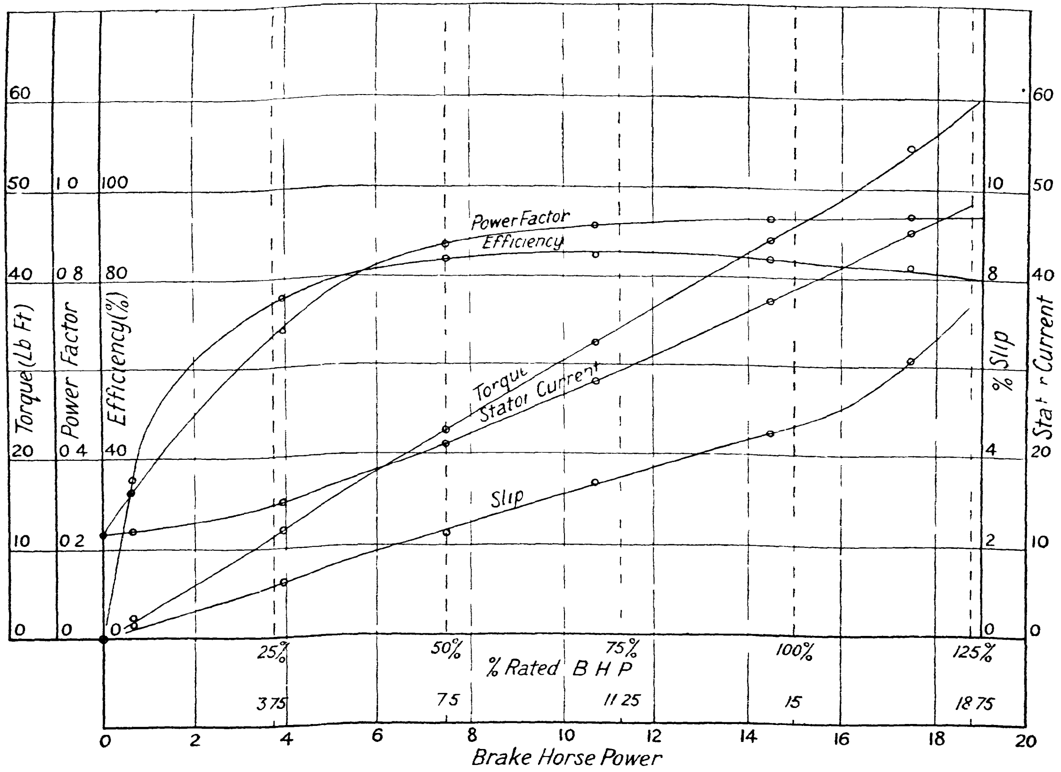


Fig. 1.16. Direct Determination of the Characteristic of a Three-phase Induction Motor

Calculations (see also Fig. 1.16)

Motor	Generator							Motor
Input watts	VI_a watts	Copper Loss watts	Core Loss watts	Total Output watts	Torque ft-lb	Output h.p.	Slip %	Efficiency %
910	0	0	0	0	0	0	0·19	0
1 400	0	203	287	490	1·9	0·66	0·35	35·0
3 895	2 410	234	—	2 930	11·6	3·93	1·22	75·3
6 650	4 980	315	—	5 580	22·4	7·48	2·22	84·0
9 430	7 280	434	—	8·000	32·4	10·70	3·34	84·8
12 850	9 880	625	—	10 790	44·2	14·50	4·45	84·0
15 930	11 960	820	—	13 070	54·5	17·50	6·12	82·0

Case 3

$$\text{Copper losses} = I_a^2R + VI_f = (20)^2(0{\cdot}0618) + (120{\cdot}5)(1{\cdot}73)$$

$$= 24{\cdot}7 + 209 \quad \hat{=} \; 234\ \text{W}$$

$$\text{Core loss} = 1400 - 910 - 203 \quad = 287\ \text{W}$$

$$\text{Generator output} = VI = (120{\cdot}5)(20) \quad = 2410\ \text{W}$$

$$\text{Motor output} = 2410 + 234 + 287 \quad = 2930\ \text{W}$$

$$\text{Torque} = \frac{(33\,000)(\text{watts output})}{(2\pi N)(746)} = \frac{(7{\cdot}05)(\text{output})}{N}$$

Full-load current = 38·5 A

$$\text{Approximate full-load output (gross)} = \sqrt{3}VI$$

$$= \sqrt{3}(220)(38{\cdot}5) = 14{\cdot}7\ \text{kW}$$

$$= 19{\cdot}7\ \text{h.p.}$$

Assume p.f. = 0·92 at full load

Assume efficiency = 0·83 at full load

Then the net full-load output = (19·7)(0·92)(0·83) = 15 h.p.

THE CIRCLE DIAGRAM OF THE INDUCTION MOTOR

Apparatus. As in previous experiment.

Procedure—

No-load Test

Volts		Amperes		Watts		$\frac{W_1}{W_2}$	P.F. Cos ϕ	θ_{NL}
1	2	1	2	1	2			
227	229	11·15	11·85	1750	− 760	− 0·434	0·225	77°

Locked Rotor Test

Volts		Amperes		Watts		$\frac{W_1}{W_2}$	P.F. Cos ϕ	θ,
1	2	1	2	1	2			
25·8	27·8	23·0	22·5	565	100	0·177	0·630	50° 00′
32·3	34·0	30·0	27·0	880	90	0·120	0·590	53° 00′
39·0	40·7	35·0	32·8	1270	200	0·157	0·610	52° 30′
45·0	47·0	40·0	38·0	1584	240	0·152	0·610	52° 30′
50·8	53·6	45·0	43·0	2040	360	0·176	0·630	50° 00′
58·0	58·2	48·4	49·3	2500	510	0·204	0·650	49° 30′
63·5	63·0	52·2	53·0	3020	688	0·228	0·675	47° 30′
							Average	50° 50′

Stator Resistance

Volts	Amps.	$2R$	R
2·34	8·35	0·28	0·14
5·70	20·50	0·28	0·14

Average resistance per phase of stator = 0·14 Ω

Calculations (see also Figs. 1.17, 1.18, 1.19)

Short-circuit current at operating voltage of previous experiment

$$= \left(\frac{50}{59{\cdot}6}\right)(228) \simeq 190 \text{ A}$$

Full-load h.p. (see previous experiment) = 15 h.p.

Full-load h.p. per phase = 5 h.p.

Increase in stator copper loss per phase

$$= R(I_s^2 - I_N^2) = (0{\cdot}14)(190^2 - 11{\cdot}5^2) \simeq 5{\cdot}0 \text{ kW}$$

$$\simeq 2{\cdot}54 \text{ kW}$$

Full Load %	θ	P.F. Cos θ	Stator Amperes	Slip %	Torque 3 phase lb-ft	Efficiency %
25	46° 00′	0·70	15·5	0	11·0	62
50	33° 00′	0·84	22·0	3·0	22·5	78
75	26° 00′	0·90	30·0	5·8	35·0	83
100	23° 00′	0·92	38·4	7·7	47·5	81
125	22° 00′	0·93	49·0	11·0	61·0	78
150	22° 30′	0·92	62·0	14·5	77·0	75

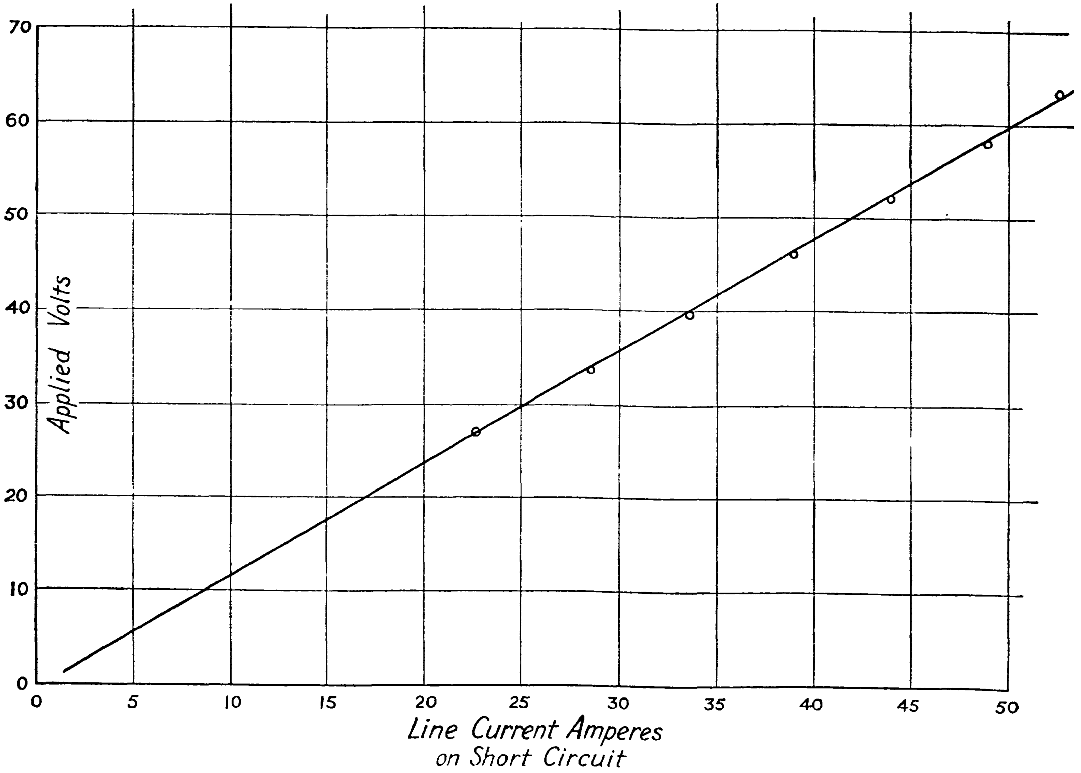


FIG. 1.17. SHORT-CIRCUIT CHARACTERISTIC FOR A THREE-PHASE INDUCTION MOTOR

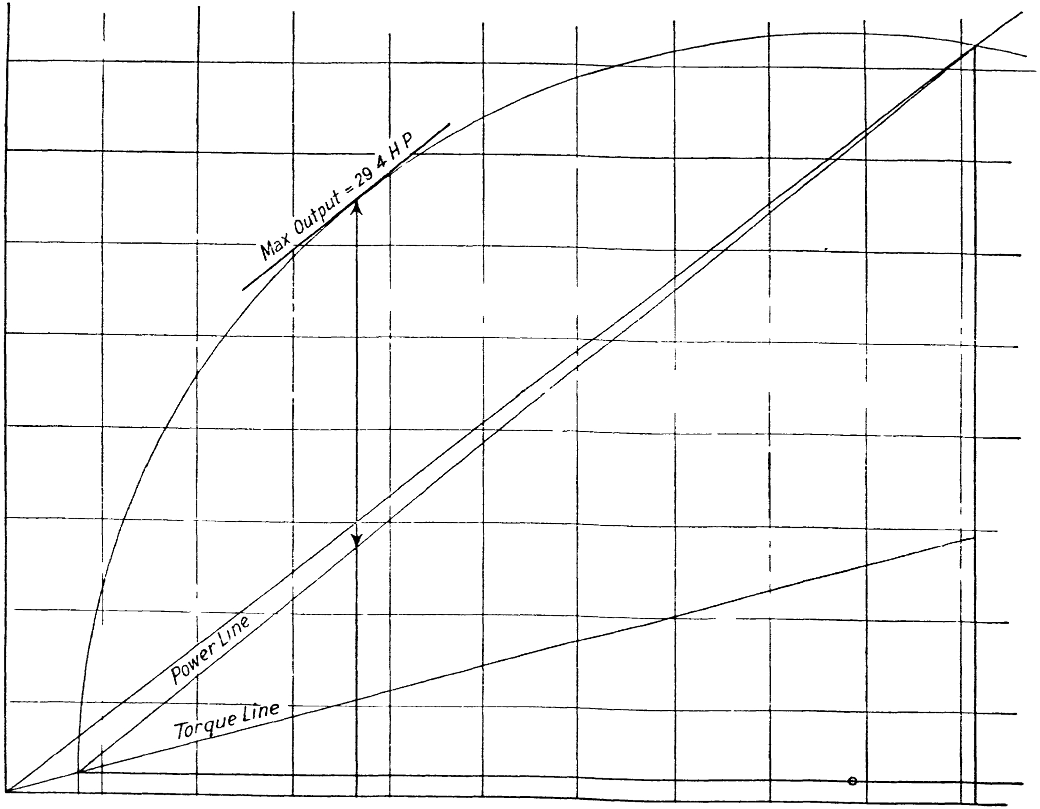


FIG. 1.18. THE CIRCLE DIAGRAM OF THE INDUCTION MOTOR

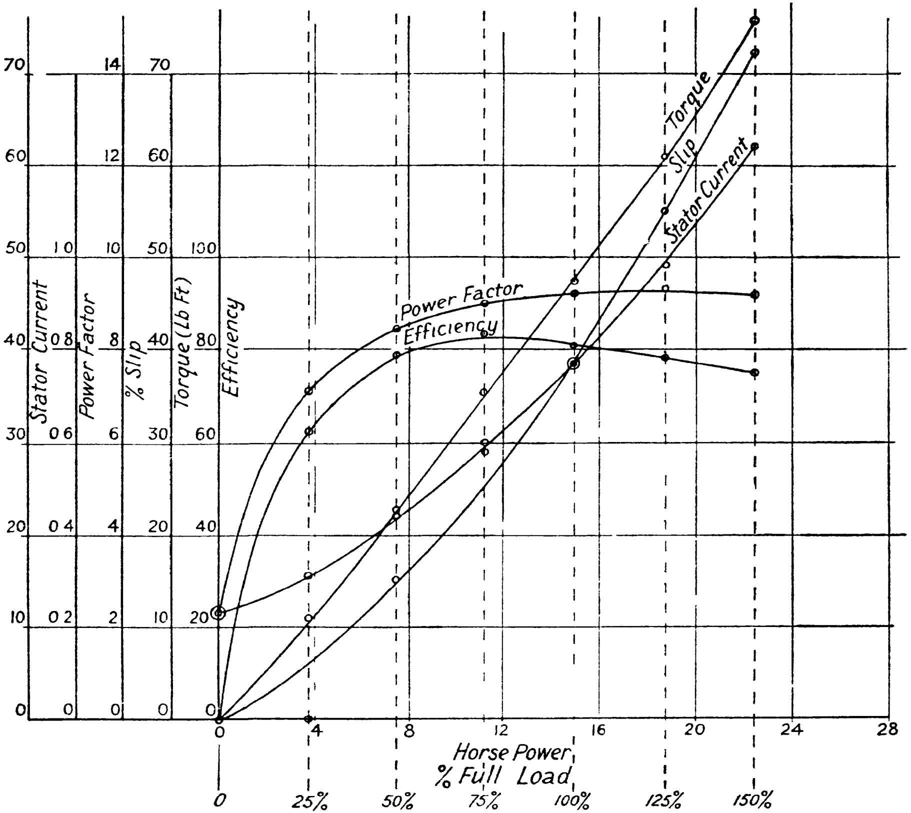


FIG. 1.19. THE CHARACTERISTIC CURVES OF THE INDUCTION MOTOR

THREE-PHASE SLIP-RING INDUCTION MOTOR

Test Equipment. Crompton and Parkinson Limited, slip-ring induction motor.

b.h.p. = 10
Volts = 220
C/s = 60

1135 r.p.m.
Mesh = 29 A
Rotor = 33 A

Observations—

Locked Rotor Test

Volts		Amperes		Watts		P.F. Cos ϕ
1	2	1	2	1	2	
34·8	35·3	14·9	15·0	370	− 100	0·32
44·0	43·0	20·2	19·8	630	− 175	0·31
61·5	61·0	30·5	30·0	1420	− 330	0·34
70·0	69·0	35·0	34·5	1760	− 375	0·35
79·0	78·0	40·3	40·0	2500	− 410	0·38
88·0	86·5	45·3	44·5	2970	− 620	0·35
					Mean P.F. 0·34	

No-load Test

Volts		Amperes		Watts		P.F. Cos ϕ
1	2	1	2	1	2	
229	230	14·6	13·8	1920	− 1280	0·11
229	230	14·8	14·0	1925	− 1305	0·11

Stator : Rotor Ratio Test

Three-phase			Single-phase		
Stator Volts	Rotor Volts	Ratio	Stator Volts	Rotor Volts	Ratio
227	134·0	1·69	228	136	1·68
228	134·5	1·69	228	135	1·69

$$\left.\begin{array}{l}\text{Resistance of stator per phase} = \dfrac{0{\cdot}3}{2} = 0{\cdot}15\ \Omega \\ \text{Resistance of rotor per phase} = \dfrac{0{\cdot}18}{2} = 0{\cdot}09\ \Omega\end{array}\right\}\ \text{Using the ductor}$$

Maximum resistance of starter per phase $= 4{\cdot}78\ \Omega$ Volt/Amp method

Starting Torque Test

Volts		Amperes		Watts		Weight lb	Scale lb	Torque lb-ft
1	2	1	2	1	2			
228	228	18·5	18·0	425	3800	11	6·5	21·8

Effective weight of Prony brake arm = 4·2 lb

Effective length of Prony brake arm = 30 in.

Calculations (see also Figs. 1.20, 1.21)

Starting torque $= (W - s)(\text{lever arm}) = (11 + 4{\cdot}2 - 6{\cdot}5)(2{\cdot}5)$

$= 21{\cdot}8$ lb-ft

Short-circuit stator current at working voltage 228

$$= \left(\frac{228}{87 - 6}\right)(45) = 125\ \text{A}$$

(This assumes that the voltmeter had a constant error of 6 V, to permit graph (1) to pass through the origin.)

P.F. at no-load $= 0{\cdot}11$ and $\therefore\ \theta_N = 83^\circ\ 30'$

Mean S.C.–P.F. $= 0{\cdot}34$ and $\therefore\ \theta_s = 70^\circ$

Increase in stator copper loss from no-load to locked rotor condition at 228 V $= R_1(I_s^2 - I_N^2)$

$= (0{\cdot}15)(125^2 - 14{\cdot}3^2) \qquad = 2310$ W

From Diagram—

Full-load rotor current $= 34{\cdot}7$ A

Full-load starting torque $= (bd)(\text{scale})$

$\therefore$ Rotor resistance for full-load starting torque

$$= \frac{(bd)(658)}{I_R^2} = 2{\cdot}1\ \Omega$$

and the extra rotor resistance required is, therefore,

$$= 2{\cdot}1 - 0{\cdot}09 \simeq 2\ \Omega$$

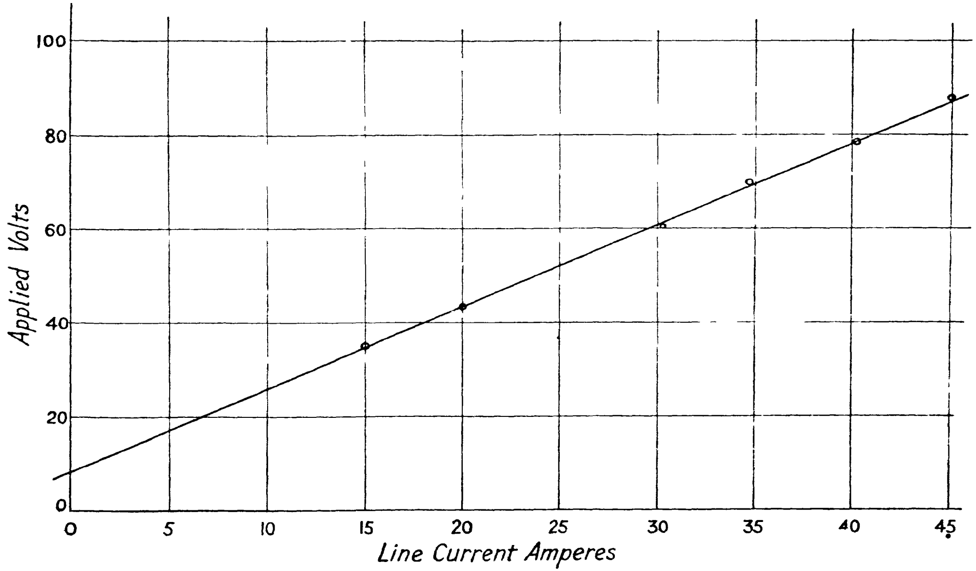


Fig. 1.20. Locked-rotor Test for Three-phase Slip-ring Induction Motor

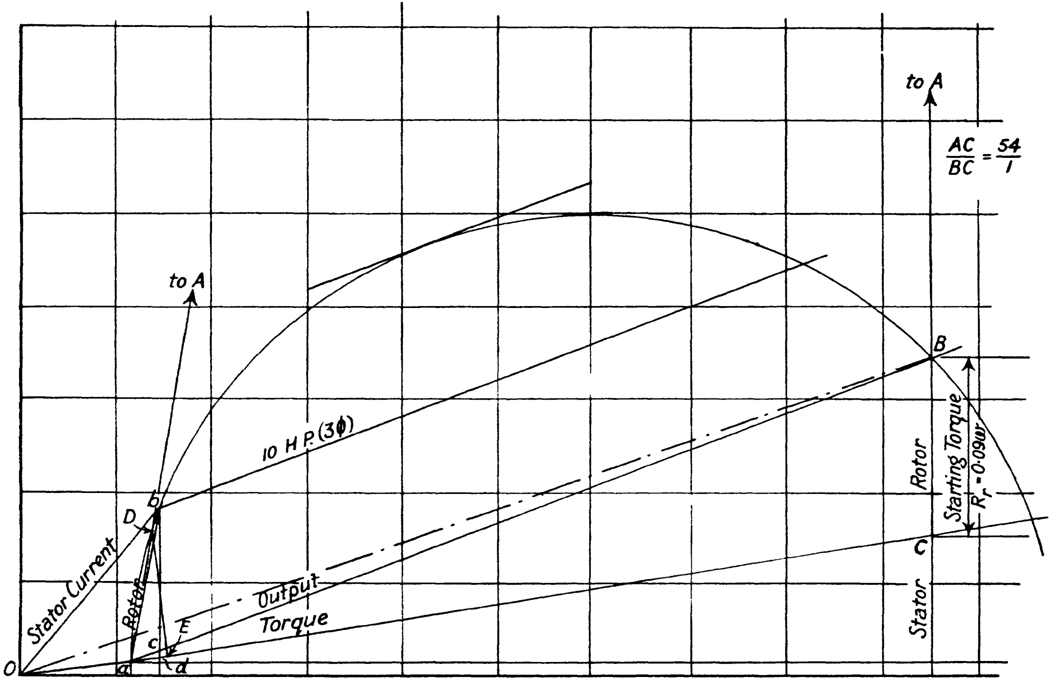


FIG. 1.21. THREE-PHASE SLIP-RING INDUCTION MOTOR

To determine the starting torque and stator and rotor current when the resistance of the rotor circuit is ($4{\cdot}78\omega + 0{\cdot}09\omega$) erect the vertical line ABC, so that $\frac{AC}{BC} = \frac{4{\cdot}78 + 0{\cdot}09}{0{\cdot}09} \bumpeq 54 : 1$—join A to a.

$\left\{\begin{array}{l}\text{Then stator current} = OD \\ \text{Measured (observed) stator current}\end{array}\right.$ $\begin{array}{l} = 18{\cdot}3\text{ A} \\ = 18{\cdot}3\text{ A}\end{array}$

Rotor current = 14·8 A

Starting torque ≏ 6·76 lb-ft per phase

$\left\{\begin{array}{l}\text{Starting torque (three-phase)} \\ \text{Observed starting torque}\end{array}\right.$ $\begin{array}{l} \bumpeq 20{\cdot}3\text{ lb-ft} \\ = 21{\cdot}8\text{ lb-ft}\end{array}$

SUMMARY OF RESULTS

It is useful to have all the important results collected together for reference. For this purpose reference will be made to Fig. 1.7 (page 18).

(*a*) The magnetizing current per phase = $O'C$

(*b*) The ideal short-circuit current per phase = $O'L$

(*c*) The no-load current per phase = $O'O$

(*d*) The actual short-circuit current per phase = $O'P$

(*e*) The dispersion co-efficient $= \dfrac{\text{magnetizing current}}{\text{ideal short-circuit current}}$

$$= \frac{O'C}{O'L} = \sigma$$

(*f*) Maximum power factor $= \dfrac{1 - \sigma}{1 + \sigma}$

(*g*) OP is the output line, and with stator current $O'B = I_1$ the output in watts $= m_1\bar{V} \times BD$

where m_1 = number of stator phases

$\bar{V}$ = applied volts per phase (r.m.s. value)

BD is the vertical intercept from B to the output line OP.

(*h*) The maximum output in watts $= m_1 \times \bar{V} \times TW$

The point T is found by drawing a line parallel to OP tangent to the circle and TW is the vertical intercept between T and the line OP.

$$\text{Maximum output in horse power} = \frac{m_1\bar{V} \times TW}{746}$$

(*i*) The output with stator current $O'B$ in h.p. $= \dfrac{m_1\bar{V} \times BD}{746}$

(*j*) The loss in the rotor, with load corresponding to $O'B$,

$$= m_1\bar{V} \times DE \text{ watts}$$

(*k*) The loss in the stator, with current $O'B = m_1\bar{V} \times EF$

(*l*) The no-load losses, namely iron, friction, and windage and no-load copper loss $= m_1\bar{V} \times FK$

(*m*) The input $= m_1\bar{V} \times BK$.

(*n*) The no-load power factor $= \cos OO'V$.

(*o*) The full-load power factor $= \cos BO'V$.

(*p*) The slip $= \dfrac{\text{rotor copper loss}}{\text{rotor input}}$

$$= \frac{\text{rotor copper loss}}{\text{torque} \times \text{synchronous speed in watts}}$$

$$= \frac{DE}{BE} = \frac{CE}{QE} \text{ in Fig. 1.10.}$$

(*q*) The torque multiplied by the synchronous speed, expressed in watts = input to the rotor circuit $= m_2 I_2{}^2 \dfrac{R_2}{s}$

where m_2 = number of rotor phases
I_2 = rotor current
R_2 = rotor resistance per phase, and
s = slip.

The torque in *lb-ft* (gross)

$$= 0{\cdot}1303 \times 10^{-8} \times p\hat{\phi} \times Z_2\bar{I}_2 \times \cos\phi_2 \times K_2 \times K_3$$

$= 0{\cdot}1303 \times 10^{-8}$ × flux per pole (max.)
× number of poles
× *total* number of rotor conductors
× rotor current per conductor
× power factor of rotor current
× breadth factor of rotor winding
× coil span factor of rotor winding

(*r*) The maximum torque in synchronous watts $= m_1\bar{V} \times VX$ in Fig. 1.7
where VX = maximum vertical intercept between the circle and the line OG

(*s*) At P, the standstill point, the slip = 1, and
$m_1\bar{V} \times PG$ = rotor copper loss
$m_1\bar{V} \times GH$ = stator copper loss with current OP

(*t*) Slip $= \dfrac{\text{synchronous speed} - \text{actual speed}}{\text{synchronous speed}} = \dfrac{\omega_0 - \omega}{\omega_0}$

actual speed $= \omega_s(1 - s)$

(*u*) OB = rotor current referred to stator = I_1'

$$= \frac{\text{actual rotor current}}{\text{ratio of transformation}} = \frac{I_2}{\alpha}$$

(*v*) The ratio of transformation $= \dfrac{T_1 \times K_1 \times K_3}{T_2 \times K_2 \times K_4} = \alpha.$

where T_1 = number of turns in series per phase in the stator

T_2 = number of turns per phase in rotor in series

K_1 = breadth factor of the stator winding for the fundamental wave

$$= \frac{\sin q_1 \frac{\lambda}{2}}{q_1 \sin \frac{\lambda}{2}}$$

where q_1 = number of stator slots per pole per phase

λ = electrical slot-pitch angle

$$= \frac{180}{\text{number of slots per pole}}$$

K_3 = coil span factor for the fundamental

$$= \cos \frac{\varepsilon}{2} \text{ or } \sin \frac{\beta}{2}$$

where ε = deficiency, from full span, of the coil in electrical degrees

β = actual span in electrical degrees

K_2 = breadth factor for the fundamental of the rotor winding

$$= \frac{\sin q_2 \frac{\lambda_2}{2}}{q_2 \sin \frac{\lambda_2}{2}}$$

$$K_4 = \cos \frac{\varepsilon_2}{2} \text{ or } \sin \frac{\beta_2}{2}$$

For the *n*th harmonic, $K_{1n} = \dfrac{\sin nq_1 \frac{\lambda}{2}}{q_1 \sin n \frac{\lambda}{2}}$

and $K_{3n} = \cos n \dfrac{\varepsilon}{2}$

If the number of phases in the stator is not the same as in the rotor, then—

$$I_1' = \frac{m_2T_2 \times K_2 \times K_4}{m_1T_1 \times K_1 \times K_3} \times I_2$$

(*w*) The rotor resistance, *referred to the stator*

$$= \frac{m_1T_1^2 \times f_1^2R_2}{m_2T_2^2 \times f_2^2}$$

where R_2 = rotor resistance per phase

$f_1 = K_1 \times K_3$

$f_2 = K_2 \times K_4$

The rotor reactance per phase, at supply frequency, *referred to the stator*

$$= X_2 \times \frac{m_1T_1^2 \times f_1^2}{m_2T_2^2 \times f_2^2}$$

where T_1f_1 = number of *effective* turns per stator phase

T_2f_2 = number of *effective* turns per rotor phase

(*x*) The slip for maximum torque

$$= s_m$$

$$= \pm \frac{R_2}{X_2}$$

$$= \pm \frac{\text{rotor resistance per phase}}{\text{rotor reactance per phase at standstill}}$$

(*y*) Maximum torque $= \dfrac{mE_2^2}{2X_2}$

(*z*) Slip for maximum output $= -\dfrac{R_2^2}{X_2^2} \pm \dfrac{R_2}{X_2}\sqrt{\dfrac{R_2^2}{X_2^2} + 1}$

$$\text{Maximum output} = \frac{m_2E_2^2}{2X_2^2}(Z_2 - R_2)$$

where $Z_2 = \sqrt{R_2^2 + X_2^2}$

CHAPTER II

The Squirrel-cage Rotor

IN the induction machine, using a squirrel-cage winding, we have a number of copper bars, of circular or rectangular section, joined together at each end by a ring. If we assume the flux to be sinusoidally distributed and rotating relatively to the rotor, it is clear that e.m.f.s will be generated in each bar, the magnitude of each e.m.f. being determined, at any instant, by the magnitude of the flux density cutting the bar. If the slip is very small, the currents will be practically in phase with the generated e.m.f.s; the distribution of current will also be sinusoidal; in other words, the curve of flux distribution will coincide with the curve of current distribution, each being a sine curve in space.

Under the north poles, the current will flow in one direction; under the south poles, the current will flow in the opposite direction. Thus the rotor currents will produce fluxes with as many north and south poles as those produced by the stator winding. Thus, the first thing to notice is that the squirrel-cage rotor adapts itself to any number of stator poles, and, of course, it is a condition for constant torque that this should be so.

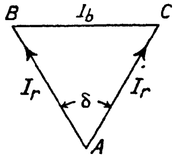


FIG. 2.1

Secondly, we notice that, since the flux is distributed sinusoidally (we confine our attention to the fundamental wave of flux), there is a constant phase difference between the e.m.f.s and currents in consecutive bars, and this phase difference is equal to the electrical slot pitch angle. This angle of phase difference

$$= \frac{p \times 180}{Q_2}$$

where p = poles

Q_2 = number of bars

If I_b = current per bar (R.M.S. value) . . (2.1)

and I_r = current per ring . . . (2.2)

Then $I_b = 2I_r \sin\frac{\delta}{2}$ . . . . . (2.3)

where $\delta = \frac{\pi p}{Q_2}$, in electrical radians

If *AB* and *AC* represent the currents in consecutive ring sections, then the current per bar

$$= 2AB \sin\frac{\delta}{2} \quad . \quad . \quad . \quad . \quad (2.4)$$

$$I_b = 2I_r \sin\frac{\delta}{2} \quad . \quad . \quad . \quad . \quad (2.5)$$

and

$$I_r = \frac{I_b}{2\sin\frac{\delta}{2}} = \frac{I_b}{2\sin\frac{\pi p}{2Q_2}} \quad . \quad . \quad . \quad (2.6)$$

If Q_2 is large, then $\sin\frac{\delta}{2}$ is nearly equal to $\frac{\delta}{2}$, and

$$I_r = \frac{I_b}{\frac{\pi p}{Q_2}} \quad . \quad . \quad . \quad . \quad (2.7)$$

The maximum current in the ring is equal to the average current per bar, namely, $\frac{2}{\pi}\hat{I}_b$ × half the bars per pole.

$$\therefore \qquad \hat{I}_r = \frac{2}{\pi}\hat{I}_b \times \frac{Q_2}{2p} = \frac{\hat{I}_b}{\frac{\pi p}{Q_2}} \quad . \quad . \quad (2.8)$$

where $\hat{I}_r$ = maximum value of I_r

and $\hat{I}_b$ = maximum value of I_b

$$\therefore \qquad I_r = \frac{I_b}{\frac{\pi p}{Q_2}} \text{ (R.M.S. values)} \quad . \quad . \quad . \quad (2.9)$$

The copper losses in the squirrel cage

$$= Q_2(I_b{}^2 r_b + 2I_r{}^2 r_r) \quad . \quad . \quad . \quad (2.10)$$

where r_b = resistance *per bar*

and r_r = resistance of a single segment of the ring, i.e. the resistance of the part of the ring between two bars.

Therefore, the copper loss in the squirrel-cage rotor

$$= Q_2\left[I_b{}^2 r_b + \frac{2I_b{}^2 r_r}{4\sin^2\frac{\pi p}{2Q_2}}\right] \quad . \quad . \quad . \quad (2.11)$$

$$= Q_2 I_b{}^2\left[r_b + \frac{r_r}{2\sin^2\frac{\pi p}{2Q_2}}\right] \quad . \quad . \quad . \quad (2.12)$$

$$= Q_2 I_b{}^2[r_b{}'] \quad . \quad . \quad . \quad . \quad . \quad (2.13)$$

where $$r_b' = r_b + \frac{r_r}{2\sin^2\dfrac{\pi p}{2Q_2}} \quad . \quad . \quad . \quad . \quad . \quad (2.14)$$

and r_b' = resistance of an equivalent bar, which gives the same loss as the actual losses in the bar and its associated segments

If the resistance of each end ring is R_r, then

$$r_r = \frac{R_r}{Q_2} \quad . \quad . \quad . \quad . \quad . \quad (2.15)$$

The copper loss in the rotor

$$= \bar{I}_b^2 \left[Q_2 r_b + \frac{Q_2 r_r}{2\sin^2\left(\dfrac{\pi p}{2Q_2}\right)} \right] \quad . \quad . \quad (2.16)$$

let R_b = resistance of *all* the bars = $Q_2 r_b$
and R_r = resistance of *one* ring = $Q_2 r_r$

Then the copper loss in the rotor

$$= \bar{I}_b^2 \left[R_b + \frac{R_r}{2\sin^2\left(\dfrac{\pi p}{2Q_2}\right)} \right] \quad . \quad . \quad . \quad (2.17)$$

The squirrel-cage is a polyphase winding, and the number of its phases

$$= m_2 = \frac{2Q_2}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.18)$$

= number of bars *per pair* of poles

Consider now the reactance of the rotor and the reactance of an equivalent bar, i.e. a bar with a reactance equal to that of one rotor bar plus the reactance of the associated segments. Apply Kirchhoff's laws to the currents at a junction of a bar and the ring, and also Law 2 to the volts drop around a loop of two bars and two ring segments.

We have $$\mathbf{I}_{b2} = \mathbf{I}_{r2} - \mathbf{I}_{r1} \quad . \quad . \quad . \quad . \quad . \quad (2.19)$$

and $$\bar{I}_{b2} = 2\bar{I}_R \sin\frac{\delta}{2} \quad . \quad . \quad . \quad . \quad . \quad (2.20)$$

We have $$2\bar{I}_b \sin\frac{\delta}{2} Z_b + \frac{2\bar{I}_b Z_r}{2\sin\dfrac{\delta}{2}} = 2\bar{E}_b \sin\frac{\delta}{2} \quad . \quad . \quad (2.21)$$

where Z_b = impedance per bar
Z_r = impedance of one ring sector between two bars

$$\bar{I}_b = \frac{\bar{E}_b}{Z'} = \frac{\bar{E}_b}{Z_b + \dfrac{Z_r}{2\sin^2\dfrac{\delta}{2}}} = \frac{\bar{E}_b}{Z_b + \dfrac{Z_r}{2\sin^2\left(\dfrac{\pi p}{2Q_2}\right)}} \quad . \quad (2.22)$$

and the equivalent impedance per bar

$$= Z' = Z_b + \frac{Z_r}{2\sin^2\left(\frac{\pi p}{2Q_2}\right)} \quad . \quad . \quad . \quad (2.23)$$

We have already shown that

$$r' = r_b + \frac{r_r}{2\sin^2\left(\frac{\pi p}{2Q_2}\right)} \quad . \quad . \quad . \quad (2.24)$$

so the equivalent reactance per bar

$$= x_b' = x_b + \frac{x_r}{2\sin^2\left(\frac{\pi p}{2Q_2}\right)} \quad . \quad . \quad . \quad (2.25)$$

where x_b = reactance per bar
x_r = reactance of one ring segment between two bars
E_b = e.m.f. generated in the bar by the slip of the field.

The copper losses in the squirrel-cage rotor

$$= Q_2 I_b^2 \left(r_b + \frac{r_r}{2\sin^2\left(\frac{\pi p}{2Q_2}\right)}\right) = Q_2 I_2^2 (r_b') \quad . \quad (2.26)$$

for $$I_B = I_2 \text{ and } r_b' = r_b + \frac{r_r}{2\sin^2\left(\frac{\pi p}{2Q_2}\right)} \quad . \quad (2.27)$$

Now $$Q_2 I_2^2 r_b' = m_1 \times (I_1')^2 \times R_2' \quad . \quad . \quad (2.28)$$

where R_2' is chosen to represent a resistance which will give the same loss in the stator as occurs in the rotor per phase
and m_1 = number of stator phases

$$\therefore \quad \frac{I_2^2}{(I_1')^2} = \frac{m_1 R_2'}{Q_2 r_b'} \quad . \quad . \quad . \quad . \quad (2.29)$$

Also $$Q_2 I_2 = m_1 \times 2T_1 \times K_1 K_3 \times I_1' \quad . \quad . \quad (2.30)$$

$$\therefore \quad R_2' = \frac{Q_2 r_b'}{m_1} \times \frac{I_2^2}{I_1'^2} \quad . \quad . \quad . \quad . \quad . \quad (2.31)$$

$$= \frac{Q_2 r_b'}{m_1} \times \frac{4T_1^2 \times K_1^2 \times K_3^2 \times m_1^2}{Q_2^2} \quad . \quad (2.32)$$

$$= r_b' \times \frac{4m_1 \times T_1^2 \times K_1^2 \times K_3^2}{Q_2} \quad . \quad . \quad (2.33)$$

Thus, the equivalent resistance of the rotor referred to the stator

$$= r_b' \times \frac{4m_1 T_1^2 \times K_1^2 \times K_3^2}{Q_2} \quad . \quad . \quad (2.34)$$

where $r_b' = r_b + \dfrac{r_r}{2\sin^2\left(\dfrac{\pi p}{2Q_2}\right)}$

r_b = resistance of one bar

$r_r = \dfrac{\text{resistance of one ring}}{Q_2}$

m_1 = number of phases in the stator

and Q_2 = number of rotor bars or slots

In the same way

$$X_2' = x_b' \times \frac{4m_1T_1^2 \times K_1^2 \times K_3^2}{Q_2} \quad . \quad . \quad (2.35)$$

where X_2' = reactance of the cage referred to the stator per phase at the supply frequency

and $x_b' = x_b + \dfrac{x_r}{2\sin^2\left(\dfrac{\pi p}{2Q_2}\right)}$

where x_b = reactance of *one* bar at the supply frequency

and $x_r = \dfrac{\text{reactance of one ring at supply frequency}}{Q_2}$

If the rotor slots are skewed, and this is done to reduce the effects of the harmonics in the flux wave, then in referring the rotor resistance and reactance to the stator, we must introduce the skew factor into equations (2.34) and (2.35).

Let K_5 = skew factor, which

$$= \frac{\sin \dfrac{C}{\tau_2} \times \dfrac{\alpha}{2}}{\dfrac{C}{\tau_2} \times \dfrac{\alpha}{2}} \quad . \quad . \quad . \quad . \quad . \quad (2.36)$$

where C = amount of skew

= BC (*see* Fig. 2.2)

τ_2 = rotor slot pitch

and $\alpha = \dfrac{p\pi}{Q_2}$

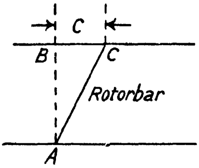


FIG. 2.2

Then for skewed rotor slots, we have

$$R_2' = r_b' \times \frac{4m_1T_1^2 \times K_1^2 \times K_3^2}{Q_2 \times K_5^2} \quad . \quad . \quad (2.37)$$

and

$$X_2' = x_b' \times \frac{4m_1T_1^2 \times K_1^2 \times K_3^2}{Q_2 \times K_5^2} \quad . \quad . \quad (2.38)$$

It will also be noticed that the squirrel-cage can be considered as a winding with m_2 phases, where $m_2 = \dfrac{Q_2 \times 2}{p}$

The Double Cage

This was introduced by Dobrowolski. In this arrangement, two separate cage windings are used; one, the outer cage, has bars of smaller cross-section than the inner cage. The outer cage, i.e. the one nearer the gap, has relatively high resistance and low leakage inductance; the inner cage has conductors of larger cross-section—usually narrow, deep bars, and it has low resistance and high leakage inductance.

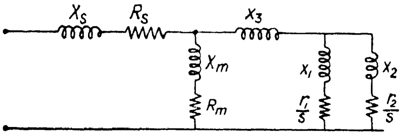


Fig. 2.3

At the start, the frequency of the rotor currents is equal to the supply frequency, and the currents in the inner cage are reduced by the large impedance of this cage. The currents in the outer cage are very effective in producing torque at the start, for the losses in this cage are large.

The outer cage is, therefore, most effective as regards starting torque. With rising speed, the slip frequency of the rotor current decreases, and the inner cage becomes effective, since its reactance becomes smaller, and its current increases. At full load the *reactances* are negligibly small compared to the resistances, and the currents in each cage are determined by the resistances almost entirely. Thus, the starting torque is increased, with smaller starting current, than in a single-cage motor, and some of the advantages of the slip-ring type machine are obtained. The inner bar is separated from the outer bar by a fairly long narrow slit, and the outer bar is usually of circular section, and the inner bar of rectangular section—narrow and deep.

Let r_1 = resistance of inner cage, referred to the stator

x_1 = leakage reactance, at supply frequency, of the inner cage, referred to the stator

r_2 = resistance of outer cage, referred to the stator

x_2 = leakage reactance of outer cage, referred to the stator, at supply frequency

x_3 = leakage reactance, referred to the stator, at supply frequency, due to leakage flux linking both cages

Then the equivalent circuit of the double-cage motor is as in Fig. 2.3.

$\mathbf{Z}_1$ = impedance of *inner* cage, referred to the stator

$$= \frac{r_1}{s} \times jx_1 \quad . \quad . \quad . \quad . \quad (2.39)$$

$\boldsymbol{Z}_2$ = impedance of outer cage, referred to the stator

$$= \frac{r_2}{s} + jx_2 \quad . \quad . \quad . \quad . \quad (2.40)$$

The combined impedance

$$= \frac{\boldsymbol{Z}_1\boldsymbol{Z}_2}{\boldsymbol{Z}_1 + \boldsymbol{Z}_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.41)$$

$$= \frac{\dfrac{r_1r_2}{s^2} + j\left(\dfrac{x_1r_2}{s} + \dfrac{x_2r_1}{s}\right) - x_1x_2}{\dfrac{r_1 + r_2}{s} + j(x_1 + x_2)} \quad . \quad . \quad (2.42)$$

Since x_2 is usually negligible

$$\boldsymbol{Z} \bumpeq \frac{\dfrac{r_1r_2}{s^2} + \dfrac{jr_2x_1}{s}}{\dfrac{r_1 + r_2}{s} + jx_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.43)$$

$$\boldsymbol{Z} \bumpeq \frac{\dfrac{r_1r_2}{s} + jr_2x_1}{r_1 + r_2 + jsx_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.44)$$

$$\bumpeq \frac{r_1r_2 + jsr_2x_1}{s\{(r_1 + r_2) \times jsx_1\}}$$

$$\bumpeq \frac{(r_1r_2)(r_1 + r_2) + s^2r_2x_1{}^2 + j\{sr_2x_1(r_1 + r_2) - sx_1r_1r_2\}}{s\{(r_1 + r_2)^2 + s^2x_1{}^2\}} \quad (2.45)$$

$$\bumpeq \frac{R_2}{s} + jX_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.46)$$

$$\text{where } R_2 \bumpeq \frac{(r_1r_2)(r_1 + r_2) + s^2r_2x_1{}^2}{(r_1 + r_2)^2 + s^2x_1{}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.47)$$

$$\text{and} \quad X_2 \bumpeq \frac{r_2{}^2x_1}{(r_1 + r_2)^2 + s^2x_1{}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.48)$$

The equivalent circuit can now be reduced to that shown in Fig. 2.4.

Clearly Fig. 2.4 can be reduced to a simple series circuit containing an equivalent reactance, and an equivalent resistance in series.

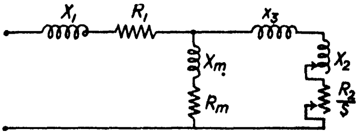


FIG. 2.4

At synchronous speed ($s = 0$)

$$\underset{(s=0)}{R_2} \bumpeq \frac{r_1r_2}{r_1 + r_2} = \underset{(s=0)}{R_2} \quad . \quad . \quad . \quad (2.49)$$

and
$$\underset{(s=0)}{X_2} = \left(\frac{r_2}{r_1 + r_2}\right)^2 x_1 = \underset{(s=0)}{X_2} \quad . \quad . \quad . \quad (2.50)$$

At standstill, $s = 1$, and

$$\underset{(s=1)}{R_2} = \frac{r_1 r_2(r_1 + r_2) + r_2 x_1^2}{(r_1 + r_2)^2 + x_1^2} \quad . \quad . \quad . \quad (2.51)$$

$$\underset{(s=1)}{X_2} = \frac{r_2^2 x_1}{(r_1 + r_2)^2 + x_1^2} \quad . \quad . \quad . \quad (2.52)$$

$R_2(s = 1)$ is a maximum at standstill when $x_1 = r_1 + r_2$. This is not the best condition for starting purposes, however, and does not correspond to maximum torque at the start.

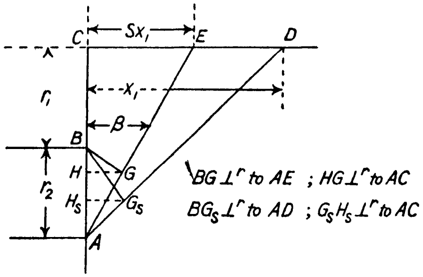

C
Sx_1
E
D
x_1
r_1
β
B
H
r_2
H_S
G
G_S
A
BG ⊥r to AE ; HG ⊥r to AC
BG_S ⊥r to AD ; $G_S H_S$ ⊥r to AC

FIG. 2.5
This construction is due to Punga and Raydt

From equation (2.47) we have

$$R_2 = \frac{r_2 r_1(r_1 + r_2) + s^2 r_2 x_1^2}{(r_1 + r_2)^2 + s^2 x_1^2} \quad . \quad . \quad . \quad . \quad (2.53)$$

$$= r_2 \left[\frac{r_1(r_1 + r_2) + s^2 x_1^2}{(r_1 + r_2)^2 + s^2 x_1^2}\right] \quad . \quad . \quad . \quad . \quad (2.54)$$

$$= r_2 \left[\frac{(r_1 + r_2)^2 + s^2 x_1^2 - r_2(r_1 + r_2)}{(r_1 + r_2)^2 + s^2 x_1^2}\right] \quad . \quad (2.55)$$

$$= r_2 \left[1 - \frac{r_2(r_1 + r_2)}{(r_1 + r_2)^2 + s^2 x_1^2}\right] \quad . \quad . \quad . \quad (2.56)$$

$$= r_2 \left[1 - \frac{AB \times AC}{AE^2}\right] \quad . \quad . \quad . \quad . \quad (2.57)$$

$$= r_2 \left[1 - \frac{AB \cos \beta}{AE}\right] \quad . \quad . \quad . \quad . \quad (2.58)$$

$$= r_2 \left[1 - \frac{AG}{AE}\right] = r_2 \left[1 - \frac{AH}{AC}\right] \quad . \quad . \quad . \quad (2.59)$$

$$= r_2 \times \frac{CH}{AC} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.60)$$

In Fig. 2.5, $AB = r_2$; $BC = r_1$; CE at right angles to $AC = sx_1$; $CD = x_1$; BG is perpendicular to AE.

Then the equivalent resistance $R_2 = r_2 \frac{CH}{AC}$

As the speed varies, the point E moves from C at synchronous speed to D at standstill, and the equivalent resistance of the two cages $= r_2 \frac{CH}{AC}$; the angle β = angle CAE.

At synchronous speed G and B coincide, and E falls on C.

At standstill $$\underset{(s=1)}{R_2} = r_2 \frac{CH_s}{AC} \quad . \quad . \quad . \quad . \quad (2.61)$$

Now, $$X_2 = \frac{r_2^2 x_1}{(r_1 + r_2)^2 + s^2 x_1^2} \quad . \quad . \quad . \quad (2.62)$$

and $$\underset{(s=0)}{X_2} = \frac{r_2^2 x_1}{(r_1 + r_2)^2} \quad . \quad . \quad . \quad (2.63)$$

$$\therefore \quad x_1 = \underset{(s=0)}{X_2} \times \frac{(r_1 + r_2)^2}{r_2^2} \quad . \quad . \quad . \quad (2.64)$$

$$\therefore \quad X_2 = \underset{(s=0)}{X_2} \times \frac{(r_1 + r_2)^2}{(r_1 + r_2)^2 + s^2 x_1^2} \quad . \quad . \quad (2.65)$$

$$\therefore \quad X_2 = \underset{(s=0)}{X_2} \times \left(\frac{AC}{AE}\right)^2 = \underset{(s=0)}{X_2} \times \cos^2 \beta \quad . \quad . \quad (2.66)$$

If AB in Fig. 2.5 represents $X_2(s = 0)$, then

$$AH = AB \cos^2 \beta \quad . \quad . \quad . \quad (2.67)$$

will represent X_2 directly.

The motion of the point H, as E moves from C to D, will give us, graphically, information with regard to increase of rotor resistance and decrease of rotor reactance during starting.

The currents in the two cages are dependent on the impedance of each cage, for the impedance drops are the same in each.

If $\mathbf{I}_1'$ and $\mathbf{I}_2'$ are the currents in the inner and outer cages, respectively, then

$$\frac{\mathbf{I}_1'}{\mathbf{I}_2'} \simeq \frac{r_2}{r_1 + jsx_1} = \frac{\mathbf{Z}_2}{\mathbf{Z}_1} = \frac{AB}{BE} \quad . \quad . \quad . \quad (2.68)$$

still assuming negligible reactance in the outer cage.

Also $$\frac{\mathbf{I}_1' + \mathbf{I}_2'}{\mathbf{I}_1'} = \frac{AE}{AB} = \frac{\mathbf{I}_2}{\mathbf{I}_1'} \quad . \quad . \quad . \quad (2.69)$$

where $\mathbf{I}_1' + \mathbf{I}_2'$ is the vector sum of the two cage currents $= \mathbf{I}_2$.

$$\therefore \quad \mathbf{I}_1' : \mathbf{I}_2' : \mathbf{I}_2 :: AB : BE : AE$$

If the secondary current I_2 is known, the currents in the two cages can be read off the diagram, and also their phase relations.

The shape of the torque-slip curve will depend entirely on the ratio x_1 to r_1, and also on $\frac{r_2}{r_1}$. The torque, at standstill, of the outer cage will be greater than at any other value of the slip, for slips from 0 to 1, and will be equal in synchronous watts to the loss in the cage.

The torque curve for the double-cage motor, and the torque curves for the separate cages are shown in Fig. 2.6.

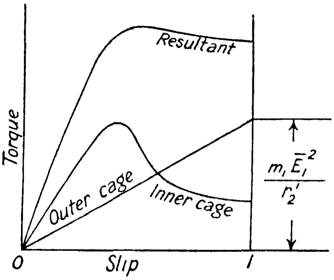


FIG. 2.6

The torque, in synchronous watts, for the inner cage

$$= \frac{s\bar{E}_1^2 r_1' m_1}{(r_1')^2 + s^2(x_1')^2} \quad . \quad . \quad . \quad (2.70)$$

All quantities are referred to the stator.

Neglecting x_2', the torque of the outer cage, in synchronous watts

$$= \frac{s\bar{E}_1^2}{r_2'} \times m_1 \text{ (approx.)} \quad . \quad . \quad . \quad (2.71)$$

The resultant torque, for the two cages, is

$$\frac{s\bar{E}_1^2 \times r_1' m_1}{(r_1')^2 + s^2(x_1')^2} + \frac{s\bar{E}_1^2 \times m_1}{r_2'} . \quad . \quad . \quad . \quad (2.72)$$

The starting torque

$$= \frac{\bar{E}_1^2 \times r_1' \times m_1}{(r_1')^2 + (x_1')^2} + \frac{\bar{E}_1^2 \times m_1}{r_2'} \quad . \quad . \quad (2.73)$$

The inner cage produces but 10 to 15 per cent (approximately) of the starting torque. By making $\frac{x_1}{r_1}$ large, one reduces the starting current taken by the inner cage, and, since this cage produces so small a part of the starting torque required, it does not matter much how one increases $\frac{x_1}{r_1}$, but it is desirable to have a reasonable value for the maximum starting torque of this cage, and this necessitates not too great a value for x_1. Clearly then one can start with the design of the outer cage, by assuming it produces 80 per cent (approximately) of the starting torque required.

From the relation

$$\frac{\text{starting torque}}{\text{full-load torque}} = \left(\frac{I_{scr}}{I_{2r}}\right)^2 \times \text{slip at full load}$$

$$\text{the ratio } \frac{I_{scr}}{I_{2r}} = \frac{\text{starting current in outer rotor}}{\text{full-load current in outer rotor}}$$

If the ratio of the $\frac{\text{starting torque}}{\text{full-load torque}}$ is given, and $\frac{I_{scr}}{I_{2r}}$ is fixed, the slip of the motor at full load, considered as a single-cage motor is determined. For small ratio of $\frac{I_{scr}}{I_{2r}}$, clearly the slip will be relatively large to produce a large starting torque. This means that r_2 must be large. The slip, however, when running under full load, with the two cages, will be relatively small, as one will see by inspection of the effective resistance of the two cages, given in equation (2.47).

SUMMARY OF CHAPTER II

(1) The squirrel-cage winding adapts itself to any number of poles.

(2) The current per ring $= \dfrac{\text{current per bar}}{2 \sin \dfrac{\pi p}{2Q_2}}$

where p = number of poles

Q_2 = total number of rotor bars

(3) The effective resistance per bar, including the effect of the end rings, $r_b' = r_b + \dfrac{r_r}{2 \sin^2 \left(\dfrac{\pi p}{2n}\right)}$

where r_b = resistance per bar

and r_r = resistance of a ring segment between two rotor bars

(4) Copper loss in the squirrel-cage rotor

$$= Q_2 I_b^2 \left\{ r_b + \frac{r_r}{2 \sin^2 \left(\dfrac{\pi p}{2Q_2}\right)} \right\}$$

(5) The effective reactance per bar, including the effect of the reactance of the segment of the rings, between two bars

$$= x_b' = x_b + \frac{x_r}{2 \sin^2 \left(\dfrac{\pi p}{2Q_2}\right)}$$

where x_b = reactance of one bar at supply frequency

x_r = reactance of the segments associated with a bar at supply frequency

(6) The effective *resistance* of the cage, *referred to the stator*, per phase

$$R_2' = \frac{r_b' \times 4m_1 \times T_1^2 \times K_1^2 \times K_3^2}{Q_2}$$

where m_1 = number of stator phase

T_1 = turns per stator phase

K_1 = breadth factor of stator winding for the fundamental

K_3 = coil span factor of stator winding for the fundamental

(7) The effective *reactance*, at supply frequency, of the cage *referred to the stator*, $X_2' = x_b' \times \frac{4m_1T_1^2 \times K_1^2 \times K_3^2}{Q_2}$

(8) The squirrel-cage winding has a number of phases

$$= \frac{\text{number of rotor bars}}{\text{pairs of poles}}$$

(9) Of the rotor currents flowing under one pole, the currents in half of the bars per pole flow in one direction in the end ring, and the currents in the other half of the bars per pole flow in the opposite direction in the ring.

(10) To reduce noise it is desirable to skew the slots a full slot pitch, and to use as large an air-gap as possible.

(11) Cogging effects must be diminished, as far as possible, by correct choice of slot ratio. The number of slots in the rotor must differ from the number in the stator, and to reduce cogging the highest common factor of the number of slots in stator and rotor must be as small as possible.

(12) The ordinary cage motor takes a large wattless current from the mains at the start, and the starting torque is not large in most motors of this type.

(13) Where large starting torque, with low starting current, is required, recourse is had to the *double*-squirrel cage. The inner cage has high reactance and low resistance at full frequency. The outer cage has high resistance and low reactance at supply frequency.

The outer cage is very effective in producing large starting torque, with low starting current. At standstill, the outer cage is the more effective. As the speed rises the reactance of the inner cage gets smaller and it becomes more effective in producing its share of the torque.

Fig. 2.7 shows the speed/torque relationships of single-cage and double-cage motors on the same graph.

The introduction of the double-cage is due to Dobrowolski.

The shape of the torque curve depends on the ratio of

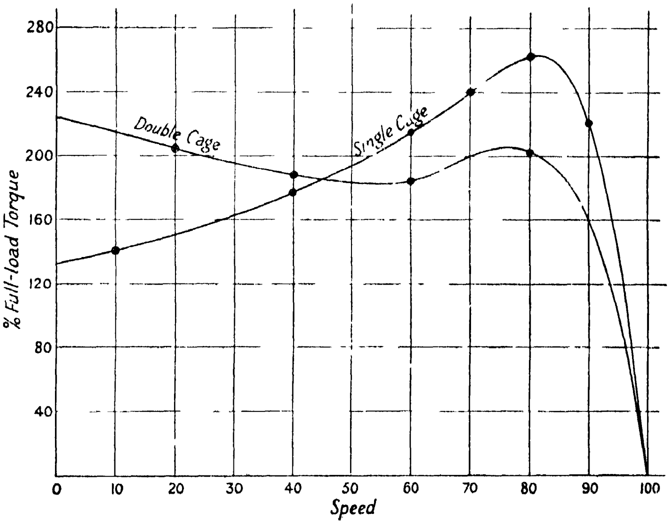


FIG. 2.7

$\frac{\text{reactance of the inner cage at full supply frequency}}{\text{resistance of inner cage}}$ and also on the ratio of $\frac{\text{resistance of outer cage}}{\text{resistance of inner cage}}$.

Usual values of $\frac{x_1}{r_1} = 4$ or 5 and $\frac{r_2}{r_1} = 2$ or 3.

Plate V (facing page 88) shows an example of a double squirrel-cage rotor (axially ventilated motor, with balance ring at non-fan end: speed 3000 r.p.m.).

CHAPTER III

The Circle Diagram

In Chapter I we developed the circle diagram by making approximations, the chief of which was the transfer of the magnetizing circuit from *C* and *D* in Fig. 1.5 to the terminals *A* and *B*. A more exact derivation will now be made.

The equivalent circuit of the polyphase motor is shown below, in Fig. 3.1. It is as follows—

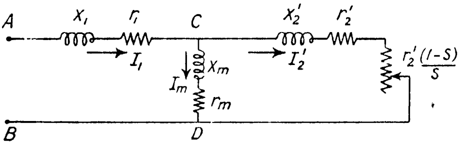


FIG. 3.1

Let the primary impedance be

$$\mathbf{Z}_1 = r_1 + jx_1 \quad . \quad . \quad . \quad . \quad (3.1)$$

The secondary impedance referred, to the primary, in the equivalent circuit, is

$$\mathbf{Z}_2' = \frac{r_2'}{s} + jx_2' \quad . \quad . \quad . \quad . \quad (3.2)$$

The magnetizing impedance

$$= \mathbf{Z}_m = r_m + jx_m \quad . \quad . \quad . \quad . \quad (3.3)$$

$$Y_m = \frac{1}{Z_m} = (g_m - jb_m) \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

$= $ admittance of magnetizing circuit

$\mathbf{V} =$ applied p.d.

$\mathbf{I}_2' =$ secondary current, referred to the primary.

Then

$$\mathbf{V} = \mathbf{Z}_1\mathbf{I}_1 + \mathbf{Z}_2'\mathbf{I}_2' \quad . \quad . \quad . \quad (3.5)$$

and

$$\mathbf{Z}_2'\mathbf{I}_2' = + \mathbf{E}_1 = (\mathbf{I}_1 - \mathbf{I}_2')\mathbf{Z}_m \quad . \quad . \quad . \quad (3.6)$$

$$\therefore \quad \frac{\mathbf{I}_1 - \mathbf{I}_2'}{\mathbf{I}_2'} = \frac{\mathbf{Z}_2'}{\mathbf{Z}_m} \quad . \quad . \quad . \quad . \quad . \quad (3.7)$$

$$\frac{\mathbf{I}_1}{\mathbf{I}_2'} - 1 = \frac{\mathbf{Z}_2'}{\mathbf{Z}_m} \quad . \quad . \quad . \quad . \quad . \quad (3.8)$$

and
$$\frac{\boldsymbol{I}_1}{\boldsymbol{I}_2'} = \frac{\boldsymbol{Z}_2' + \boldsymbol{Z}_m}{\boldsymbol{Z}_m} \qquad . \quad . \quad . \quad . \quad (3{\cdot}9)$$

and
$$\boldsymbol{I}_2' = \boldsymbol{I}_1 \frac{\boldsymbol{Z}_m}{\boldsymbol{Z}_2' + \boldsymbol{Z}_m} \qquad . \quad . \quad . \quad (3{\cdot}10)$$

Therefore, equation (3.5) becomes

$$\boldsymbol{V} = \boldsymbol{Z}_1\boldsymbol{I}_1 + \frac{\boldsymbol{Z}_2'\boldsymbol{Z}_m}{\boldsymbol{Z}_2' + \boldsymbol{Z}_m}\boldsymbol{I}_1 \qquad . \quad . \quad . \quad (3{\cdot}11)$$

$$\therefore \qquad \boldsymbol{V} = \boldsymbol{I}_1\left\{\frac{\boldsymbol{Z}_2'\boldsymbol{Z}_1 + \boldsymbol{Z}_1\boldsymbol{Z}_m + \boldsymbol{Z}_2'\boldsymbol{Z}_m}{\boldsymbol{Z}_2' + \boldsymbol{Z}_m}\right\} \qquad . \quad . \quad (3{\cdot}12)$$

$$\therefore \qquad \boldsymbol{I}_1 = \frac{\boldsymbol{Z}_2' + \boldsymbol{Z}_m}{\boldsymbol{Z}_2'\boldsymbol{Z}_1 + \boldsymbol{Z}_1\boldsymbol{Z}_m + \boldsymbol{Z}_2'\boldsymbol{Z}_m} \times \boldsymbol{V} \qquad . \quad . \quad (3{\cdot}13)$$

$$= \boldsymbol{V}\left\{\frac{1 + \boldsymbol{Z}_2'\boldsymbol{Y}_m}{\boldsymbol{Z}_1 + \boldsymbol{Z}_2' + \boldsymbol{Z}_1\boldsymbol{Z}_2'\boldsymbol{Y}_m}\right\} \qquad . \quad . \quad . \quad (3{\cdot}14)$$

And from equation (3.10), we have

$$\boldsymbol{I}_2' = \boldsymbol{I}_1 \times \frac{1}{\boldsymbol{Z}_2'\boldsymbol{Y}_m + 1}$$

$$\therefore \qquad \boldsymbol{I}_2' = \boldsymbol{V} \times \frac{1}{\boldsymbol{Z}_1 + \boldsymbol{Z}_2' + \boldsymbol{Z}_1\boldsymbol{Z}_2'\boldsymbol{Y}_m} \qquad . \quad . \quad (3{\cdot}15)$$

By substituting the values for $\boldsymbol{Z}_2'$, $\boldsymbol{Y}_m$, $\boldsymbol{Z}_1$ we can obtain the primary and secondary currents.

Now
$$\boldsymbol{Z}_2'\boldsymbol{Y}_m = \left(\frac{r_2'}{s} + jx_2'\right)(g_m - jb_m) \qquad . \quad . \quad . \quad (3{\cdot}16)$$

$$= \frac{r_2'}{s}g_m + x_2'b_m + j\left(x_2'g_m - \frac{r_2'}{s}b_m\right) \qquad . \quad (3{\cdot}17)$$

$$1 + \boldsymbol{Z}_2'\boldsymbol{Y}_m = 1 + \frac{r_2'}{s}g_m + x_2'b_m + j\left(x_2'g_m - \frac{r_2'}{s}b_m\right) \quad (3{\cdot}18)$$

$$= a + jb \qquad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}19)$$

where
$$a = 1 + \frac{r_2'}{s}g_m + x_2'b_m \qquad . \quad . \quad . \quad (3{\cdot}20)$$

and
$$b = x_2'g_m - \frac{r_2'}{s}b_m \qquad . \quad . \quad . \quad (3{\cdot}21)$$

Now $\boldsymbol{Z}_1 + \boldsymbol{Z}_2' + \boldsymbol{Z}_1\boldsymbol{Z}_2'\boldsymbol{Y}_m$

$$= (r_1 + jx_1) + \left(\frac{r_2'}{s} + jx_2'\right) + (r_1 + jx_1)\left(\frac{r_2'}{s} + jx_2'\right)\left(g_m - jb_m\right) \quad (3{\cdot}22)$$

$$= r_1 + \frac{r_2'}{s} + j(x_1 + x_2') + \frac{r_1 r_2'}{s} g_m + b_m x_2' r_1 + \frac{x_1 r_2'}{s} b_m - x_1 x_2' g_m$$

$$+ j\left\{x_1 \frac{r_2'}{s} g_m + x_2' r_1 g_m - \frac{r_1 r_2'}{s} b_m + x_1 x_2' b_m\right\} \quad (3\cdot23)$$

$$= r_1 + \frac{r_2'}{s} + \frac{r_1 r_2'}{s} g_m + b_m x_2' r_1 + \frac{x_1 r_2'}{s} b_m - x_1 x_2' g_m$$

$$+ j\left\{x_1 + x_2' + \frac{x_1 r_2' g_m}{s} + x_2' r_1 g_m - \frac{r_1 r_2' b_m}{s} + x_1 x_2' b_m\right\} \quad (3\cdot24)$$

$$= c + jd$$

$$\therefore \qquad \mathbf{I}_1 = \mathbf{V}\left\{\frac{a + jb}{c + jd}\right\} \quad . \quad . \quad . \quad (3\cdot25)$$

$$\underset{\text{absolute}}{I_1} = \bar{V}\sqrt{\frac{a^2 + b^2}{c^2 + d^2}} \quad . \quad . \quad . \quad . \quad (3\cdot26)$$

$$I_2' = \bar{V} \times \frac{1}{\sqrt{c^2 + d^2}} \quad . \quad . \quad (3\cdot27)$$

The angle between the primary applied volts and primary current

$$= \tan^{-1}\frac{b}{a} - \tan^{-1}\frac{d}{c} \quad . \quad . \quad (3\cdot28)$$

Now

$$a - 1 + \frac{r_2'}{s} g_m + x_2' b_m \quad . \quad . \quad (3\cdot29)$$

$$b = x_2' g_m - \frac{r_2' b_m}{s} \quad . \quad . \quad . \quad (3\cdot30)$$

$$c = r_1 + \frac{r_2'}{s} + \frac{r_1 r_2'}{s} g_m + b_m x_2' r_1 + \frac{x_1 r_2'}{s} b_m - x_1 x_2' g_m \quad (3\cdot31)$$

$$d = x_1 + x_2' + \frac{x_1 r_2' g_m}{s} + x_2' r_1 g_m - \frac{r_1 r_2' b_m}{s} + x_1 x_2' b_m \quad (3\cdot32)$$

$$\frac{a + jb}{c + jd} = \frac{(r_2' g_m - j r_2' b_m) + 1\,(1 + x_2' b_m + j x_2' g_m)s}{r_2' + r_1 r_2' g_m + x_1 r_2' b_m + j(x_1 r_2' g_m - r_1 r_2' b_m) + \{r_1 + b_m x_2' r_1 - x_1 x_2' g_m + j(x_1 + x_2' + x_2' r_1 g_m + x_1 x_2' b_m\}s} \quad . \quad . \quad (3\cdot33)$$

$$= \frac{\mathbf{E} + \mathbf{F}s}{\mathbf{G} + \mathbf{H}s}$$

where $\mathbf{E} = r_2' g_m - j r_2' b_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\cdot34)$

$\mathbf{F} = 1 + x_2' b_m + j x_2' g_m \quad . \quad . \quad . \quad . \quad . \quad (3\cdot35)$

$\mathbf{G} = r_2' + r_1 r_2' g_m + x_1 r_2' b_m + j(x_1 r_2' g_m - r_1 r_2' b_m) \quad . \quad (3\cdot36)$

$\mathbf{H} = r_1 + b_m x_2' r_1 - x_1 x_2' g_m + j(x_1 + x_2' + x_2' r_1 g_m + x_1 x_2' b_m) \quad (3\cdot37)$

Now
$$\frac{\mathbf{E}+\mathbf{F}s}{\mathbf{G}+\mathbf{H}s}=\frac{\mathbf{F}}{\mathbf{H}}+\frac{\mathbf{E}-\dfrac{\mathbf{FG}}{\mathbf{H}}}{\mathbf{G}+\mathbf{H}s} \quad . \quad . \quad . \quad (3.38)$$

$$=\mathbf{A}+\frac{\mathbf{B}}{\mathbf{G}+\mathbf{H}s} \quad . \quad . \quad . \quad (3.39)$$

Now $\mathbf{A}=\dfrac{\mathbf{F}}{\mathbf{H}}$ = a complex constant

$$=\frac{1+x_2'b_m+jx_2'g_m}{r_1+b_mx_2'r_1-x_1x_2'g_m+j(x_1+x_2'+x_2'r_1g_m+x_1x_2'b_m)} \quad (3.40)$$

i.e.
$$\mathbf{A}=\frac{\{1+x_2'b_m+jx_2'g_m\}\times\{r_1+b_mx_2'r_1-x_1x_2'g_m-j(x_1+x_2'+x_2'r_1g_m+x_1x_2'b_m)\}}{(r_1+b_mx_2'r_1-x_1x_2'g_m)^2+(x_1+x_2'+x_2'r_1g_m+x_1x_2'b_m)^2} \quad . \quad . \quad (3.41)$$

$$\mathbf{A}=f+jg \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.42)$$

where
$$f=\frac{(r_1+b_mx_2'r_1-x_1x_2'g_m)(1+x_2'b_m)+(x_1+x_2'+x_2'r_1g_m+x_1x_2'b_m)x_2'g_m}{(r_1+b_mx_2'r_1-x_1x_2'g_m)^2+(x_1+x_2'+x_2'r_1g_m+x_1x_2'b_m)^2} \quad (3.43)$$

$$g=\frac{x_2'g_m\{r_1+b_mx_2'r_1-x_1x_2'g_m\}-\{1+x_2'b_m\}\times\{x_1+x_2'+x_2^1r_1g_m+x_1x_2'b_m\}}{(r_1+b_mx_2'r_1-x_1x_2'g_m)^2+(x_1+x_2'+x_2'r_1g_m+x_1x_2'b_m)^2} \quad (3.44)$$

$$\mathbf{B}=\mathbf{E}-\frac{\mathbf{FG}}{\mathbf{H}}$$

also a complex constant.

$$\mathbf{B}=r_2'g_m-jr_2'b_m-\frac{\{1+x_2'b_m+jx_2'g_m\}\{r_2'+r_1r_2'g_m+x_1r_2'b_m+j(x_1r_2'g_m-r_1r_2'b_m)\}}{r_1+b_mx_2'r_1-x_1x_2'g_m+j(x_1+x_2'+x_2'r_1g_m+x_1x_2'b_m)} \quad . \quad . \quad (3.45)$$

$$=h+jk \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.46)$$

where h is the real part of $\mathbf{B}$ and k is the imaginary part of $\mathbf{B}$.

Let
$$\left.\begin{aligned}\mathbf{E}&=q+jr\\ \mathbf{G}&=y+jz\\ \mathbf{F}&=s_1+jt\\ \mathbf{H}&=u+jv\end{aligned}\right\} \quad . \quad (3.47)$$

then $$\mathbf{B} = q + jr - \frac{(s_1 + jt)(y + jz)}{u + jv} \tag{3.48}$$

$$= \frac{uq + jur + jqv - vr - s_1y + tz - jty - js_1z}{u + jv} \tag{3.49}$$

$$= \frac{uq - vr - s_1y + tz + j(ur - ty - s_1z + qv)}{u + jv} \tag{3.50}$$

$$= \frac{(uq - vr - s_1y + tz) + j(ur + qv - ty - s_1z)}{u + jv}$$

Rationalizing the denominator, we have

$$\mathbf{B} = \frac{\{(uq - vr - s_1y + tz) + j(ur + qv - ty - s_1z)\}\{u - jv\}}{u^2 + v^2} \tag{3.51}$$

$$\begin{aligned} &= \frac{u^2q - uvr - us_1y + utz + vur + v^2q - vty - vs_1z}{u^2 + v^2} \\ &\qquad + \frac{j(u^2r + uqv - uty - us_1z - uvq + v^2r + vs_1y - vtz)}{u^2 + v^2} \end{aligned} \tag{3.52}$$

$$= h + jk$$

$$h = \frac{u^2q - uvr - us_1y + utz + vur + v^2q - vty - vs_1z}{u^2 + v^2} \tag{3.53}$$

$$k = \frac{u^2r + uqv - uty - us_1z - uvq + v^2r + vs_1y - vtz}{u^2 + v^2} \tag{3.54}$$

Also $$\mathbf{A} = \frac{\mathbf{F}}{\mathbf{H}} = \frac{s_1 + jt}{u + jv} = \frac{(s_1 + jt)(u - jv)}{u^2 + v^2} \tag{3.55}$$

$$= \frac{us_1 + tv + j(tu - s_1v)}{u^2 + v^2} \tag{3.56}$$

$$= \frac{us_1 + tv}{u^2 + v^2} + j\frac{(tu - s_1v)}{u^2 + v^2} \tag{3.57}$$

$$= f + jg \tag{3.58}$$

where $$f = \frac{us_1 + tv}{u^2 + v^2} \tag{3.59}$$

$$g = \frac{tu - s_1v}{u^2 + v^2} \tag{3.60}$$

Now $$\mathbf{I}_1 = \mathbf{V}\left[\frac{\mathbf{E} + \mathbf{F}s}{\mathbf{G} + \mathbf{H}s}\right] = \mathbf{V}\left[\mathbf{A} + \frac{\mathbf{B}}{\mathbf{G} + \mathbf{H}s}\right] \tag{3.61}$$

$$= \mathbf{V}\,(f + jg) + \mathbf{V}\frac{(h + jk)}{\mathbf{G} + \mathbf{H}s} \tag{3.62}$$

Let us take a system of rectangular co-ordinates, in which the vertical axis represents real quantities, which we will call the X-axis, and the horizontal axis represents imaginary quantities, i.e. the quantities which are multiplied by j or $-j$.

Then $\boldsymbol{G} + \boldsymbol{H}s$ represents a straight line (Fig. 3.2).

$$\begin{aligned}\boldsymbol{G} + \boldsymbol{H}s &= r_2' + r_1r_2'g_m + x_1r_2'b_m + j(x_1r_2'g_m - r_1r_2'b_m) \\ &\quad + \{r_1 + b_mx_2'r_1 - x_1x_2'g_m + j(x_1 + x_2' + x_2'r_1g_m + x_1x_2'b_m)\}s \\ &= y + jz + (u + jv)s\,. \qquad (3.63)\end{aligned}$$

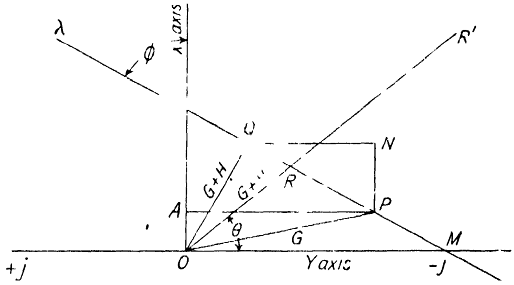


FIG. 3.2

Set off $OA = y = r_2' + r_1r_2'g_m + x_1r_2'b_m$ vertically along the X-axis, and $z = AP = x_1r_2'g_m - r_1r_2'b_m$ parallel to the $-j$ axis, for AP is negative.

Then $OP = \boldsymbol{G}$.

Now draw PN vertically parallel to the X-axis and make it $= u$, and NQ vertically parallel to the Y-axis $= v$.

Then $PQ = \boldsymbol{H}$ for $s = 1$. Therefore, $OQ = \boldsymbol{G} + \boldsymbol{H}$ and represents $\boldsymbol{G} + \boldsymbol{H}s$ for $s = 1$.

As s varies from 0 to 1, R moves from P to Q. Take any point R between P and Q, and join OR and produce to R', where $OR' = \frac{1}{OR}$.

Then R' is said to be the inverse point of R.

Repeat this construction, i.e. find the various inverse points corresponding to the various positions of R on the line PQ and PQ produced. The loci of the various inverse points is a circle, which passes through the origin O.

We notice also that since $OR = \boldsymbol{G} + \boldsymbol{H}s$ is a complex quantity, we may represent it by $re^{j\theta}$, then if r' represents OR' in magnitude

$$\frac{1}{\boldsymbol{G} + \boldsymbol{H}s} = \frac{1}{r}e^{-j\theta} = r'e^{-j\theta} \qquad (3.64)$$

Thus, if the vector OR makes an angle θ with the horizontal, the point R' (i.e. the inverse point) should be equal to $\frac{1}{r}$ and be below the horizontal by the angle $-\theta$.

Thus, the complete inversion is a circle, passing through the origin O, and is an image of the circle above the horizontal axis in the horizontal axis. In other words our inverse circle is below the Y-axis. Let y and x represent the co-ordinates of point R and y' and x' represent the co-ordinates of point R'.

$$OR = r,\ OR' = r'.$$

$$\text{Then } x' = r' \sin\theta = \frac{1}{r}\sin\theta = \frac{r\sin\theta}{r^2} = \frac{x}{r^2} = \frac{x}{x^2+y^2} \quad . \quad (3.65)$$

$$y' = r'\cos\theta = \frac{1}{r}\cos\theta = \frac{r\cos\theta}{r^2} = \frac{y}{r^2} = \frac{y}{x^2+y^2} \quad . \quad (3.66)$$

Thus, for the inverse of any locus $F(x, y) = 0$, we have the locus $F\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) = 0$.

Therefore, to obtain the inverse, we substitute in the equation of the curve $\frac{x}{x^2+y^2}$ for x and $\frac{y}{x^2+y^2}$ for y.

Now we saw that $\boldsymbol{G} + \boldsymbol{H}s$ represents the line PQT. Let the equation of this line be represented by

$$y = mx + n \quad . \quad . \quad . \quad . \quad (3.67)$$

The inverse of this line is

$$\frac{y}{x^2+y^2} = \frac{mx}{x^2+y^2} + n \quad . \quad . \quad . \quad (3.68)$$

i.e.

$$-mx - n(x^2+y^2) + y = 0 \quad . \quad . \quad (3.69)$$

i.e.

$$x^2 + y^2 + \frac{mx}{n} - \frac{y}{n} = 0 \quad . \quad . \quad . \quad (3.70)$$

i.e.

$$\left(x + \frac{m}{2n}\right)^2 + \left(y - \frac{1}{2n}\right)^2 = \frac{m^2}{4n^2} + \frac{1}{4n^2} \quad . \quad . \quad (3.71)$$

$$= \frac{m^2+1}{4n^2} \quad . \quad . \quad (3.72)$$

This is the equation of a circle. The radius

$$= \frac{\sqrt{m^2+1}}{2n} \quad . \quad . \quad . \quad . \quad (3.73)$$

The x_c co-ordinate of the centre, with respect to O,

$$= \frac{-m}{2n} \quad . \quad . \quad . \quad . \quad (3.74)$$

and

$$y_c = \frac{1}{2n} \quad . \quad . \quad . \quad . \quad (3.75)$$

Now we need the radius in amps, and the other co-ordinates in amps.

Now
$$\mathbf{I}_1 = \mathbf{V}\{f + jg\} + \mathbf{V}\frac{(h + jk)}{\mathbf{G} + \mathbf{H}s} \quad . \quad . \quad . \quad (3.76)$$

The radius of the circle in amps

$$= V\sqrt{h^2 + k^2}\,\frac{\sqrt{m^2 + 1}}{2n} \quad . \quad . \quad . \quad (3.77)$$

The true origin will be displaced from O.

The real co-ordinate of the centre, i.e. the x-co ordinate, will be

$$Vf + V\sqrt{h^2 + k^2}\left(\frac{-m}{2n}\right) \quad . \quad . \quad . \quad (3.78)$$

i.e. x_c is multiplied by $V\sqrt{h^2 + k^2}$

The new y-co-ordinate of the centre

$$= Vg + V\sqrt{h^2 + k^2}\,\frac{1}{2n} \quad . \quad . \quad . \quad (3.79)$$

i.e. y_c is multiplied by $\sqrt{h^2 + k^2}$ and the radius is turned through an angle $\alpha = \tan^{-1}\frac{k}{h}$

To determine the radius, we need to know the values of m and n.

Now m, the slope of the line $PQT = \tan\phi$.

$$m = \frac{x_1 + x_2' + x_2'r_1g_m + x_1x_2'b_m}{r_1 + b_mx_2'r_1 - x_1x_2'g_m} \quad . \quad . \quad (3.80)$$

And n is the intercept OM on the $-j$ axis.

We have
$$y - y_1 = m(x - x_1)$$

where x_1 and y_1 are the co-ordinates of P in Fig. 3.2,

i.e.
$$y = mx - mx_1 + y_1 \quad . \quad . \quad . \quad (3.81)$$

When $x = 0, y = OM$,

$$\therefore \qquad OM = y_1 - mx_1 = n$$

$$\therefore n = x_1r_2'g_m - r_1r_2'b_m - \frac{(x_1 + x_2' + x_2'r_1g_m + x_1x_2'b_m)(r_2' + r_1r_2'g_m + x_1r_2'b_m)}{r_1 + b_mx_2'r_1 - x_1x_2'g_m}$$

We have seen that

$$\mathbf{I}_1 = \mathbf{V}\left[\frac{\mathbf{E} + \mathbf{F}s}{\mathbf{G} + \mathbf{H}s}\right] \quad . \quad . \quad . \quad (3.82)$$

At synchronous speed, $s = 0$

$$\text{and } \underset{(s=0)}{\mathbf{I}_1} = \mathbf{V} \times \frac{\mathbf{E}}{\mathbf{G}} = \mathbf{V} \frac{(r_2'g_m - jr_2'b_m)}{r_2' + r_1r_2'g_m + x_1r_2'b_m + j(x_1r_2'g_m - r_1r_2'b_m)}$$

$$\therefore \underset{(s=0)}{I_1} = \bar{V}\sqrt{\frac{(r_2'g_m)^2 + (r_2'b_m)^2}{(r_2' + r_1'r_2'g_m + x_1r_2'b_m)^2 + (x_1r_2'g_m - r_1r_2'b_m)^2}} \quad (3.83)$$

The angle between the primary current and primary volts at synchronous speed

$$= -\tan^{-1}\frac{r_2'b_m}{r_2'g_m} - \tan^{-1}\frac{x_1r_2'g_m - r_1r_2'b_m}{r_2' + r_1r_2'g_m + x_1r_2'b_m} \quad . \quad (3.84)$$

At standstill, $s = 1$,

$$\underset{(s=1)}{\mathbf{I}_1} = \mathbf{V}\left[\frac{\mathbf{E} + \mathbf{F}}{\mathbf{G} + \mathbf{H}}\right]$$

$$\therefore \underset{(s=1)}{\mathbf{I}_1} = \mathbf{V}\left[\frac{r_2'g_m - jr_2'b_m + 1 + x_2'b_m + jx_2'g_m}{\begin{array}{l} r_2' + r_1r_2'g_m + x_1r_2'b_m + r_1 + b_mx_2'r_1 - x_1x_2'g_m \\ \quad + j(x_1r_2'g_m - r_1r_2'b_m + x_1 + x_2' + x_2'r_1g_m + x_1x_2'b_m) \end{array}}\right]$$

$$\therefore \underset{(s=1)}{I_1} = \bar{V}\sqrt{\frac{(r_2'g_m + 1 + x_2'b_m)^2 + (x_2'g_m - r_2'b_m)^2}{\begin{array}{l} (r_2' + r_1r_2'g_m + x_1r_2'b_m + r_1 + b_mx_2'r_1 - x_1x_2'g_m)^2 \\ \quad + (x_1r_2'g_m - r_1r_2'b_m + x_1 + x_2' + x_2'r_1g_m + x_1x_2'b_m)^2 \end{array}}} \quad . \quad . \quad (3.85)$$

The angle between the primary current and primary volts

$$= \tan^{-1}\frac{x_2'g_m - r_2'b_m}{1 + r_2'g_m + x_2'b_m} - \tan^{-1}\frac{x_1r_2'g_m - r_1r_2'b_m + x_1 + x_2' + x_2'r_1g_m + x_1x_2'b_m}{r_2' + r_1r_2'g_m + x_1r_2'b_m + r_1 + b_mx_2'r_1 - x_1x_2'g_m} \quad (3.86)$$

At $s = \infty$

$$\underset{(s=\infty)}{\mathbf{I}_1} = \mathbf{V}\frac{\mathbf{F}}{\mathbf{H}} = \frac{\mathbf{V} \times \{1 + x_2'b_m + jx_2'g_m\}}{r_1 + b_mx_2'r_1 - x_1x_2'g_m + j(x_1 + x_2' + x_2'r_1g_m + x_1x_2'b_m)} \quad . \quad . \quad (3.87)$$

and

$$\underset{(s=\infty)}{\mathbf{I}_1} = \bar{V}\sqrt{\frac{(1 + x_2'b_m)^2 + (x_2'g_m)^2}{(r_1 + b_mx_2'r_1 - x_1x_2'g_m)^2 + (x_1 + x_2' + x_2'r_1g_m + x_1x_2'b_m)^2}} \quad . \quad . \quad (3.88)$$

and the angle between primary volts and primary current for $s = \infty$

$$= \tan^{-1}\frac{x_2'g_m}{1 + x_2'b_m} - \tan^{-1}\frac{x_1 + x_2' + x_2'r_1g_m + x_1x_2'b_m}{r_1 + b_mx_2' - x_1x_2'g_m} \quad (3.89)$$

Thus, three points on the circle can be calculated and the centre and the radius found by geometry or by calculation.

Some of these expressions are rather cumbrous, but in any given example, we shall be able to discard, without sensible error, those which are negligibly small.

Thus, $r_2'b_m \simeq \frac{r_2'}{x_m}$ is small, and $x_1 r_2 g_m$ very small, and other similar terms.

The expressions can then be considerably simplified by neglecting these small terms.

The Torque Line

We will now show that the torque is represented by the vertical intercept between the circle and a straight line, called the torque line.

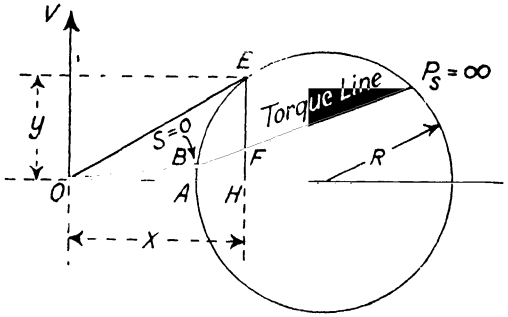


FIG. 3.3

Let B be the point at which the slip $s = 0$, and, therefore, the speed is synchronous. At B no torque is developed in the rotor. The primary current at B is OB.

Let P be the point where $s = \infty$. Since the torque in synchronous watts per phase $= (I_2')^2 \frac{r_2'}{s}$, the torque is zero when $s = \infty$.

Let x and y be the co-ordinates of point E. Then

$$(x - x_c)^2 + (y - y_c)^2 = R^2$$

where x_c and y_c are the co-ordinates of the centre and R = radius of the circle.

$$\therefore \quad x^2 + y^2 = R^2 + 2xx_c + 2yy_c - x_c^2 - y_c^2$$

The torque in synchronous watts = input to the rotor and the input to the rotor = input to the stator — iron loss — stator copper loss

$$= V \times EH - (x^2 + y^2)r_1 - C \quad . \quad . \quad (3.90)$$

where C = iron loss

and OE = primary amps $= \sqrt{x^2 + y^2}$

$$\therefore \quad Vy - (x^2 + y^2)r_1 - C = \text{rotor input per phase}$$

$$\therefore \quad Vy - r_1[R^2 + 2xx_c + 2yy_c - x_c^2 - y_c^2) - C = 0 \quad . \quad (3.91)$$

i.e. $$y(V - 2y_c r_1) - 2xx_c r_1 - r_1[R^2 - x_c^2 - y_c^2) - C = 0 \quad . \quad (3.92)$$

At B and P, the torque is zero, since at B, $I_2' = 0$ and at P, $s = \infty$, so at B also the torque is zero. But this condition implies the truth of equation (3.92). This equation is linear in y and x, and, therefore, represents a straight line.

We may also arrive at the circle and its centre co-ordinates in another way—

We have $$\mathbf{I}_1 = \mathbf{V}\left\{\frac{a + jb}{c + jd}\right\} = \mathbf{V}\left\{\frac{\mathbf{E} + \mathbf{F}s}{\mathbf{G} + \mathbf{H}s}\right\} \quad . \quad . \quad (3.33)$$

Let $\mathbf{I}_1 = \alpha - j\beta$

also $\mathbf{E} = r_2'g_m - jr_2'b_m = q + jr$

$\mathbf{F} = 1 + x_2'b_m + jx_2'g_m = s_1 + jt$

$\mathbf{G} = r_2' + r_1r_2'g_m + x_1r_2'b_m + j(x_1r_2'g_m - r_1r_2'b_m) = y + jz$

$\mathbf{H} = r_1 + b_mx_2'r_1 - x_1x_2'g_m + j(x_1 + x_2' + x_2'r_1g_m + x_1x_2'b_m)$
$= u + jv$

From equation (3.33) we have

$$\mathbf{I}_1\mathbf{G} + \mathbf{I}_1\mathbf{H}s = V\mathbf{E} + V\mathbf{F}s \quad . \quad . \quad . \quad (3.93)$$

i.e. $$\mathbf{I}_1\mathbf{G} - V\mathbf{E} = s(V\mathbf{F} - \mathbf{I}_1\mathbf{H}) \quad . \quad . \quad . \quad (3.94)$$

$$\therefore \quad (\alpha - j\beta)(y + jz) - V(q + jr) = s[Vs_1 + Vjt - (\alpha - j\beta)(u + jv)] \quad . \quad . \quad (3.95)$$

$$\therefore \quad \alpha y + \beta z + j(z\alpha - y\beta) - Vq - Vjr = s(Vs_1 - \alpha u - \beta v) + js[Vt + \beta u - \alpha v] \quad (3.96)$$

$$\therefore \quad \alpha y + \beta z - Vq = s(Vs_1 - \alpha u - \beta v) \quad . \quad . \quad (3.97)$$

$$\alpha z - y\beta - Vr = s(Vt + \beta u - \alpha v) \quad . \quad . \quad (3.98)$$

$$\therefore \quad \frac{\alpha y + \beta z - Vq}{\alpha z - y\beta - Vr} = \frac{Vs_1 - \alpha u - \beta v}{Vt + \beta u - \alpha v} \quad . \quad . \quad (3.99)$$

$$\alpha yVt + \alpha y\beta u - \alpha^2 yv + \beta zVt + \beta^2 zu - \beta z\alpha v - V^2qt - V\beta qu + \alpha vVq$$
$$= \alpha zVs_1 - \alpha^2 zu - \alpha\beta zv - y\beta Vs_1 + y\beta\alpha u + y\beta^2 v - V^2rs_1 + V\alpha ru + V\beta rv$$

i.e. $$\alpha^2[zu - yv] + \beta^2[zu - yv] + \beta[zVt - Vqu + yVs_1 - Vrv]$$
$$+ \alpha[yVt + vVq - zVs_1 - Vru] + V^2rs_1 - V^2qt = 0 \quad . \quad (3.100)$$

i.e. that is

$$\alpha^2 + \beta^2 + \alpha\left[\frac{yVt + vVq - zVs_1 - Vru}{zu - yv}\right]$$
$$+ \beta\left[\frac{zVt - Vqu + yVs_1 - Vrv}{zu - yv}\right] + V^2\left[\frac{rs_1 - qt}{zu - yv}\right] = 0 \quad (3.101)$$

Let $$K = \frac{yVt + vVq - zVs_1 - Vru}{zu - yv} \quad . \quad . \quad . \quad . \quad (3.102)$$

$$l = \frac{zVt - Vqu + yVs_1 - Vrv}{zu - yv} \quad . \quad . \quad . \quad . \quad (3.103)$$

$$N = V^2 \left(\frac{rs_1 - qt}{zu - yv}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.104)$$

Our equation is, therefore,

$$\alpha^2 + \beta^2 + K\alpha + l\beta + N = 0 \quad . \quad . \quad (3.105)$$

i.e. $$\left(\alpha + \frac{1}{2}K\right)^2 + \left(\beta + \frac{l}{2}\right)^2 = \frac{K^2}{4} + \frac{l^2}{4} - N \quad . \quad (3.106)$$

The co-ordinates of the centre are

$$-\frac{1}{2}K \text{ and } -\frac{l}{2}$$

and the radius of the circle

$$= \sqrt{\frac{K^2}{4} + \frac{l^2}{4} - N} \quad . \quad . \quad . \quad (3.107)$$

CHAPTER IV

The Single-Phase Induction Motor

THIS motor is in very wide use in industry, especially in the fractional horse-power field, and is extensively used for oil-burners, fans, blowers, office appliances, and certain types of small tools. It is also widely used for refrigerators, pumps, compressors, washing and ironing machines. For use on the farm, it is used in sizes of from 1 to 7·5 h.p., for threshing, feed-grinding, corn-shelling, grain-drying, corn-husking, hay-baling, milking, water-pumping, bottle-washing, milk-cooling, and also for silo work. It is a very useful motor in relatively small outputs. For large powers it suffers from disadvantages, which are inherent in its characteristics, and is never used in cases where a polyphase motor can be adopted. Chief among these disadvantages are: (*a*) output only about 50 per cent of the three-phase motor, for a given frame size and temperature rise, (*b*) lower power factor, (*c*) lower efficiency, and (*c*) has no inherent starting torque, and, therefore, requires a starting winding, with a phase-splitting device. In spite of these drawbacks, it is admirably adapted for small outputs. Whilst the simplest in construction of all a.c. machines, its theory is more complicated than any. Its characteristics are deduced by two very different methods. The methods are known as: (1) the cross-field theory and (2) the two revolving fields theory.

THE CROSS-FIELD THEORY*

This theory is due to Potier and H. Goerges. Since this offers some difficulty to the average student, it will help, perhaps, if a few simple fundamental principles are enunciated. Briefly these are—

(*a*) In any transformer, the secondary ampere-turns oppose the primary ampere-turns, and tend to reduce the flux.

(*b*) To determine the direction of the e.m.f. in any conductor, place the *right* hand along the conductor, in such a position that the flux passes from the palm to the back and the thumb points in the direction of motion of the conductor relative to the field, then the direction of *the e.m.f. is from the wrist to the finger tips.*

(*c*) To determine the direction of the force on a conductor, carrying current, and situated in a magnetic field, place the *left hand* along the conductor, in such a position that the flux passes from the palm to the back, and the current flows in the direction from the wrist to the finger tips, then the thumb points to the direction of the force and motion, due to the force.

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* The following paragraphs follow Behrend closely

(*d*) Another variant of (*c*) is to imagine the flux to be like a heavy curtain, the field surrounding the current-carrying conductors has a general direction at right angles to the main flux, and tends to move the curtain to the right or left, depending on its direction. The force acting on the conductor is in the opposite direction to the movement of the curtain, due to the field of the conductor. This is shown in Fig. 4.1.

The single-phase motor has usually, a distributed, drum winding, which usually fills two-thirds of the pole pitch in each pole arc.

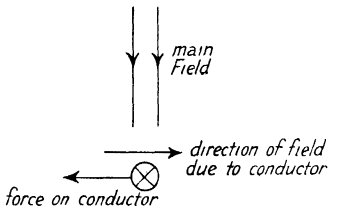


FIG. 4.1

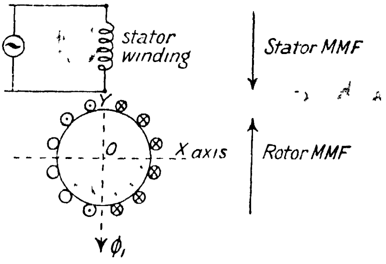


FIG. 4.2

The remaining one-third of the pole arc, accommodates another winding, which is used for starting purposes.

In Fig. 4.2 we represent the stator winding by a single coil, whose magnetic axis is the vertical axis. With the direction of current, at the instant chosen, our magnetic flux is shown. We will call this flux ϕ_1. Assume there is a squirrel-cage rotor.

At standstill our rotor behaves like a short-circuited transformer, so our rotor currents, at standstill, must flow in such a direction that the rotor m.m.f. opposes that of the stator. The magnetic axis of the rotor, due to these currents, generated in it by transformer action, is the vertical or *Y*-axis or *transformer axis.*

It will be noted that, *at standstill,* there is no resultant torque on the rotor, for the interaction of the currents on the right-hand side of the rotor with the flux ϕ_1 is neutralized by the action of the currents on the left-hand side of the rotor with ϕ_1.

Thus, the single-phase induction motor, unaided by a starting winding, has no starting torque.

Such a motor, however, if given motion in any direction, either clockwise or anti-clockwise, will develop a torque and continue to run in that direction. Now let us assume it is running in a clockwise direction. Rotation in the flux ϕ_1 will generate in the rotor conductors e.m.f.s, and currents will flow in such a direction as to produce another flux along the horizontal axis, i.e. the X-axis. This is known as the *excitation axis.*

Fig. 4.3 shows the directions of e.m.f.s and currents in the rotor due to rotation in the transformer flux during the time ϕ_1 acts downwards. We will concentrate our attention on the actions in the two axes X and Y.

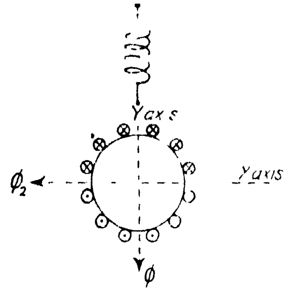


FIG. 4.3

Consider first the X-axis. We have e.m.f.s set up by rotation in ϕ_1, in the direction shown in Fig. 4.3. Due to pulsation of ϕ_2, we have a transformer e.m.f. in the X-axis, which lags 90° behind ϕ_2.

We shall call the e.m.f.s in the X-axis—

(*a*) Rotational e.m.f., e_{xr}.

(*b*) Transformer e.m.f., e_{xt}.

These two e.m.f.s in the X-axis oppose one another, and are nearly in phase opposition. The direction of the currents set up is determined by the rotational voltage e_{xr}.

In the vertical axis, the Y-axis, we have also two e.m.f.s—

(*a*) A transformer e.m.f., e_{vt}, due to pulsation of ϕ_1.

(*b*) A rotational e.m.f., e_{vr}, due to rotation in the flux ϕ_2.

$e_{vt} > e_{vr}$ and the direction of the currents in the Y-axis is determined by the resultant of e_{vt} and e_{vr}.

Now in the Y-axis e_{vt} and e_{vr} are opposed to one another, or more strictly, the phase angle between them is greater than 90°.

Now torque will be produced in the following manner—

(*a*) By the interaction of the excitation flux ϕ_2 and the currents in the Y-axis. This is the *driving torque*, produced by the interaction of *the currents in the Y-axis* with *the excitation flux* ϕ_2.

(*b*) By the interaction of the currents in the excitation axis, i.e. the X-axis currents, with the transformer flux ϕ_1. This is a *retarding torque.*

Thus, we note that in the single-phase motor there are two torques produced—

(1) The driving torque, due to the action of the currents i_v and ϕ_2, the excitation flux.

(2) The retarding torque, due to the action of the currents i_x and ϕ_1, the transformer flux.

Now the excitation flux ϕ_2 is displaced by $\frac{\pi}{2}$ in space from ϕ_1, but its time phase angle will differ from $\frac{\pi}{2}$.

The current produced in the X-axis by rotation in ϕ_1 will lag behind the rotational e.m.f. in the X-axis by an angle whose tangent $= \frac{\text{rotor reactance}}{\text{resistance}}$, and since the rotational voltage is in phase with the flux ϕ_1, it follows that the time phase angle of ϕ_1 and ϕ_2 will be given by $\tan^{-1}\frac{x_2}{r_2} = \tan^{-1}\frac{\text{rotor reactance}}{\text{rotor resistance}}$

We shall be concerned with the relation of ϕ_1 and ϕ_2 very closely in what follows. It should be noted that ϕ_2 is in phase, neglecting hysteresis effects, with i_x.

At synchronous speed $\phi_1 = \phi_2$, and if the time phase angle were $\frac{\pi}{2}$, we should have a true rotating wave of flux of constant amplitude, as in the polyphase motor.

Since the time phase angle $= \tan^{-1}\frac{x_2}{r_2}$, it follows that the angle will approach $\frac{\pi}{2}$ the smaller r_2, i.e. the smaller the *rotor resistance.* Since, also, x_2 depends on the leakage flux of the rotor, it follows that, since the leakage flux is inversely proportional to the reluctance, this reluctance must be small also. *A low resistance of rotor* is required for approximate quadrature time relations of the fluxes ϕ_1 and ϕ_2. Whatever the resistance be, the phase relation between ϕ_1 and ϕ_2 remains approximately constant as the load and speed vary.

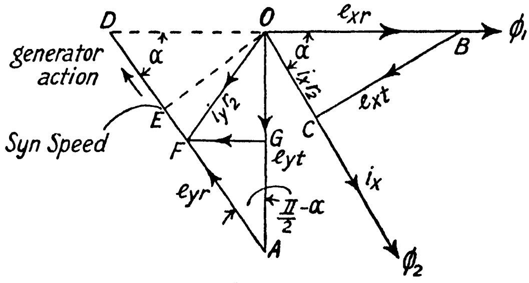


FIG. 4.4

The vector diagram, Fig. 4.4, is very simple. We shall adopt the device of replacing leakage reactance voltage drops by the corresponding leakage flux. In that case, only resistance drops are present.

$OB = e_{xr}$ = rotational e.m.f. in X-axis in phase with ϕ_1

$\alpha = \angle BOC = \tan^{-1}\frac{x_2}{r_2}$ = constant angle between ϕ_1 and ϕ_2

$OC = i_x r_2$ = resistance voltage in x-axis of the rotor

$BC = e_{xt}$ = transformer voltage in X-axis lagging $\frac{\pi}{2}$ behind ϕ_2

$OA = e_{yt}$ = transformer voltage in Y-axis due to ϕ_1 and lagging $\frac{\pi}{2}$ behind ϕ_1

$AF = e_{yr}$ = rotational voltage in Y-axis due to rotation in ϕ_2

$OF = i_y r_2$ = resistance voltage in the vertical axis

OBC = triangle of voltages in horizontal or excitation axis (= X-axis)

OAF = triangle of voltages in vertical Y-axis of rotor

As the load varies, the point F moves along the line AD. At the point E, where OE is perpendicular to AD, synchronous speed is reached.

From E to D and beyond we have generator action.

Relation of Fluxes ϕ_1 and ϕ_2

Let f = supply frequency = pairs of poles × sychronous revolutions per second

f_1 = frequency due to rotation = pairs of poles × revolutions per second

Then
$$e_{xr} \propto \phi_1 \times f_1 \qquad . \quad . \quad . \quad . \quad (4.1)$$

and
$$e_{xt} \propto \phi_2 \times f = e_{xr} \sin \alpha \qquad . \quad . \quad . \quad (4.2)$$

$$\therefore \quad \phi_2 \times f = \phi_1 \times f_1 \sin \alpha \qquad . \quad . \quad . \quad (4.3)$$

$$\therefore \quad \frac{\phi_2}{\phi_1} = \frac{f_1}{f} \sin \alpha = k \sin \alpha \qquad . \quad . \quad . \quad (4.4)$$

where $k = \frac{f_1}{f}$

Also
$$e_{yt} \propto \phi_1 \times f \qquad . \quad . \quad . \quad (4.5)$$

$$e_{yr} \propto \phi_2 \times f_1 \propto \phi_1 \times \left(\frac{f_1}{f}\right)^2 \times f \sin \alpha \qquad . \quad . \quad (4.6)$$

$$AE = OA \cos\left(\frac{\pi}{2} - \alpha\right) = OA \sin \alpha = e_{yt} \sin \alpha \qquad . \quad (4.7)$$

i.e.
$$AE = e_{yt} \sin \alpha \propto \phi_1 f \sin \alpha \qquad . \quad . \quad . \quad (4.8)$$

$$AF = e_{yr} \propto \phi_1 \times \left(\frac{f_1}{f}\right)^2 f \sin \alpha \qquad . \quad . \quad (4.9)$$

$AF = AE$ when $f_1 = f$, i.e. at synchronous speed. Therefore, the point E corresponds to synchronous speed. Also it is clear that, beyond the point E, the current i_y makes an angle less than 90° with e_{yr}, i.e. the machine is a generator beyond E towards D.

The driving torque, multiplied by the speed, expressed in watts

$$= e_{yr} \times i_y \cos \angle OFA \propto AF \times EF \qquad . \quad . \quad (4.10)$$

At synchronous speed the $\triangle OCB = \triangle OAE$.

Now
$$OE = OA \cos \alpha = e_{yt} \cos \alpha . \qquad . \quad . \quad (4.11)$$

and $$OC = OB \cos\alpha = e_{xr} \cos\alpha \quad . \quad . \quad . \quad (4.12)$$

but $e_{yt} = e_{xr}$ at synchronous speed (*see* equations (4.1) and (4.5)).

Also $$OE^2 = AE \, . \, DE \quad . \quad . \quad . \quad (4.13)$$

The negative power

$$= e_{xr} i_x \cos\alpha \quad . \quad . \quad . \quad . \quad (4.14)$$

$$= i_x^2 r_2 \propto \frac{OC^2}{r_2} \quad . \quad . \quad . \quad . \quad (4.15)$$

Now $$OC = e_{xr} \cos\alpha \propto \phi_1 \times f_1 \cos\alpha$$

$$OE = OA \cos\alpha \propto \phi_1 f \cos\alpha \quad . \quad . \quad (4.16)$$

$$\therefore \quad OC = OE \times \frac{f_1}{f} \quad . \quad . \quad . \quad . \quad (4.17)$$

Now $$AF = e_{yr} \propto \phi_2 \times f_1 \propto \phi_1 \times \left(\frac{f_1}{f}\right)^2 f \sin\alpha . \quad . \quad (4.18)$$

$$AE = e_{yt} \sin\alpha \propto \phi_1 \times f \sin\alpha \quad . \quad . \quad (4.19)$$

$$\therefore \quad \frac{AF}{AE} = \left(\frac{f_1}{f}\right)^2 \quad . \quad . \quad . \quad . \quad (4.20)$$

Now the copper loss $\propto OC^2 \propto OE^2 \times \left(\dfrac{f_1}{f}\right)^2$

i.e. $$OC^2 \propto AE \, . \, ED \times \frac{AF}{AE} \quad . \quad . \quad . \quad (4.21)$$

for $OE^2 = AE \, . \, ED$

$$\therefore \quad OC^2 \propto AF \, . \, ED$$

Therefore, the negative power is proportional to

$AF \times ED$ in Fig. 4.4 . . . (4.22)

Therefore, the net power = driving power − negative power

$\propto AF(EF - ED)$ in Fig. 4.4 . . (4.23)

Equivalent Circuit

The current in the *Y*-axis, $i_y = \dfrac{OF}{r_2}$, is resolved into two components, $\dfrac{OG}{r_2}$ is in phase with *OA*, i.e. with e_{yt}. It is, therefore, a watt component, and a second component $\dfrac{FG}{r_2}$, which is wattless, and subtracting in its magnetizing action from the flux ϕ_1. (*See* Fig. 4.4.)

Now $OG = OA - AG = e_{vt} - AG = e_{vt} - AF \sin \alpha$. (4·24)

$$= e_{vt} - e_{vr} \sin \alpha \quad . \quad (4·25)$$

Let R = resistance, which when acted upon by e_{vt} will have a current OG flowing in it, i.e. $\frac{e_{vt}}{R}$ = the watt component of the current.

$$GF = AF \cos \alpha = e_{vr} \cos \alpha \quad . \quad . \quad . \quad (4·26)$$

but
$$e_{vr} = \left(\frac{f_1}{f}\right)^2 \sin \alpha \, e_{vt}$$

$$\therefore \quad GF = e_{vt} \left(\frac{f_1}{f}\right)^2 \sin \alpha \cos \alpha$$

and
$$OG = e_{vt} - \left(\frac{f_1}{f}\right)^2 \sin^2 \alpha \, e_{vt} = e_{vt} \left\{1 - \left(\frac{f_1}{f}\right)^2 \sin^2 \alpha\right\} \quad . \quad (4·27)$$

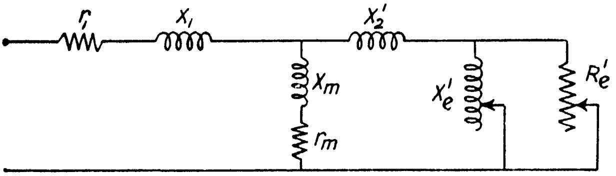


FIG. 4·5

$$\therefore \quad \frac{e_{vt}}{R} = \frac{e_{vt}\left\{1 - \left(\frac{f_1}{f}\right)^2 \sin^2 \alpha\right\}}{r_2} \quad . \quad . \quad . \quad (4·28)$$

$$\therefore \quad R_e = \frac{r_2}{1 - \left(\frac{f_1}{f}\right)^2 \sin^2 \alpha} \quad . \quad . \quad . \quad (4·29)$$

Therefore, the equivalent resistance

$$= \frac{r_2}{1 - \left(\frac{f_1}{f}\right)^2 \sin^2 \alpha} \quad . \quad . \quad (4·30)$$

The wattless component of current in the Y-axis

$$= \frac{FG}{r_2} = e_{vt} \left(\frac{f_1}{f}\right)^2 \frac{\sin \alpha \cos \alpha}{r_2} \quad . \quad . \quad (4·31)$$

Therefore, let X_e = the equivalent reactance.

Then
$$\frac{e_{vt}}{X_e} = e_{vt} \left(\frac{f_1}{f}\right)^2 \frac{\sin \alpha \cos \alpha}{r_2} \quad . \quad . \quad . \quad (4·32)$$

$$\therefore \quad X_e = \frac{r_2}{\left(\frac{f_1}{f}\right)^2 \sin\alpha\cos\alpha} \quad . \quad . \quad . \quad . \quad (4\cdot33)$$

The equivalent reactance

$$= \frac{r_2}{\left(\frac{f_1}{f}\right)^2 \sin\alpha\cos\alpha} \quad . \quad . \quad . \quad . \quad (4\cdot34)$$

$$= \frac{r_2}{k^2 \sin\alpha\cos\alpha}$$

The total current in the Y-axis, $i_y = \frac{OF}{r_2}$

$$= \frac{\sqrt{OG^2 + GF^2}}{r_2} \quad . \quad . \quad . \quad . \quad (4\cdot35)$$

$$OG^2 = e_{yt}^2\,[1 - 2k^2\sin^2\alpha + k^4\sin^4\alpha]. \quad . \quad (4\cdot36)$$

where $$k = \frac{f_1}{f} = \frac{\text{actual speed}}{\text{synchronous speed}} \quad . \quad . \quad (4\cdot37)$$

$$GF^2 = e_{yt}^2\, k^4 \sin^2\alpha\cos^2\alpha$$

$$\therefore i_y = \frac{e_{yt}\sqrt{1 - 2k^2\sin^2\alpha + k^4\sin^4\alpha + k^4\sin^2\alpha\cos^2\alpha}}{r_2} \quad (4\cdot38)$$

$$= \frac{e_{yt}\sqrt{1 - 2k^2\sin^2\alpha + k^4\sin^2\alpha(\sin^2\alpha + \cos^2\alpha)}}{r_2} \quad . \quad (4\cdot39)$$

$$= \frac{e_{yt}\sqrt{1 - 2k^2\sin^2\alpha + k^4\sin^2\alpha}}{r_2} \quad . \quad . \quad . \quad . \quad (4\cdot40)$$

$$= \frac{e_{yt}\sqrt{1 + k^2\sin^2\alpha(k^2 - 2)}}{r_2} \quad . \quad . \quad . \quad . \quad (4\cdot41)$$

When F is at A, i.e. $k = \frac{f_1}{f} = 0$, i.e. at standstill

$$\underset{(k=0)}{i_y} = \frac{e_{yt}}{r_2} \quad . \quad . \quad . \quad . \quad (4\cdot42)$$

At E, when F coincides with E, i.e. at synchronous speed

$$\underset{(k=1)}{i_y} = \frac{e_{yt}\cos\alpha}{r_2} \quad . \quad . \quad . \quad . \quad (4\cdot43)$$

This is the minimum secondary current as will be seen clearly from Fig. 4.4, since OE is the shortest distance from O to AD.

The equivalent reactance of the rotor circuit X_c (*see* equation (4·34))

$$= \frac{r_2}{\left(\frac{f_1}{f}\right)^2 \sin\alpha\cos\alpha} = \frac{r_2}{k^2 \sin\alpha\cos\alpha} \quad . \quad . \quad . \quad (4{\cdot}44)$$

The equivalent resistance

$$= \frac{r_2}{1 - k^2 \sin^2\alpha} \quad . \quad . \quad . \quad . \quad (4{\cdot}45)$$

Since the locus of the rotor current in the *Y*-axis is a circle, it follows that the locus of the primary current is a circle, for the primary current will consist of the magnetizing current *OM* (in Fig. 4.6), which is necessary to produce the flux ϕ_s and a component, compounded with *OM*, which is parallel to I_y, the current in the *Y*-rotor circuit, and drawn in the opposite direction to I_y, and the magnetizing action of this component, flowing in the stator must equal the magnetizing action of I_y. This component is shown as *MP* in Fig. 4.6.

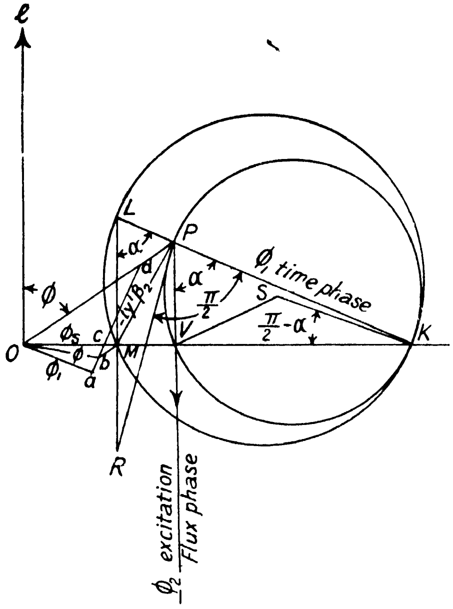


FIG. 4.6

$$MP = -I_y \times \frac{\text{rotor turns per pole (effective)}}{\text{stator turns per pole (effective)}} \times \beta_2 \quad (4{\cdot}46)$$

At synchronous speed, the transformer flux ϕ_1 and the excitation flux ϕ_2 are equal and $OM = MV$.

The ideal short-circuit current $= OK$.

As the load varies, the locus of the primary current is the circle *KPV*.

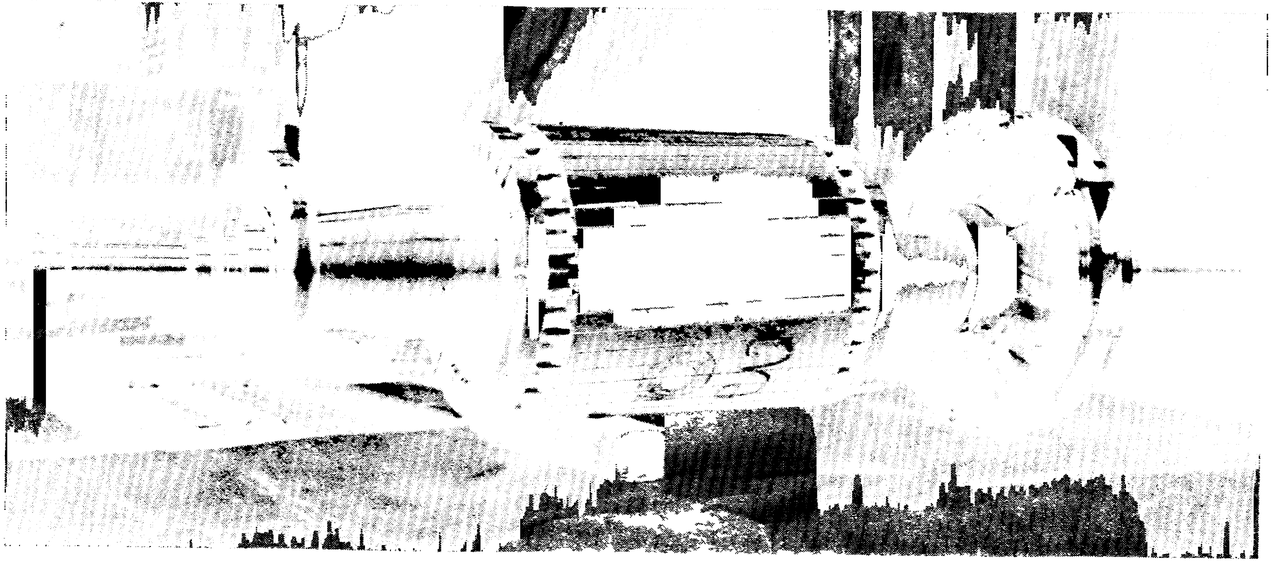


PLATE V DOUBLE SQUIRREL-CAGE ROTOR

This rotor has axially ventilated core with a balance ring at the non-fan end. Speed 3[illegible] r.p.m.

[illegible]

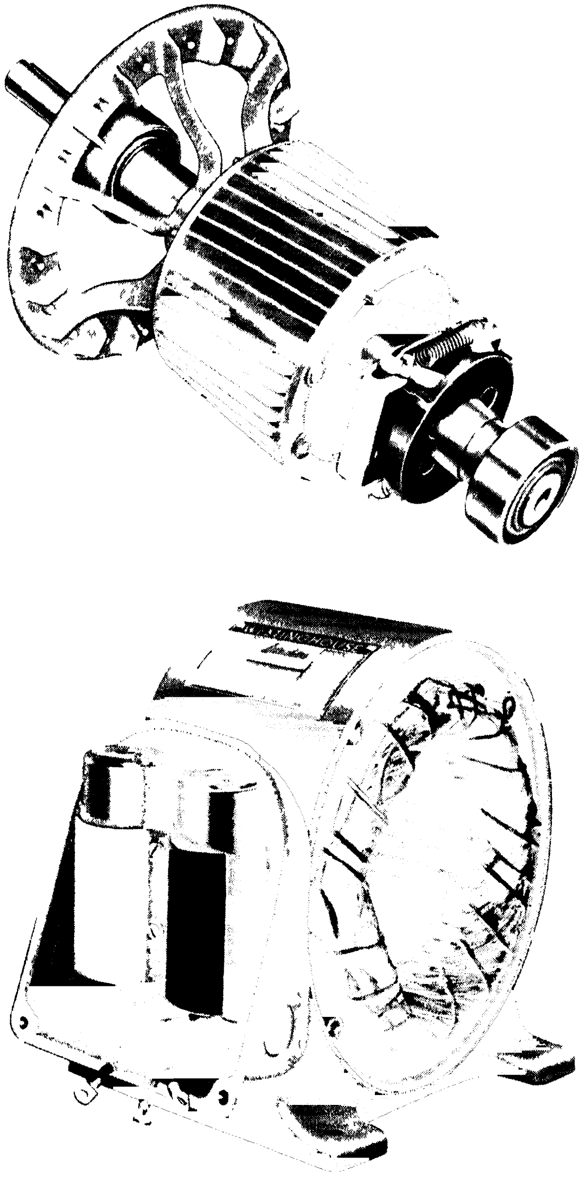


PLATE VI

(*Upper*) SQUIRREL-CAGE ROTOR

Capacitor conveniently located inside the protecting conduit box

(*Courtesy Westinghouse Electric Corp of America*)

(*Lower*) SINGLE-PHASE CAPACITOR MOTOR STATOR

This had a cast aluminum fan centrifugal starting switch and sealed prelubricated bearings Note balance washers on end ring

(*Courtesy Westinghouse Electric Corp of America*)

OP = primary current, and represents to another scale the fictitious flux linking the primary due to current OP. Of this total flux linking the primary, the part Od crosses the gap and links the secondary.

ad = rotor current I_y referred to the primary, and represents to another scale the flux produced by the secondary current. Of this flux the part ab is secondary leakage flux, and the part bd or MP combines with OP to produce OM the resultant flux linking the primary winding, OM.

The resultant of Od, i.e. that part of the primary flux which crosses the air-gap, and da the rotor flux due to rotor current MP, gives us the flux linking the secondary, namely $Oa = \phi_1$—our transformer flux.

The resultant of ϕ_1 and ab, the secondary leakage flux, gives Ob, the resultant flux crossing the gap and linking both primary and secondary. bM parallel to OP and equal to dP represents the primary leakage flux. The vector sum of ϕ and bM gives ϕ_s, the flux linking the primary, to which the voltage $-e$, lagging 90° behind ϕ_s, is due.

Now PK is parallel to Oa or ϕ_1, the transformer flux, and MP is parallel to ac. The triangles Oac and MPK are similar. PV is parallel to ϕ_2, the excitation flux, and $\angle KPV = \angle KLM$.

The $\triangle MPR$ is the $\triangle AOF$ in Fig. 4.4.

$PR = OA = e_{yt}$; $PM = i_y r_2$; $MR = FA = e_{yr}$ drawn vertically parallel to ϕ_2.

The $\angle VSK = 2\alpha$, and since $SK = SV$ = radius of the circle, it follows that $\angle SKV = \angle SVK = \frac{\pi}{2} - \alpha$.

So the centre of the circle is found by drawing lines from K and V, making angles of $\frac{\pi}{2} - \alpha$ with the horizontal.

Let $ad = \beta_2 bd$

and $OP = \beta_1 Od$

Then ab (Fig. 4.6) $= (\beta_2 - 1)bd$

and

$$bc = bd - cd$$

$$= bd\left(1 - \frac{1}{\beta_1}\right)$$

$$ab + bc = \left(\beta_2 - \frac{1}{\beta_1}\right) bd = \left(\frac{\beta_1\beta_2 - 1}{\beta_1}\right) bd = ac \quad . \quad (4.47)$$

$$\therefore \quad \frac{ac}{cd} = \left(\frac{\beta_1\beta_2 - 1}{\beta_1}\right) \frac{bd \times \beta_1}{bd} = \beta_1\beta_2 - 1 \quad . \quad . \quad (4.48)$$

but from similar triangles Oac and MPK

$$\frac{Oc}{ca} = \frac{MK}{MP}$$

$$\therefore \quad \frac{Oc}{MK} = \frac{ca}{MP}$$

$$\therefore \quad \frac{OM}{\beta_1 MK} = \frac{ca}{\beta_1 cd}$$

i.e.

$$\frac{OM}{MK} = \frac{ca}{cd} = \beta_1\beta_2 - 1$$

If we call this ratio σ, we have

$$\frac{OM}{MK} = \sigma \quad . \quad . \quad . \quad . \quad . \quad (4\cdot49)$$

Since

$$\frac{MV}{VK} = \frac{LP}{PK} = \frac{\phi_1}{\beta_2} \Big/ MK\beta_1 \frac{\phi_1}{\phi_s} \quad . \quad . \quad . \quad (4\cdot50)$$

$$= \frac{1}{\beta_1\beta_2}\left(\frac{\phi_s}{MK}\right) \quad . \quad . \quad . \quad . \quad (4\cdot51)$$

$$= \frac{1}{\beta_1\beta_2}(\beta_1\beta_2 - 1) \quad . \quad . \quad . \quad (4\cdot52)$$

$$= \frac{\sigma}{\sigma + 1} \quad . \quad . \quad . \quad . \quad . \quad (4\cdot53)$$

and

$$\frac{MK}{VK} = \frac{2\sigma + 1}{\sigma + 1} = \frac{MV + VK}{VK} = 1 + \frac{MV}{VK} \quad . \quad (4\cdot54)$$

$$= 1 + \frac{\sigma}{\sigma + 1} \quad . \quad . \quad . \quad . \quad . \quad (4\cdot55)$$

$$\frac{OM}{MV} = \frac{OM}{MK} \times \frac{MK}{MV} = \sigma \times \frac{MK}{VK} \times \frac{VK}{MV} \quad . \quad . \quad (4\cdot56)$$

$$= \sigma \times \frac{2\sigma + 1}{\sigma + 1} \times \frac{\sigma + 1}{\sigma}$$

$$= 2\sigma + 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4\cdot57)$$

Also

$$\frac{OV}{OM} = \frac{MV}{OM} + 1 = \frac{1}{2\sigma + 1} + 1 = \frac{2\sigma + 2}{2\sigma + 1} = \frac{2\{\sigma + 1\}}{2\sigma + 1} \quad (4\cdot58)$$

Note that

$$\frac{MV}{OM} = \frac{1}{2\sigma + 1} \quad (4\cdot59)$$

i.e. the magnetizing current for the excitation field demands an extra wattless current in the Y-circuit of the stator = the magnetizing current for $\phi_s \times \dfrac{1}{1 + 2\sigma}$

Since σ is usually small the *magnetizing current* for the single-phase induction motor is nearly double that necessary to produce ϕ_1, the transformer field.

Note, we have used for σ here the ratio $\frac{OM}{MK}$. When considering the polyphase motor we used for σ the ratio

$$\frac{OM}{OK} = \frac{\text{magnetizing current}}{\text{ideal short-circuit current}} = \frac{i_\mu}{i_{sc}}$$

We shall call the ratio $\frac{OM}{OK} = \sigma_1$.

Then
$$\frac{OM}{MK} = \frac{OM}{OK - OM} = \frac{\frac{OM}{OK}}{1 - \frac{OM}{OK}} = \frac{\sigma_1}{1 - \sigma_1} \qquad . \quad (4.60)$$

$$\therefore \qquad \sigma = \frac{\sigma_1}{1 - \sigma_1} \qquad . \quad . \quad . \quad . \quad (4.61)$$

Thus, if we desire to keep to our old definition, the relation is

$$\sigma = \frac{\sigma_1}{1 - \sigma_1}$$

where $\sigma = \frac{OM}{MK}$

and $\sigma_1 = \frac{OM}{OK}$

Finally, we have

$$\begin{aligned}
\frac{OV}{VK} &= \frac{OM + MV}{VK} = \frac{OM}{VK} + \frac{MV}{VK} \\
&= \frac{OM}{MV} \times \frac{MV}{VK} + \frac{MV}{VK} \\
&= 2\sigma + 1 \times \frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} \\
&= \frac{\sigma}{\sigma + 1}\{2\sigma + 1 + 1\} = 2\sigma \qquad . \quad . \quad (4.62)
\end{aligned}$$

Thus, collecting our results—

$$\frac{OM}{MK} = \sigma = \beta_1\beta_2 - 1 \qquad . \quad . \quad . \quad (4.63)$$

where β_1 = stator leakage factor
and β_2 = rotor leakage factor

$$\frac{OM}{MK} = \sigma = \frac{\sigma_1}{1 - \sigma_1} \qquad . \quad . \quad . \quad (4.64)$$

where $\sigma_1 = \frac{OM}{OK}$

$$\frac{MV}{VK} = \frac{\sigma}{\sigma + 1} = \sigma_1 \quad . \quad . \quad . \quad (4.65)$$

$$\frac{OM}{MV} = 2\sigma + 1 = \frac{\sigma_1 + 1}{1 - \sigma_1} \quad . \quad . \quad . \quad (4.66)$$

$$\frac{OV}{OM} = 2\left\{\frac{\sigma + 1}{2\sigma + 1}\right\} = \frac{2}{1 + \sigma_1} \quad . \quad . \quad (4.67)$$

$$\frac{MV}{OM} = \frac{1}{2\sigma + 1} = \frac{1 - \sigma_1}{1 + \sigma_1} \quad . \quad . \quad . \quad (4.68)$$

$$\frac{OV}{VK} = 2\sigma = \frac{2\sigma_1}{1 - \sigma_1} \quad . \quad . \quad . \quad (4.69)$$

To make clear these various equations, it should be noted that

$$\left.\begin{array}{l} \beta_1 = \dfrac{OP}{od} = \dfrac{\text{total flux produced by stator current}}{\text{that part of the stator flux which crosses the gap and links the secondary}} \\[2ex] \text{and } \beta_2 = \dfrac{ad}{bd} = \dfrac{\text{total flux produced by rotor current}}{\text{flux produced by rotor current which crosses the gap and links the primary}} \end{array}\right\} (4.70)$$

It will be abundantly clear now why the power factor of this machine is lower than the three- or two-phase motor

We now have an additional magnetizing current MV, which causes a raising of the dispersion coefficient σ_1, and as we have seen this is one of the main causes of low power factor.

FURTHER ANALYSIS*

Consider a coil AB on the rotor, the plane of which makes an angle θ with the X-axis at any time t.

Let us consider two axes, viz. the X-axis or transformer axis and the Y-axis or excitation axis.

Let ϕ_c = resolved part of the flux in the direction of the coil axis.

Then $$\phi_c = \phi_{RX} \sin\theta - \phi_{RY} \cos\theta \quad . \quad . \quad (4.71)$$

where ϕ_{RX} = flux in the rotor along the X-axis . . . (4.72)

ϕ_{RY} = flux in the rotor along the Y-axis . . . (4.73)

---

* This analysis is from a paper by the author in the *Journal of the American Institute of Electrical Engineers*, vol. 67 (1948)

The e.m.f. induced in the coil *AB*, due to variation of ϕ_{RX} and ϕ_{RY} with time, and also due to variation of θ with time is obtained by differentiating equation (4.71) with respect to time

$$\therefore \phi_c' = \omega \cos \theta \phi_{RX} + \omega \sin \theta \phi_{RY} + \sin \theta \phi_{RX}' - \cos \theta \phi_{RY}' \quad (4{\cdot}74)$$

where $\omega = \dfrac{d\theta}{dt}$

The e.m.f. in the coil $= -\phi_c'$ and the current in the coil

$$= \frac{-\phi_c'}{r \times 10^8} \text{ amps} = i_c \quad . \quad . \quad . \quad (4{\cdot}75)$$

In this analysis, the flux linking the coil, is the resultant of the flux crossing the air-gap into the rotor and the leakage flux. In other words the leakage reactance drops are taken care of, and the only voltage drop to consider is that used in overcoming resistance.

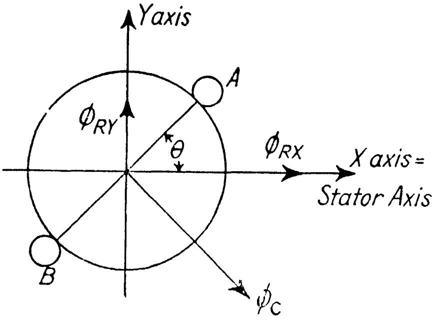


Fig. 4.7

r = resistance of one coil, in ohms.

The m.m.f. of the coil

$$= 0{\cdot}4\pi i_c = F_c \quad . \quad . \quad . \quad . \quad (4{\cdot}76)$$

$$\therefore F_c = \left(\frac{-4\pi}{10^9 r}\right)(\omega\phi_{RX} \cos \theta - \phi_{RY}' \cos \theta + \omega\phi_{RY} \sin \theta + \phi_{RX}' \sin \theta) \quad . \quad . \quad (4{\cdot}77)$$

The resolved part of F_c along the *X*-axis, which we may call

$$\delta F_{cRX} = \left(\frac{-4\pi}{10^9 r}\right)(\omega\phi_{RX} \sin \theta \cos \theta - \phi_{RY}' \sin \theta \cos \theta + \omega\phi_{RY} \sin^2 \theta + \phi_{RX}' \sin^2 \theta) \quad . \quad (4{\cdot}78)$$

To find the total m.m.f. along the *X*-axis, due to N_2 coils or $2N_2$ rotor bars, we must sum up the m.m.f.s due to these N_2 coils along the *X*-axis.

$$\therefore \quad F_{RX} = \left(\frac{-4\pi}{10^9 r}\right) N_2 \int_0^{\pi} [\sin^2 \theta\{\omega\phi_{RY} + \phi_{RX}'\} + \sin \theta \cos \theta(\omega\phi_{RX} - \phi_{RY}')] \frac{d\theta}{\pi} \quad . \quad (4{\cdot}79)$$

$$= \frac{-2\pi N_2\{\omega\phi_{RY} + \phi_{RX}'\}}{10^9 r} \quad . \quad . \quad . \quad . \quad . \quad (4.80)$$

This is the m.m.f. along the X-axis due to all the coils on the rotor.

Similarly, resolving F_c along the Y-axis and summing up as before, over the coils of the rotor, we get

$$F_{RY} = \frac{2\pi N_2}{10^9 r}\{\omega\phi_{RX} - \phi_{RY}'\} \quad . \quad . \quad (4.81)$$

F_{RY} is the m.m.f. along the Y-axis, due to all the rotor coils.

Let $k_2 = \frac{2\pi N_2}{10^9 r}$, then equations (4.80) and (4.81) may be written

$$F_{RX} = -k_2\omega\phi_{RY} - k_2\phi_{RX}' \quad . \quad (4.82)$$

and

$$F_{RY} = +k_2\omega\phi_{RX} - k_2\phi_{RY}' \quad (4.83)$$

Let us call the flux produced in the X-axis of the stator ϕ_{SX}: this ϕ_{SX} is the transformer flux produced by the combined m.m.f.s of stator and rotor in the X-axis, which is the magnetic axis of the stator.

Let ϕ_{SY} = stator flux in the Y-axis, i.e. the excitation axis

and ϕ_{RY} = rotor flux in the Y-axis

Then

$$\phi_{SY} = \frac{\omega\hat{\phi}_{SX}\sin(\omega_0 t - \lambda)}{\sqrt{\left[\left\{(1+\sigma_r)k_2\left(\frac{\sigma_r + \sigma_r\sigma_s + \sigma_s}{R}\right)\right\}\{\omega^2 - \omega_0^2\} + \frac{R(1+\sigma_s)}{k_2}\right]^2 + [1 + 2\sigma_r + 2\sigma_s + 2\sigma_r\sigma_s]^2\omega_0^2}} \quad . \quad . \quad (4.84)$$

$$\text{and } \tan\lambda = \frac{(1 + 2\sigma_r + 2\sigma_s + 2\sigma_r\sigma_s)\omega_0}{\left\{(1+\sigma_r)k_2\left(\frac{\sigma_r + \sigma_r\sigma_s + \sigma_s}{R}\right)\right\}\left\{(\omega^2 - \omega_0^2) + \frac{R(1+\sigma_s)}{k_2}\right\}} \quad . \quad . \quad (4.85)$$

where λ = angle of lag of ϕ_{SY} behind ϕ_{SX}

ω = angular velocity of rotor in radians per second

$\omega_0 = 2\pi \times$ supply frequency

$\sigma_r = \frac{\text{leakage flux in rotor}}{\text{useful flux in rotor}}$

$\sigma_s = \frac{\text{leakage flux in the stator}}{\text{useful flux in the stator}}$

R = air-gap reluctance

Proof of the Equation for ϕ_{SY} in Terms of ϕ_{SX}

$$F_{RX} = -k_2\omega\phi_{RY} - k_2\phi_{RX}' \quad . \quad . \quad . \quad (4.82)$$

and

$$F_{RY} = k_2\omega\phi_{RX} - k_2\phi_{RY}' \quad . \quad . \quad . \quad (4.83)$$

ϕ_{LX} = stator plus rotor leakage flux in the X-axis

$$= F_{SX}P_{SX} - F_{RX}P_{rx} \quad . \quad . \quad (4.86)$$

P_{SX} = equivalent permeance of stator leakage path.
P_{rx} = equivalent permeance of rotor leakage path.

Now F_{RX} measured along the X-axis will be negative, or opposed to F_{SX}.

$$\phi_{RY} = \phi_{SY} + F_{RY}P_{RY} = \phi_{SY} + F_{RY}P_{rx} \quad . \quad (4.87)$$

$$= \frac{F_{RY}}{R} + F_{RY}P_{rx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.88)$$

$$= F_{RY}\frac{(1+\sigma_r)}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.89)$$

$$\therefore \quad \frac{\phi_{RY}R}{1+\sigma_r} = F_{RY}$$

$$= k_2\omega[\phi_{SX} - F_{SX}P_{SX} + F_{RX}P_{rx}] - k_2\phi_{RY}' \quad (4.90)$$

$$\phi_{SX} = \frac{(F_{SX} + F_{RX})}{R}(1+\sigma_s) \quad . \quad . \quad (4.91)$$

$$\frac{R\phi_{SX}}{(1+\sigma_s)} - F_{RX} - F_{SX} \quad . \quad . \quad . \quad (4.92)$$

$$\therefore \quad \frac{\phi_{RY}R}{1+\sigma_r} = k_2\omega\left[\phi_{SX} - \frac{R\phi_{SX}P_{SX}}{1+\sigma_s} + F_{RX}P_{SX} + F_{RX}P_{rx}\right] - k_2\phi_{RY}' \quad . \quad . \quad (4.93)$$

$$= k_2\omega\left[\phi_{SX} - \frac{\sigma_s\phi_{SX}}{1+\sigma_s} + F_{rx}\left(\frac{\sigma_s}{R} + \frac{\sigma_r}{R}\right)\right] - k_2\phi_{RY} \quad . \quad (4.94)$$

$$= \frac{k_2\omega\phi_{SX}}{1+\sigma_s} + \left\{k_2\omega\frac{\sigma_s}{R} + k_2\omega\frac{\sigma_r}{R}\right\}F_{RX} - k_2\phi_{RY}' \quad . \quad (4.95)$$

Note $\quad P_{SX}R = \sigma_s$ and $P_{rx}R = \sigma_r \quad . \quad . \quad (4.96)$

$$\therefore \quad \frac{\phi_{RY}R}{1+\sigma_r} = \frac{k_2\omega\phi_{SX}}{1+\sigma_s} + \left\{k_2\omega\frac{\sigma_s}{R} + k_2\omega\frac{\sigma_r}{R}\right\}$$

$$\{-k_2\omega\phi_{RY} - k_2\phi_{RX}'\} - k_2\phi_{RY}' \quad . \quad (4.97)$$

$$= A\phi_{SX} - B\phi_{RY} - C\phi_{RX}' - k_2\phi_{RY}' \quad . \quad . \quad (4.98)$$

where
$$A = \frac{k_2\omega}{1 + \sigma_s} \quad . \quad . \quad . \quad . \quad . \quad (4.99)$$

$$B = k_2\omega\left[\frac{[k_2\omega\sigma_s}{R} + \frac{k_2\omega\sigma_r}{R}\right] \quad . \quad . \quad (4.100)$$

$$C = \frac{B}{\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.101)$$

Now substitute the value of ϕ_{RX} (derived from equation (4.83)),

viz.
$$\phi_{RX} = \frac{F_{RY}}{k_2\omega} + \frac{\phi_{RY}'}{\omega} \qquad \text{(from equation (4.83))}$$

$$= \frac{\phi_{RY}R}{(1 + \sigma_r)k_2\omega} + \frac{\phi_{RY}'}{\omega}$$

$$\therefore \quad \phi_{RX}' = \frac{\phi_{RY}'R}{(1 + \sigma_r)(k_2\omega)} + \frac{\phi_{RY}''}{\omega} \quad . \quad . \quad . \quad (4.102)$$

Therefore, equation (4.100) becomes

$$A\phi_{SX} = \frac{C\phi_{RY}''}{\omega} + \phi_{RY}'\left\{\frac{CR}{k_2\omega(1 + \sigma_r)} + k_2\right\} + \phi_{RY}\left\{\frac{R}{1 + \sigma_r} + B\right\} \quad (4.103)$$

$$\therefore \; A\phi_{SX} = \left[\frac{CD^2}{\omega} + \left\{\frac{CR}{k_2\omega(1 + \sigma_r)} + k_2\right\}D + \left(\frac{R}{1 + \sigma_r} + B\right)\right]\phi_{RY} \quad (4.104)$$

$$\therefore \; \phi_{RY} = \frac{A\omega\phi_{SX}}{CD^2 + \left(\frac{CR}{k_2(1 + \sigma_r)} + k_2\omega\right)D + \frac{R\omega}{1 + \sigma_r} + B\omega} \quad (4.105)$$

here $\phi' = \frac{d}{dt}\phi = D\phi$.

$$D^2\phi_{RY} = \frac{d^2}{dt^2}(\phi_{RY})$$

ϕ_{SX} is assumed to vary sinusoidally,

i.e.
$$\phi_{SX} = \hat{\phi}_{SX}\sin\omega_0 t$$

$$\phi_{RY} = \frac{\omega\hat{\phi}_{SX}\sin\omega_0 t}{\frac{C}{A}D^2 + \left(\frac{CR}{Ak_2(1 + \sigma_r)} + \frac{k_2\omega}{A}\right)D + \frac{1}{A}\left\{\frac{R\omega}{1 + \sigma_r} + B\omega\right\}} \quad (4.106)$$

Now
$$\frac{C}{A} = \frac{B}{\omega A} = \frac{k_2\omega\left[\frac{k_2\omega\sigma_s}{R} + \frac{k_2\omega\sigma_r}{R}\right](1 + \sigma_s)}{k_2\omega \times \omega} \quad . \quad (4.107)$$

$$\simeq \frac{k_2\{\sigma_r + \sigma_r\sigma_s + \sigma_s\}}{R} \quad . \quad . \quad . \quad . \quad (4.108)$$

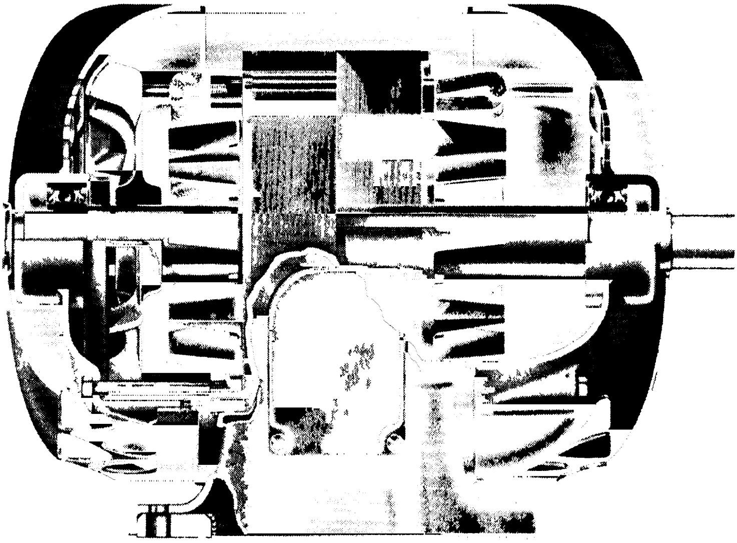


PLATE VII

(*Upper*) SINGLE-PHASE MOTOR TYPE CAP

(*Courtesy Westinghouse Electric Corp of America*)

(*Lower*) SPLASH-PROOF TYPE LIFE-LINE MOTOR

(*Courtesy Westinghouse Electric Corp of America*)

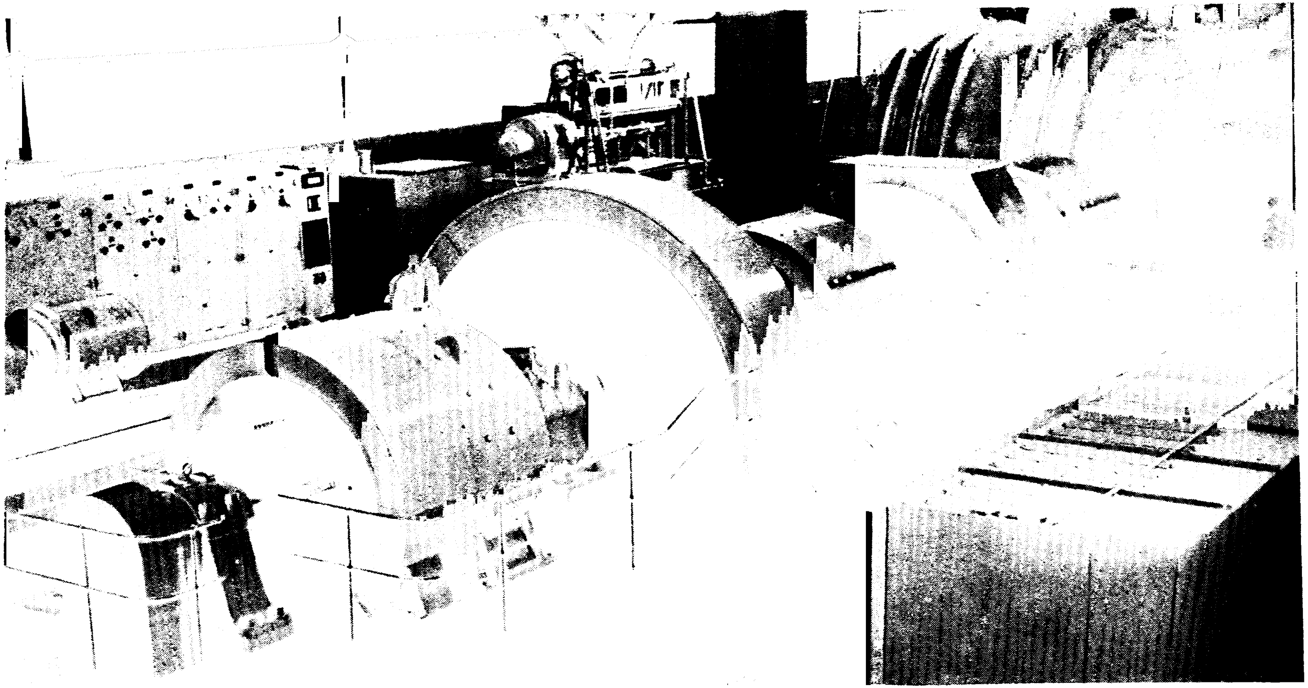


PLATE VIII. ILGNER SET FOR ROLLING-MILL DRIVE

Note 5400 h.p. reversing plate-mill motor in background

(*Courtesy English Electric Co., Ltd.*)

Now
$$\phi_{SY} = \frac{\phi_{RY}}{1 + \sigma_r} \quad . \quad . \quad . \quad (4.109)$$

ϕ_{RY} can be obtained in the usual way by solving the second-order differential equation (4.106). The solution is given in equation (4.84).

It should be noted that
$$\sigma_s = RP_s = \frac{\text{leakage flux in stator}}{\text{useful flux in stator}}$$
and
$$\sigma_r = RP_r = \frac{\text{leakage flux in rotor}}{\text{useful flux in rotor}}$$

Both σ_s and σ_r are small, so their products and squares can be neglected.

Under this assumption,
$$\phi_{SY} \triangleq \frac{\omega\hat{\phi}_{SX} \sin(\omega_0 t - \lambda)}{\sqrt{\left\{(\sigma_r + \sigma_s)(\omega^2 - \omega_0^2)\frac{k_2}{R} + \frac{R(1 + \sigma_s)}{k_2}\right\}^2 + [1 + 2\sigma_r + 2\sigma_s]^2\omega_0^2}} \quad . \quad . \quad (4.110)$$
and
$$\tan\lambda \triangleq -\frac{(1 + 2\sigma_r + 2\sigma_s)\omega_0}{(\sigma_r + \sigma_s)(\omega^2 - \omega_0^2)\frac{k_2}{R} + \frac{R}{k_2}(1 + \sigma_s)} \quad . \quad (4.111)$$

In order to predict the performance of the motor, we need to determine the magneto-motive force in the X-axis of the rotor and also the m.m.f. in the Y-axis. From the m.m.f.s we shall be able to determine the stator and rotor currents. All these quantities will be determined in terms of ϕ_{SX}, which is determined also from the applied voltage and frequency.

Determination of the m.m.f. in the X-axis of the Rotor, F_{RX}

$$F_{RX} = -k_2\omega\phi_{RY} - k_2\phi_{RX}' \quad . \quad . \quad . \quad (4.112)$$
$$= -k_2\omega\phi_{RY} - k_2\left[\frac{F_{RY}' + k_2\phi_{RY}''}{k_2\omega}\right]$$
$$= -k_2\omega\phi_{RY} - \frac{R\phi_{SY}'}{\omega} - \frac{k_2\phi_{RY}''}{\omega} \quad . \quad . \quad (4.113)$$
since $F_{RY}' = \phi_{SY}'R$.
$$\therefore\ F_{RX} = -k_2\omega(1 + \sigma_r)\phi_{SY} - \frac{R\phi_{SY}'}{\omega} - \frac{k_2(1 + \sigma_r)\phi_{SY}''}{\omega} \quad . \quad (4.114)$$
$$= \left[-k_2\omega(1 + \sigma_r) - \frac{RD}{\omega} - \frac{k_2(1 + \sigma_r)D^2}{\omega}\right]\phi_{SY} \quad . \quad (4.115)$$

Equation (4.115) may also be expressed thus—

$$F_{RX} = [-k_2\omega^2(1+\sigma_r) - RD - k_2(1+\sigma_r)D^2]\frac{\phi_{SY}}{\omega} \qquad (4.116)$$

where for ϕ_{SY} is substituted its value given by equation (4.110). The rotor current is determined from the m.m.f.

Determination of F_{SX}, the m.m.f. in the X-axis of the Stator

We have

$$\phi_{SX} = \frac{F_{SX} + F_{RX}}{R} + F_{SX}P_{SX} \qquad . \qquad . \qquad (4.117)$$

i.e. the total flux in the X-axis of the stator is equal to the flux in the X-axis, which crosses the air-gap, plus the stator leakage flux. F_{RX} will be negative in any numerical example. It is the component of the rotor m.m.f. measured in the *positive* direction of the X-axis.

$$F_{SX} = \frac{(R\phi_{SX} - F_{RX})}{1+\sigma_s} \qquad . \qquad . \qquad . \qquad (4.118)$$

since $RP_{SX} = \sigma_s$

but

$$\phi_{SX} = \frac{\left[\left\{(1+\sigma_r)(\sigma_r+\sigma_r\sigma_s+\sigma_s)\frac{k_2}{R}\right\}\{\omega^2+D^2\} + D(1+2\sigma_r+2\sigma_s+2\sigma_r\sigma_s) + \frac{R}{k_2}(1+\sigma_s)\right]\phi_{SY}}{\omega} \qquad (4.119)$$

and

$$F_{SX} = R\left[\frac{(\omega^2+D^2)(2\sigma_r+1)}{R} \times k_2 + 2D(1+\sigma_r) + \frac{R}{k_2}\right]\frac{\phi_{SY}}{\omega} \qquad (4.120)$$

$$= R\left[\frac{(\omega^2+D^2)(2\sigma_r+1)}{R}k_2 + 2D(1+\sigma_r) + \frac{R}{k_2}\right] \times \frac{\hat{\phi}_{SX}\sin(\omega_0 t - \alpha)}{\sqrt{\left\{(\sigma_r+\sigma_s)(\omega^2-\omega_0^2)\frac{k_2}{R} + \frac{R(1+\sigma_s)}{k_2}\right\}^2 + \{1+2\sigma_r+2\sigma_s\}^2\omega_0^2}} \qquad . \qquad . \qquad (4.121)$$

where ϕ_{SY} is substituted in terms of ϕ_{SX} from equation (4.110).

Determination of ϕ_{SX} in Terms of the Applied Voltage per Phase

Let $v = \hat{V}\sin(\omega_0 t + \delta)$ = applied volts to stator per phase

N_1 = number of turns in the stator winding per phase in series

k_1 = breadth factor for the fundamental

k_3 = coil span factor for the fundamental

I = r.m.s. value of current in the stator

For a single-phase winding, the amplitude of the m.m.f. wave in ampere-turns

$$\hat{F}_{SX} = 1{\cdot}8N_1 \times k_1k_3 \times \frac{I}{p} \qquad (4.122)$$

where p = number of poles

Now F_{SX} will be obtained in the form $a\hat{\phi}_{SX} \sin(\omega_0 t + \beta)$

$$\therefore\ v = N_1k_1k_3\omega_0\hat{\phi}_{SX}\frac{\cos\omega_0 t}{10^8} + \frac{(R_S a\hat{\phi}_{SX}\sin(\omega_0 t + \beta)p}{1{\cdot}8N_1k_1k_3} \qquad (4.123)$$

R_s = stator resistance per phase.

Let

$$c = \frac{N_1k_1k_3\omega_0}{10^8} \qquad (4.124)$$

and

$$d = \frac{R_S ap}{1{\cdot}8N_1 \times k_1 \times k_3} \qquad (4.125)$$

then

$$v = c\hat{\phi}_{SX}\cos\omega_0 t + d\hat{\phi}_{SX}\sin(\omega_0 t + \beta) \qquad (4.126)$$

$$= c\hat{\phi}_{SX}\cos\omega_0 t + d\hat{\phi}_{SX}\sin\omega_0 t\cos\beta + d\hat{\phi}_{SX}\cos\omega_0 t\sin\beta \qquad (4.127)$$

$$= \hat{\phi}_{SX}[(c + d\sin\beta)\cos\omega_0 t + d\cos\beta\sin\omega_0 t] \qquad (4.128)$$

$$= \hat{\phi}_{SX}[f\cos\omega_0 t + g\sin\omega_0 t] \qquad (4.129)$$

where

$$f = c + d\sin\beta \qquad (4.130)$$

and

$$g = d\cos\beta$$

Then

$$v = \hat{\phi}_{SX}k\sin(\omega_0 t + \delta) \qquad (4.131)$$

where

$$k\cos\delta = g \qquad (4.132)$$

$$k\sin\delta = f \qquad (4.133)$$

$$k = \sqrt{g^2 + f^2} \qquad (4.134)$$

and

$$\tan\delta = \frac{f}{g} \qquad (4.135)$$

$$\therefore\quad k = \sqrt{d^2\cos^2\beta + c^2 + 2cd\sin\beta + d^2\sin^2\beta} \qquad (4.136)$$

and

$$\tan\delta = \frac{c + d\sin\beta}{d\cos\beta} \qquad (4.137)$$

Now

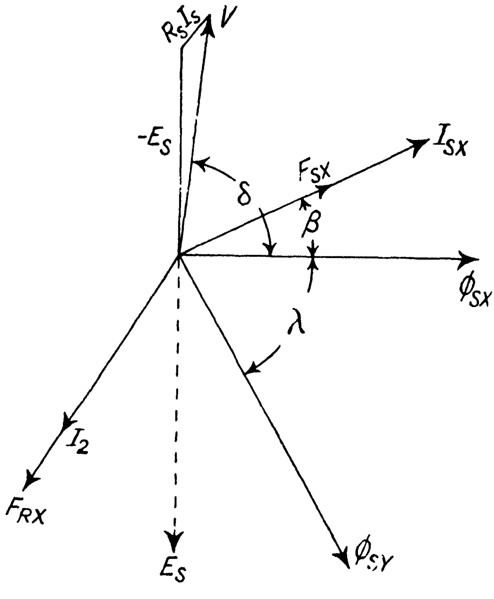
$$\hat{V} = \hat{\phi}_{SX} \times k \qquad (4.138)$$

$$\therefore\quad \hat{\phi}_{SX} = \frac{\hat{V}}{k} = \frac{\hat{V}}{\sqrt{c^2 + d^2 + 2cd\sin\beta}} \qquad (4.139)$$

where $\hat{V}$ = maximum value of V

We thus have $\hat{\phi}_{SX}$ determined in terms of $\hat{V}$ and also its phase angle δ with respect to $\hat{V}$. All the other quantities are determined with respect to $\hat{\phi}_{SX}$. They are, therefore, known and their phase relations. The m.m.f. F_{SX} is known in terms of $\hat{\phi}_{SX}$ and its phase relation. The stator current is known, and its phase relation. ϕ_{RY} is known, and F_{RX} and F_{RY} are known, and, therefore, the rotor currents in the X- and Y-axes are known.

Since all quantities are known in terms of ϕ_{SX}, and ϕ_{SX} is known in terms of V, the whole characteristics of the motor are determined. We may now set out the vector diagram (Fig. 4.8).



ϕ_{SX} flux in stator in the X-axis, which is the stator axis
F_{SX} — m.m.f. of stator
I_{SX} - current in stator
E_S = back e.m.f. in stator, 90° behind ϕ_{SX}
V = supply volts, and
F_{RX} rotor m.m.f.

FIG. 4.8

The following test results on a three-phase machine running (*a*) as a three-phase machine, and (*b*) as a single-phase machine are of interest to the student.

CHARACTERISTICS OF A SINGLE-PHASE INDUCTION MOTOR

Test Machine. Brook motor. No. 38496.

220 V Speed 1200 r.p.m.
10 h.p. Cycles 60 c/s

Length of Prony brake arm = 30 in.

Effective weight of Prony brake arm = 4 lb.

Observations—

Three-phase Operation

Volts	Amperes	Kilowatts			P.F. Cos ϕ	R.p.m.	Load			Torque lb-ft	H.p.	Efficiency (%)	Losses Watts	Slip (%)
		W_1	W_2	$W_1 + W_2$			W	S	Net					
230	29·2	6·60	3·55	10·15	0·89	1150	22	4·5	21·5	53·0	11·70	86·0	1420	4·20
230	24·2	5·60	2·80	8.40	0·87	1160	18	4·2	17·8	44·0	9·75	86·5	1120	3·30
231	20·0	4·60	1·96	6·56	0·82	1170	14	3·9	14·1	35·0	7·76	88·3	760	2·50
232	16·4	3·75	1·25	5·00	0·75	1180	10	3·3	10·7	26·5	5·94	88·6	560	1·66
232	13·3	2·90	0·55	3·45	0·64	1187	6	2·8	7·2	18·0	4·04	87·3	430	1·08
233	10·6	2·06	0·00	2·06	0·50	1195	2	2·1	3·9	9·6	2·19	79·2	420	0·42
231	8·0	1·15	− 0·65	0·50	0·16	1200	0	0·0	4·0	9·8	2·20	0·0	—	0·00

Single-phase Operation

Volts	Amperes	Kilowatts	P.F. Cos ϕ	R.p.m.	Load			Torque lb-ft	H.p.	Efficiency (%)	Losses Watts	Slip (%)
					W	S	Net					
234	16·5	2·10	0·540	1190	2	2·25	3·75	9·26	2·1	74·6	530	0·83
232	22·3	3·70	0·715	1185	6	2·90	7·10	17·80	3·9	80·0	740	1·25
232	30·0	5·58	0·800	1181	10	3·60	10·40	25·70	5·8	77·2	1260	1·58
231	41·2	7·80	0·820	1150	14	4·10	13·90	34·30	7·5	71·8	2100	4·17
231	36·2	6·90	0·825	1155	12	3·60	12·40	30·60	6·7	73·0	1860	3·75
232	26·2	4·72	0·775	1170	8	3·10	8·90	22·00	4·9	77·5	1060	2·50
232	19·1	2·90	0·655	1185	4	2·50	5·50	13·60	3·1	79·0	610	1·25
232	13·2	0·40	—	1200	0	0·00	4·00	9·90	2·3	0·0	—	—

Calculations

Case 1

$$\text{Torque} = (30)(\tfrac{1}{12})(22 - 4\cdot 5 + 4) \qquad \hat{=}\ 53\ \text{lb-ft}$$

$$\text{Total losses} = \text{input} - \text{output}$$

$$= 10\,150 - (11\cdot 7)(746) \qquad \hat{=}\ 1420\ \text{W}$$

$$\text{Percentage efficiency} = \frac{\text{output}}{\text{input}} = \frac{(11\cdot 7)(746)}{10\,150} \qquad = 86\ \text{per cent}$$

$$\text{Percentage slip} = \frac{\text{synchronous speed} - \text{actual speed}}{\text{synchronous speed}}$$

$$= \frac{1200 - 1150}{1200} \qquad = 4\cdot 16\ \text{per cent}$$

From graph (Fig. 4.9)—

Losses when run as an induction motor (three-phase) at 10 h.p. $\hat{=}$ 1100 W

Single-phase output with the same loss (Fig. 4.10) = 5·1 h.p.

The characteristics of the motor under various conditions are shown in the graphs of Figs. 4.9, 4.10. and 4.11.

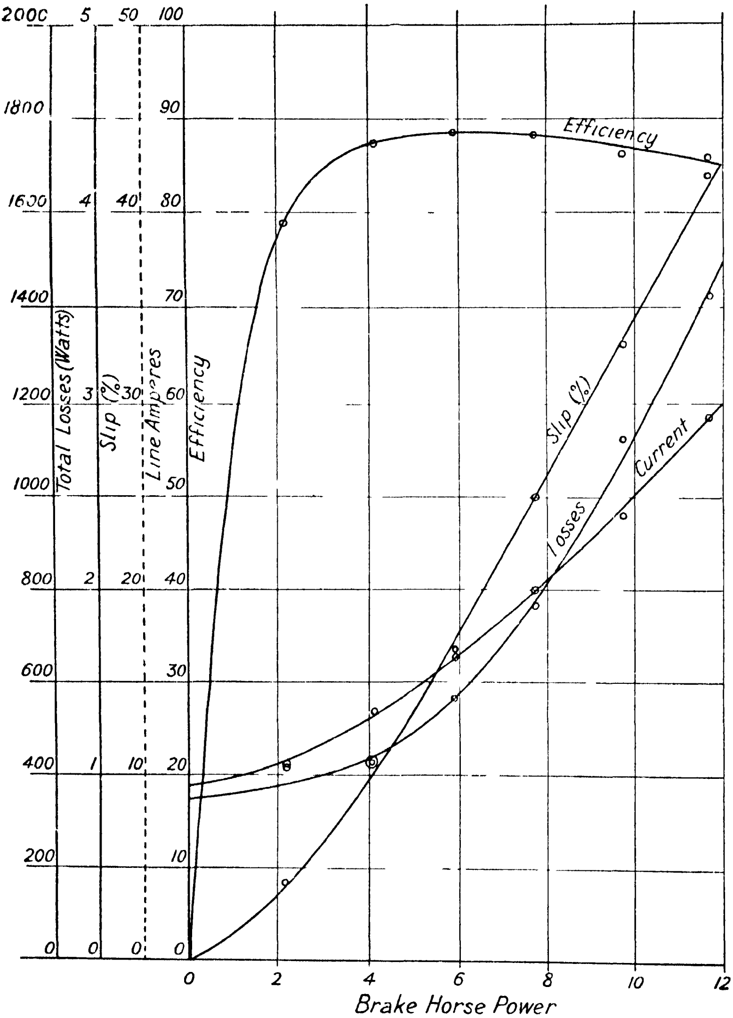


FIG. 49. CHARACTERISTICS OF A SINGLE-PHASE INDUCTION MOTOR ON THREE-PHASE OPERATION

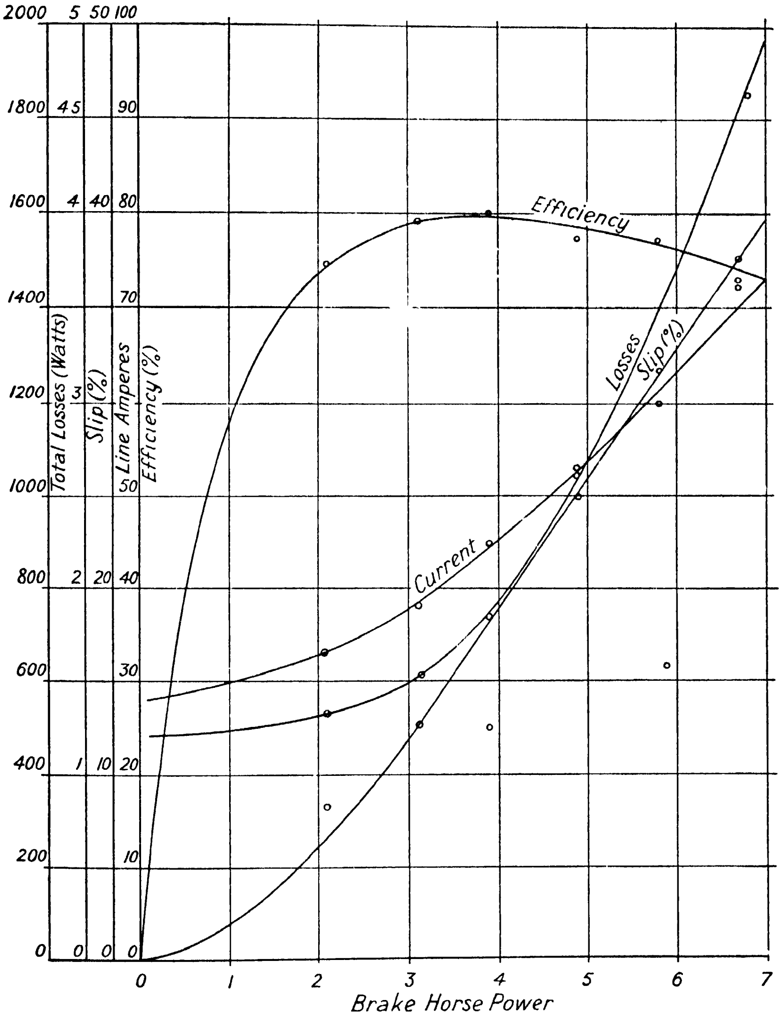


FIG. 4.10. CHARACTERISTICS OF A SINGLE-PHASE INDUCTION MOTOR ON SINGLE-PHASE OPERATION

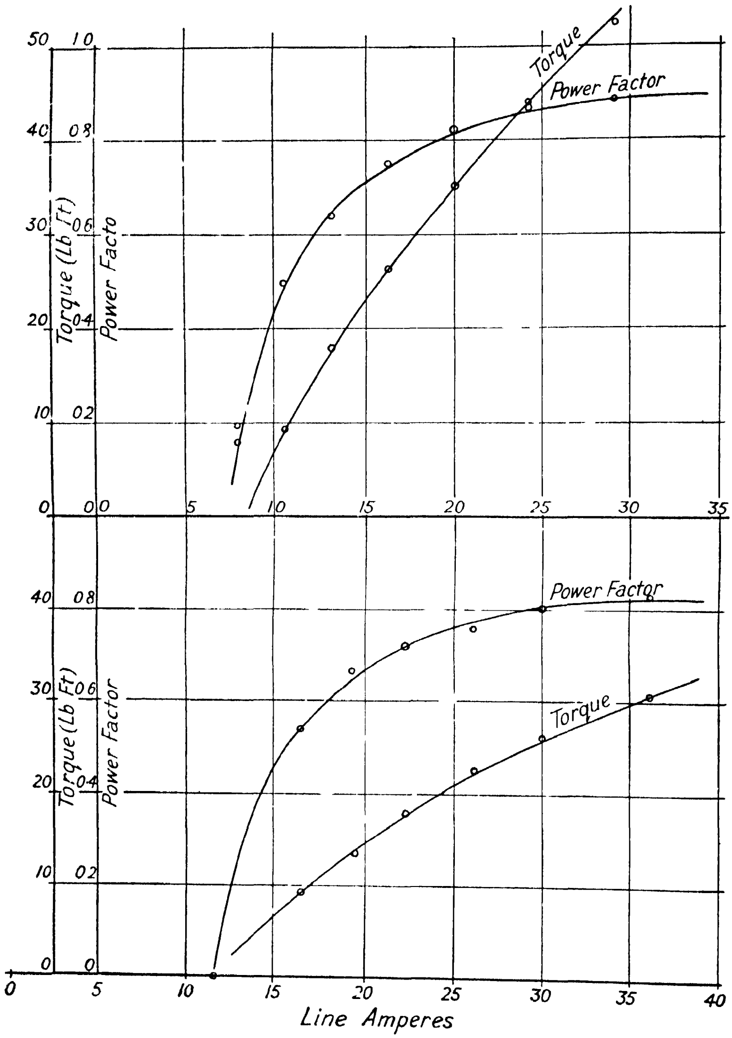


FIG. 4.11. TORQUE AND POWER-FACTOR CHARACTERISTICS FOR THE SINGLE-PHASE INDUCTION MOTOR OPERATED UNDER THREE-PHASE AND SINGLE-PHASE CONDITIONS

CHAPTER V

The Two Revolving Field Theory

THIS theory is based on the simple conception that a pulsating sinusoidal wave of flux or m.m.f. can be resolved into two rotating waves, each of half the amplitude of the pulsating wave, and rotating at synchronous speed in opposite directions.

Thus, consider a coil, or several coils, constituting a single-phase winding, and carrying alternating current of frequency f.

The m.m.f. curve of such a winding consists of several rectangular waves, displaced in space by the slot-pitch angle. The resultant wave, for the phase, is a stepped rectangular wave, which can be replaced by a series of sine waves of different frequency. Confining attention to the fundamental wave of flux density in the air-gap, due to the fundamental m.m.f., we have

$$B = a\sqrt{2}I \sin \omega t \sin x \quad . \quad . \quad . \quad (5.1)$$

This is a wave distributed sinusoidally in space, as shown by the factor sin x, and the maximum value, in space, varies, also sinusoidally, with time, as shown by the factor sin ωt. This curve remains stationary in space, but its values over the pole pitch pulsate, but still remain sinusoidal in space.

Now $$\sin \omega t \sin x = \tfrac{1}{2}[\cos(x - \omega t) - \cos(x + \omega t)] \quad . \quad (5.2)$$

$$\therefore \qquad B = a\frac{I}{\sqrt{2}}[\cos(x - \omega t) - \cos(x + \omega t)] \quad . \quad . \quad (5.3)$$

Equation (5.3) represents two rotating waves, each having half the amplitude of the pulsating wave. Each of these rotating waves will generate e.m.f.s in the rotor. At standstill the e.m.f.s will produce currents in the rotor which produce equal driving and retarding torques, so there is no starting torque. If the rotor is given rotation in any given direction it will continue to run in that direction. If ω is the angular velocity of the rotor, and ω_0 that of the rotating fields, then the relative angular velocity of the field and the rotor, for one wave, is $\omega_0 - \omega$ and the relative angular velocity of the second rotating field with respect to the rotor $= \omega_0 + \omega$.

The slip of the rotor with respect to the field running in the same direction as the rotor

$$= s = \frac{\omega_0 - \omega}{\omega_0} \quad . \quad . \quad . \quad . \quad (5.4)$$

and $$\omega = \omega_0(1 - s) \quad . \quad . \quad . \quad . \quad (5.5)$$

The slip of the rotor with respect to the field which runs in the opposite direction to that of the rotor

$$= \frac{\omega_0 + \omega}{\omega_0} = \frac{\omega_0 + \omega_0(1 - s)}{\omega_0}$$

$$= 2 - s \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.6)$$

Clearly the slip of the rotor, with regard to the reverse field, is nearly 2 for the normal range of operation of the motor. The resistance corresponding to the rotor input in the equivalent circuit, namely $\frac{r_2}{s}$ for the forward field and $\frac{r_2}{2 - s}$ for the reverse field, shows at once that the rotor input of the motor, for the reverse field, is small compared to that for the forward field, and the torque is a retarding torque due to it.

The behaviour of the motor is similar, *in some respects*, to two polyphase machines, whose stator windings are in series, and which have the same rotor, but the connections of the second motor are made in such a manner that the fields rotate in opposite directions in the two machines. This representation is not true to the facts, for in a polyphase machine, the torque is a constant torque, whereas in the single-phase machine, the torque is a pulsating torque. It will be clear also that the forward rotating field absorbs the larger share of the supply volts—in fact, much the larger share. Since the slip for the reverse field is $2 - s$, it follows that, for this field, the machine acts as if it were on short-circuit. The effect of this field can be taken into account by an *increase* in the primary leakage of the motor.

Let $\boldsymbol{V}$ = applied volts = vector of reference = V (Fig. 5.1)

r_1 = stator resistance of running winding

x_1 = stator *leakage* reactance at supply frequency

$\boldsymbol{I}_1$ = stator current

$\boldsymbol{I}_2'$ = rotor current, referred to the stator, due to forward rotating field, i.e. the field which rotates in the same direction as the rotor

$\boldsymbol{I}_{2R}'$ = rotor current, referred to the stator, due to the reverse field

x_m = reactance of the magnetizing circuit for each field at full supply frequency; note *this is one half the reactance of the pulsating field*

Then

$$\boldsymbol{V} = (r_1 + jx_1)\boldsymbol{I}_1 + jx_m(\boldsymbol{I}_2' + \boldsymbol{I}_1) + jx_m(\boldsymbol{I}_{2R}' + \boldsymbol{I}_1) \quad . \quad (5.7)$$

This is the equation for the stator.

For the rotor, we have

$$\left(\frac{r_2'}{s} + jx_2'\right) I_2' + jx_m(I_1 + I_2') = 0 \quad . \quad . \quad (5.8)$$

and

$$\left(\frac{r_2'}{2-s} + jx_2'\right) I_{2R}' + jx_m(I_1 + I_{2R}') = 0 \quad . \quad . \quad (5.9)$$

where r_2' and x_2' are the rotor resistance and reactance, referred to the stator, *divided by* 2.

Equations (5.8) and (5.9) refer to the rotor circuit voltages for the forward and reverse fields respectively.

From equation (5.8) we can obtain I_2' in terms of I_1 and from equation (5.9) we can obtain I_{2R}' in terms of I_1.

If we substitute the values, so obtained, for these quantities in equation (5.7), we have a relation between V and I_1, from which we

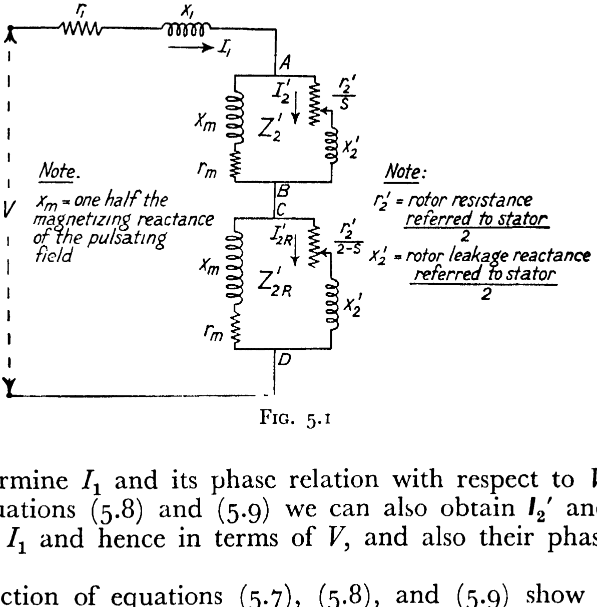

r_1
x_1
I_1
A
I_2'
$\frac{r_2'}{s}$
x_m
Z_2'
x_2'
Note.
r_m
Note:
B
C
V
x_m = one half the magnetizing reactance of the pulsating field
r_2' = rotor resistance referred to stator / 2
I_{2R}'
$\frac{r_2'}{2-s}$
x_2' = rotor leakage reactance referred to stator / 2
x_m
Z_{2R}'
x_2'
r_m
D

FIG. 5.1

can determine I_1 and its phase relation with respect to V. Then from equations (5.8) and (5.9) we can also obtain I_2' and I_{2R}' in terms of I_1 and hence in terms of V, and also their phase angles with V.

Inspection of equations (5.7), (5.8), and (5.9) show that the equivalent circuit for the single-phase motor is as shown in Fig. 5.1.

From equation (5.8) we have, neglecting r_m,

$$I_2' = \frac{-jx_m I_1}{\frac{r_2'}{s} + j(x_2' + x_m)} \quad . \quad . \quad . \quad (5.10)$$

and from equation (5.9), we have,

$$I_{2R}' = \frac{-jx_m I_1}{\frac{r_2'}{2-s} + j(x_2' + x_m)} \quad . \quad . \quad . \quad (5.11)$$

Therefore, equation (5.7) becomes

$$V = (r_1 + jx_1)\mathbf{I}_1 + jx_m\mathbf{I}_1 + \frac{x_m{}^2\mathbf{I}_1}{\frac{r_2'}{s} + j(x_2' + x_m)} + jx_m\mathbf{I}_1 + \frac{x_m{}^2 I_1}{\frac{r_2'}{2-s} + j(x_2' + x_m)} \qquad (5.12)$$

$$\therefore V = \mathbf{I}_1\left[r_1 + x_m{}^2\left\{\frac{1}{\frac{r_2'}{s} + j(x_2' + x_m)} + \frac{1}{\frac{r_2'}{2-s} + j(x_2' + x_m)}\right\} + j(x_1 + 2x_m)\right] \qquad (5.13)$$

$$= \mathbf{I}_1\left[r_1 + \frac{x_m{}^2\frac{r_2'}{s}}{\left(\frac{r_2'}{s}\right)^2 + (x_2' + x_m)^2} - \frac{jx_m{}^2 \times (x_2' + x_m)}{\left(\frac{r_2'}{s}\right)^2 + (x_2' + x_m)^2}\right.$$

$$\left. + \frac{x_m{}^2 \times \frac{r_2'}{2-s}}{\left(\frac{r_2'}{2-s}\right)^2 + (x_2' + x_m)^2} - \frac{jx_m{}^2(x_2' + x_m)}{\left(\frac{r_2'}{2-s}\right)^2 + (x_2' + x_m)^2} + j(x_1 + 2x_m)\right] \qquad (5.14)$$

$$\therefore \qquad V = \mathbf{I}_1[c + jd] \qquad (5.15)$$

$$I_1 = \frac{V}{\sqrt{c^2 + d^2}} \qquad (5.16)$$

where
$$c = r_1 + \frac{x_m{}^2\frac{r_2'}{s}}{\left(\frac{r_2'}{s}\right)^2 + (x_2' + x_m)^2} + \frac{x_m{}^2\frac{r_2'}{2-s}}{\left(\frac{r_2'}{2-s}\right)^2 + (x_2' + x_m)^2} \qquad (5.17)$$

and
$$d = x_1 + 2x_m - \frac{x_m{}^2(x_2' + x_m)}{\left(\frac{r_2'}{s}\right)^2 + (x_2' + x_m)^2} - \frac{x_m{}^2(x_2' + x_m)}{\left(\frac{r_2'}{2-s}\right)^2 + (x_2' + x_m)^2} \qquad (5.18)$$

and
$$\tan\phi_1{}^* = \frac{d}{c} \qquad (5.19)$$

From equation (5.10)

$$\mathbf{I}_2' = \frac{-jx_m\mathbf{I}_1}{\frac{r_2'}{s} + j(x_2' + x_m)} \qquad (5.20)$$

---

* ϕ_1 — angle of lag of $\mathbf{I}_1$ behind $\mathbf{V}$

$$= \frac{-jx_m}{\dfrac{r_2'}{s} + j(x_2' + x_m)} \times \frac{V}{(c + jd)} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (5.21)$$

$$= \frac{-Vjx_m}{\dfrac{r_2'c}{s} - d(x_2' + x_m) + j\left\{\dfrac{dr_2'}{s} + c(x_2' + x_m)\right\}} \quad \cdot \quad \cdot \quad (5.22)$$

$$\boldsymbol{I}_2' = \frac{Vx_m}{-\left\{\dfrac{dr_2'}{s} + c(x_2' + x_m)\right\} + j\left\{\dfrac{r_2'c}{s} - d(x_2' + x_m)\right\}} \quad \cdot \quad (5.23)$$

and $$\bar{I}_2' = \frac{\bar{V}x_m}{\sqrt{\left\{\dfrac{dr_2'}{s} + c(x_2' + x_m)\right\}^2 + \left\{\dfrac{r_2{}^2c}{s} - d(x_2' + x_m)\right\}^2}} \quad \cdot \quad (5.24)$$

Also $$\boldsymbol{I}_{2R}' = \frac{-jx_m\boldsymbol{I}_1}{\dfrac{r_2'}{2 - s} + j(x_2' + x_m)}$$

$$= \frac{-jx_m}{\dfrac{r_2'}{2 - s} + j(x_2' + x_m)} \times \frac{V}{(c + jd)} \quad \cdot \quad \cdot \quad \cdot \quad (5.25)$$

$$= \frac{-jx_m \times V}{\dfrac{cr_2'}{2 - s} - d(x_2' + x_m) + j\left\{c(x_2' + x_m) + \dfrac{dr_2'}{2 - s}\right\}} \quad \cdot \quad (5.26)$$

$$= \frac{x_mV}{-\left\{c(x_2' + x_m) + \dfrac{dr_2'}{2 - s}\right\} + j\left\{\dfrac{cr_2'}{2 - s} - d(x_2' + x_m)\right\}} \quad \cdot \quad (5.27)$$

$$\therefore \quad \bar{I}_{2R}' = \frac{x_m\bar{V}}{\sqrt{\left\{c(x_2' + x_m) + \dfrac{dr_2'}{2 - s}\right\}^2 + \left\{\dfrac{cr_2'}{2 - s} - d(x_2' + x_m)\right\}^2}} \quad (5.28)$$

Note that when $s = 1$, i.e. at standstill

$$\boldsymbol{I}_2' = \boldsymbol{I}_{2R}'$$

and the torques due to the two currents are equal and opposite, since the field producing $\boldsymbol{I}_2'$ rotates in the opposite direction to that producing $\boldsymbol{I}_{2R}'$. Thus, at standstill, the torques are equal and opposite and there is no starting torque.

When $s = 1$, neglecting iron loss, that is neglecting r_m

$$\boldsymbol{Z}_2' = \frac{(r_2' + jx_2')(jx_m)}{r_2' + j(x_2' + x_m)} = \frac{jr_2'x_m - x_2'x_m}{r_2' + j(x_2' + x_m)} \quad \cdot \quad \cdot \quad \cdot \quad (5.29)$$

$$= \frac{-x_2'x_m + jr_2'x_m}{r_2' + j(x_2' + x_m)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.30)$$

$$= \frac{-r_2'x_2'x_m + j(r_2')^2x_m + jx_2'x_m(x_2' + x_m) + r_2'x_m(x_2' + x_m)}{(r_2')^2 + (x_2' + x_m)^2} \quad (5.31)$$

and $\mathbf{Z}_{2R}'$ has the same value at $s = 1$.

So that at standstill

$$\mathbf{I}_1 = \frac{V}{r_1 + jx_1 + 2\underset{s=1}{\mathbf{Z}_2'}} \quad . \quad . \quad . \quad (5.32)$$

Z_2' may be written $e + jf$,

where
$$e = -\frac{r_2'x_2'x_m + r_2'x_m(x_2' + x_m)}{(r_2')^2 + (x_2' + x_m)^2} \quad . \quad . \quad (5.33)$$

and
$$f = \frac{(r_2')^2x_m + x_2'x_m(x_2' + x_m)}{(r_2')^2 + (x_2' + x_m)^2} \quad . \quad . \quad (5.34)$$

and
$$\underset{(s=1)}{\mathbf{I}_1} = \frac{V}{r_1 + jx_1 + 2e + 2jf} \cdot \quad . \quad . \quad (5.35)$$

$$= \frac{V}{r_1 + 2e + j(x_1 + 2f)} \quad . \quad . \quad (5.36)$$

$$\underset{(s=1)}{I_1} = \frac{V}{\sqrt{(r_1 + 2e)^2 + (x_1 + 2f)^2}} \quad . \quad . \quad (5.37)$$

and
$$\tan \phi_{sc} = \frac{x_1 + 2f}{r_1 + 2e} \quad . \quad . \quad . \quad (5.38)$$

Equation (5.37) gives the short-circuit current and equation (5.38) its phase angle with respect to V.

Now look at Fig. 5.1 and note that Z_2' and Z_{2R}' change values when $s = 0$ and when $s = 2$, so that the primary current is the same when $s = 0$ and when $s = 2$. When $s = 0$ or $s = 2$, by substituting in equation (5.14), we get the stator current and its power factor.

Thus, $s = 0$ or $s = 2$,

$$V = (r_1 + jx_1)\mathbf{I}_1 + 2jx_m\mathbf{I}_1 + \frac{x_m^2\frac{r_2'}{2}\mathbf{I}_1 - jx_m^2(x_2' + x_m)\mathbf{I}_1}{\left(\frac{r_2'}{2}\right)^2 + (x_2' + x_m)^2} \quad . \quad (5.39)$$

At $s = 0$ or $s = 2$,

$$V = I_1\left\{\left[r_1 + \frac{x_m^2\frac{r_2'}{2}}{\left(\frac{r_2'}{2}\right)^2 + (x_2' + x_m)^2}\right] + j\left[2x_m + x_1 - \frac{x_m^2(x_2' + x_m)}{\left(\frac{r_2'}{2}\right)^2 + (x_2' + x_m)^2}\right]\right\} \quad (5{\cdot}40)$$

$$= I_1(g + jh) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}41)$$

where
$$g = r_1 + \frac{x_m^2\frac{r_2'}{2}}{\left(\frac{r_2'}{2}\right)^2 + (x_2' + x_m)^2} \quad . \quad . \quad . \quad (5{\cdot}42)$$

and
$$h = 2x_m - \frac{x_m^2(x_2' + x_m)}{\left(\frac{r_2'}{2}\right)^2 + (x_2' + x_m)^2} + x_1 \quad . \quad . \quad (5{\cdot}43)$$

$$\therefore \quad \underset{(s = 0 \text{ or } s = 2)}{\bar{I}_1} = \frac{\bar{V}}{\sqrt{g^2 + h^2}} \quad . \quad . \quad . \quad (5{\cdot}44)$$

and
$$\underset{(s = 0 \text{ or } s = 2)}{\tan \phi} = \frac{h}{g} \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}45)$$

With the rotor circuit open at $s = 0$,

$$\bar{I}_1 = \frac{\bar{V}}{\sqrt{r_1 + (x_1 + 2x_m)^2}} \quad . \quad . \quad . \quad (5{\cdot}46)$$

The ratio of equations (5·46) to (5·44), i.e. the

$$\frac{\text{primary current when rotor is open when } s = 0}{\text{primary current when the rotor is closed}} \bumpeq \frac{1}{2}$$

The value of I_1 when $s = \infty$,

$$V = \underset{(s = \infty)}{I_1}\left[r_1 - \frac{jx_m^2}{x_2' + x_m} - \frac{jx_m^2}{x_2' + x_m} + j(x_1 + 2x_m)\right] \quad (5{\cdot}47)$$

$$\therefore \quad \underset{(s = \infty)}{I_1} = \frac{V}{r_1 - \frac{2jx_m^2}{x_2' + x_m} + j(x_1 + 2x_m)} \quad . \quad . \quad (5{\cdot}48)$$

and
$$\underset{(s = \infty)}{\bar{I}_1} = \frac{V}{\sqrt{r_1^2 + \left(x_1 + 2x_m - \frac{2x_m^2}{x_2' + x_m}\right)^2}} \quad . \quad (5{\cdot}49)$$

and
$$\tan \phi_{sc} = \frac{x_1 + 2x_m - \dfrac{2x_m^2}{x_2' + x_m}}{r_1} \quad . \quad . \quad . \quad (5{\cdot}50)$$

Thus, we have determined points on the current locus for $s = 0$, $s = \infty$, $s = 1$, and $s = 2$ and the phase angles made by the current, at these points, with respect to the applied volts.

By the Principle of Inversion, our equation (5.14) can be shown to represent a circle, for it is of the form

$$\frac{\mathbf{I}_1}{\mathbf{V}_1} = \frac{C + Ds(2 - s)}{E + Fs(2 - s)} \quad . \quad . \quad . \quad (5{\cdot}51)$$

where C, D, E, and F are complex constants and s = slip of the motor with respect to the forward field.

We may obtain all the information we require about the performance of our motor from the equivalent circuit shown in Fig. 5.1.

Thus, the torque in synchronous watts = input to the rotor. The torque in synchronous watts

$$= (I_2')^2 \frac{r_2'}{s} - (I_{2R}')^2 \frac{r_2'}{2 - s} \quad . \quad . \quad . \quad (5{\cdot}52)$$

The term $(I_2')^2 \dfrac{r_2'}{s}$ represents the forward torque in synchronous watts and $(I_{2R}')^2 \dfrac{r_2'}{2 - s}$ represents the backward torque due to the reverse field.

The torque in lb-ft

$$= \frac{7{\cdot}04}{\text{r.p.m.}_{\text{syn}}} \left\{ (I_2')^2 \frac{r_2'}{s} - (I_{2R}')^2 \frac{r_2'}{2 - s} \right\} \quad . \quad . \quad (5{\cdot}53)$$

where $\text{r.p.m.}_{\text{syn}}$ = synchronous speed in revolutions per minute.

The gross mechanical power in horse-power

$$= \frac{2\pi\, \text{r.p.m.}_{\text{syn}}(1 - s) \times 7{\cdot}04}{\text{r.p.m.}_{\text{syn}} \times 33\,000} \left\{ (I_2')^2 \frac{r_2'}{s} - (I_{2R}')^2 \frac{r_2'}{2 - s} \right\} \quad (5{\cdot}54)$$

$$= \frac{(1 - s)}{746} \left\{ (I_2')^2 \frac{r_2'}{s} - (I_{2R}')^2 \frac{r_2'}{2 - s} \right\} \text{h.p.} \quad . \quad . \quad . \quad (5{\cdot}55)$$

It is interesting to note that the copper loss in the rotor, due to I_{2R}', viz. $(I_{2R}')^2 r_2'$ is greater than the electrical input to the rotor from the reverse field. It follows that part of the copper loss must be supplied mechanically, i.e. the difference between $(I_{2R}')^2 r_2'$ and $(I_{2R}')^2 \dfrac{r_2'}{2 - s}$ must be supplied mechanically through the shaft.

It will be clear, from inspection of Fig. 5.1, that the impedance Z_2', for the normal range of operation of the motor, is very much greater than Z_{2R}', so that most of the voltage is impressed across Z_2', and in fact usually only about 10 per cent of the voltage is impressed on $\mathbf{Z}_{2R}'$. Also the impedance of $\mathbf{Z}_{2R}'$ is nearly equal to $\frac{r_2'}{2-s}+jx_2'$ for normal values of the slip, for

$$\mathbf{Z}_{2R}' = \frac{\left(\frac{r_2'}{2-s}+jx_2'\right)(jx_m)}{\frac{r_2'}{2-s}+j(x_2'+x_m)} \quad . \quad . \quad (5.56)$$

$$= \frac{\frac{r_2'}{2-s}+jx_2'}{\frac{r_2'}{j(x_m)(2-s)}+\frac{x_2'+x_m}{x_m}} \quad . \quad . \quad (5.57)$$

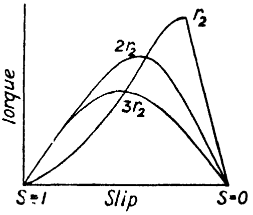


FIG. 5.2. TORQUE/SLIP CURVE FOR DIFFERENT VALUES OF ROTOR RESISTANCE

and since x_m is much greater than $\frac{r_2'}{2-s}$ and also much greater than x_2', it follows that $\mathbf{Z}_{2R}'$ is nearly equal to $\frac{r_2'}{2-s}+jx_2'$, so it follows that the performance of the motor in the usual range of operation can be determined approximately by increasing the leakage impedance of the stator by the amount $\frac{r_2'}{2-s}+jx_2'$.

Effect of Secondary Resistance and Reactance on the Torque/Speed Curve

In the polyphase motor the maximum torque is independent of the rotor resistance, which is only effective in determining the slip at which maximum torque occurs. In the single-phase motor, the secondary resistance not only determines the slip for maximum torque, but also determines its magnitude (*see* Fig. 5.2).

This can be clearly seen by inspecting equation (5.53), which gives us the torque in synchronous watts,

$$= \left\{(I_2')^2\frac{r_2'}{s}-(I_{2R}')^2\frac{r_2'}{2-s}\right\}$$

Now I_2' is given by equation (5.23) and I_{2R}' by equation (5.28). If these values are substituted, and then the expression differentiated with respect to s and equated to zero, we shall get the slip for maximum torque, and then by substitution of this slip, we shall obtain the maximum torque. We see that the maximum torque is reduced by increasing the secondary resistance. Also at low values of the slip, the value of $\mathbf{Z}_{2R}'$ is nearly x_2', and this is in series with the forward impedance and reduces the maximum torque.

Iron Loss in Single-phase Motors

In Fig. 5.1 these losses are represented by a resistance r_m inserted in series with the magnetizing reactance x_m. If the impedance of the magnetizing circuit is represented by $Z_m = r_m + jx_m$, then the admittance of this circuit

$$Y_m = \frac{1}{r_m + jx_m} = \frac{r_m}{r_m^2 + x_m^2} - \frac{jx_m}{r_m^2 + x_m^2} = g_m - jb_m \quad (5.58)$$

where g_m = conductance of the magnetizing circuit
and b_m = susceptance of the magnetizing circuit

$$r_m = \frac{g_m}{g_m^2 + b_m^2} \simeq \frac{g_m}{b_m^2} \text{ and } x_m = \frac{b_m}{g_m^2 + b_m^2} \simeq \frac{1}{b_m} \quad . \quad (5.59)$$

The current in the magnetizing circuits

$$= \text{volts across } Z_m \times (g_m - jb_m)$$

If W_i = iron loss due to the flux,

$$g_m = \frac{W_i}{E^2}$$

where E = r.m.s. volts across Z_m

There are, of course, iron losses due to each of the rotating fields, so it will be necessary to calculate the voltages across the two magnetizing circuits in Fig. 5.1.

We have already obtained expressions for I_2' and I_{2R}', so these voltages are

$$I_2' \sqrt{\left(\frac{r_2'}{s}\right)^2 + (x_2')^2} \text{ and } (I_{2R}')^2 \sqrt{\left(\frac{r_2'}{2-s}\right)^2 + (x_2')^2}$$

The iron losses are determined from the flux densities in the various parts of the machines, and, of course, these are determined by the fluxes and areas of the parts. The fluxes are determined from the voltages across $\mathbf{Z}_2'$ and $\mathbf{Z}_{2R}'$ in Fig. 5.1. Again iron-loss curves, appropriate to the frequency, must be used. The iron loss in the rotor must also be evaluated, and this is fairly large due to the reverse field. The current at zero slip is given in equation (5.44). If the no-load current is measured, it is possible to obtain an approximate expression for x_m. Otherwise, it may be obtained from the expression

$$x_m = 3{\cdot}2 \times \frac{\tau \times L}{\delta} \times K^2 \times \frac{N^2}{p} \times f \times 10^{-8} \text{ ohm} \quad . \quad (5.60)$$

where τ = pole pitch in centimetres
L = core length in centimetres
δ = air-gap length in centimetres, corrected for slots and saturation

K = product of breadth factor and coil-span factor for the stator winding

N = total number of turns on the stator for running phase

f = supply frequency

and p = poles

Referred Quantities

In Fig. 5.1. *all quantities are referred to the stator.*

If r_2 = actual resistance of the rotor

r_2' = rotor resistance, *referred to the stator*

x_2 = rotor reactance at supply frequency

x_2' = rotor reactance, at supply frequency, *referred to the stator*

and m_2 = number of rotor phases

Then

$$r_2' = \frac{1}{m_2}\left(\frac{N_1K_1}{N_2K_2}\right)^2 \times r_2 \quad . \quad . \quad . \quad (5.61)$$

where N_1 = turns per phase in stator

K_1 = winding factor for stator = breadth factor × coil-span factor

and N_2 = number of turns per phase in rotor

Also

$$x_2' = \frac{1}{m_2}\left(\frac{N_1K_1}{N_2K_2}\right)^2 x_2 \quad . \quad . \quad . \quad (5.62)$$

where K_2 = winding factor for the rotor

For the squirrel-cage rotor

$$m_2 = \frac{\text{number of rotor bars}}{\text{pairs of poles}}$$

and

$$N_2 = \tfrac{1}{2};\ K_2 = 1$$

$$\underset{\text{(referred current)}}{I_2'} = I_2 \times \frac{m_2N_2K_2}{N_1 \times K_1} \quad . \quad . \quad . \quad (5.63)$$

$$\underset{\text{(referred current)}}{I_{2R}'} = I_{2R} \times \frac{m_2N_2K_2}{N_1K_1} \quad . \quad . \quad . \quad (5.64)$$

All the quantities in Fig. 5.1, can now be determined. Given an actual motor, r_m and x_m can be determined from the no-load test.

It has been demonstrated that the single-phase motor, unaided, has no starting torque. There are several methods adopted to make it useful as a motor.

The most common method is to use a second phase, known as the starting phase, and to ensure, as far as possible, that this starting phase shall have quadrature relation in space, i.e. it shall be displaced by 90° from the running phase, and further that the currents in the running and starting phases shall be displaced in time by 90°, or as near 90° as possible. If true quadrature relations are maintained in *both space* and *time* for the two phases, and if also the ampere-turns for the two phases are equal, then a true rotating field will be produced of constant amplitude, and constant speed. Since both running and starting windings are fed from a single-phase supply source, some form of phase splitting device is necessary. This can be done in various ways.

In one very common method used, the running phase occupies two-thirds of the slots per pole, and the starting phase the remaining one-third of the slots. The turns per coil in the starting phase are made double those for the running phase. This gives the same number of turns per phase for both phases, but the reactance of the starting phase is double that of the running phase, for the reactance of each coil, for the starting phase, is four times that of each coil in the running phase, but there are only half the number of coils in the starting phase, and so the reactance is doubled. One tries to make the reactance of one phase high and the reactance of the other phase low, and by adding resistance to the low reactance phase, one approximates to the condition for a time phase displacement of the currents sufficiently great to produce the starting torque required. If there is inequality of ampere-turns in both phases, and a phase displacement of less than 90° in time, then an elliptical field will be produced, rotating at a variable speed. For the analysis of such a motor, the methods of symmetrical components is necessary.

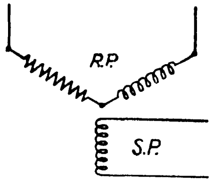


Fig. 5.3

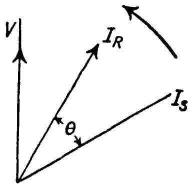


Fig. 5.4

The starting torque in such a machine will be proportional to the product of the fluxes produced by the two phases multiplied by the sine of the angle between them. This method of phase splitting is only used in those cases where the starting conditions are easy, i.e. light load or no-load.

Sometimes a three-phase winding is used, with two of the phases connected in series for the running phase. The third phase is used for the starting phase. This method is shown in Fig. 5.3.

Fig. 5.4 shows the relation of the currents in the running phase I_R and in the starting phase I_S.

In the fractional horse-power field, split-phase motors are used from about $\frac{1}{20}$ to $\frac{1}{3}$ h.p., and usually for speeds of 2890, 1430, and 960 r.p.m. for 50 c/s, and 1425 r.p.m. for 25 c/s.

Capacitator-start Motors

In all these machines a squirrel-cage rotor is used and a capacitor is used in the starting phase, as shown in Fig. 5.5.

The relation of the currents in the phases to the supply volts is shown in Fig. 5.6. The current in the running winding lags the supply volts by the angle ϕ_1; the current in the starting phase leads V by ϕ_2. Then $\phi_1 + \phi_2 = 90°$ by adjustment.

This method is used in the smaller sizes of from $\frac{1}{6}$ to $\frac{3}{4}$ h.p., and up to 10 h.p. A centrifugal switch is usually fitted, cutting out the starting phase when up to speed.

With this method of starting torques up to 350 per cent of full-load torque are obtained, with a pull-out torque as single-phase motor of 250 per cent. The size of capacitance required varies with the output, being about 80 μF for $\frac{1}{8}$ h.p. to about 350 μF for 1 h.p.

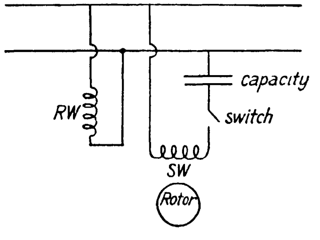


FIG. 5.5

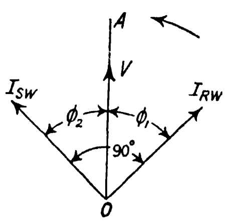


FIG. 5.6

In some cases, the motor runs with the second phase and capacitor permanently in circuit, i.e. as a two-phase unbalanced motor. This is the case in machines operating dental lathes, and in other applications. In this case a small capacitance is necessary for good operation, varying from 3 to 20 μF. The starting torque is only about 50 per cent of full-load torque with these low capacitances.

The electrolytic capacitor is used for starting purposes, but for running purposes, paper capacitors are used. These, oil insulated, paper capacitors are bigger and more expensive than the electrolytic types.

The Motor with Capacitance in Series with the Auxiliary Phase Vector Diagram

$V_M = OA$ = voltage across the main or running winding (Fig. 5.7)

$E_M = OC$ = voltage component to overcome the back e.m.f. in the main winding

CB = voltage component to overcome the leakage reactance drop of the main winding

BA = voltage component to overcome resistance drop in the main winding

$E_S = OD$ = voltage component to overcome back e.m.f. in starting winding

DE = voltage component to overcome resistance volts in starting winding

EF = voltage component to overcome leakage reactance drop in starting winding

FA = voltage component to overcome the capacitor reactive drop in starting winding

I_M = current in the main winding

I_S = current in starting winding, in quadrature with I_M

$V_S = OF$ = voltage across the starting winding

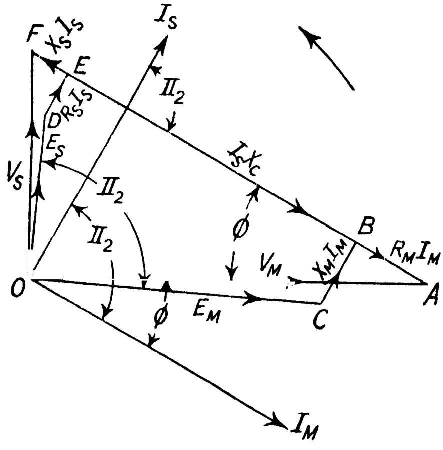


FIG. 5·7

The number of turns in the windings may not be equal, but the motor should act as a symmetrical two-phase machine. To that end the currents must be in time quadrature, and further the effective ampere-turns of each winding should be equal and in time quadrature. When that condition is satisfied the fluxes will be equal and in time quadrature, and it follows that the resistance and reactance of the auxiliary winding, when referred to the main winding, will be equal to those of the main winding.

Thus, r_s' = resistance of auxiliary winding referred to the main winding

$$= r_s \times \alpha^2 = r_m$$

If T_1 = number of turns in main winding

T_2 = number of turns in auxiliary winding

$f_1 = K_1 \times K_3$ = breadth factor × coil-span factor for main winding

$f_2 = K_2 \times K_4$ = breadth factor × coil-span factor for auxiliary winding

I_M = current in main winding

and I_S = current in starting winding

$$I_M T_t \times f_1 = I_S \times T_2 \times f_2$$

Then
$$I_S = I_M \times \frac{T_1 \times f_1}{T_2 \times f_2} = I_M \times \alpha \qquad . \quad . \quad (5{\cdot}65)$$

Since the ampere-turns are equal, the fluxes are equal and the back e.m.f.s of both windings are proportional to the number of effective turns,

i.e.
$$\frac{E_S}{E_M} = \frac{T_2 \times f_2}{T_1 \times f_1} = \frac{1}{\alpha} \qquad . \quad . \quad . \quad (5{\cdot}66)$$

$$\therefore \qquad E_S = \frac{E_M}{\alpha} \qquad . \quad . \quad . \quad . \quad (5{\cdot}67)$$

and, of course, $\bar{E}_S$ is in quadrature with $\bar{E}_M$.

Also
$$x_s' = \alpha^2 x_s = x_m \qquad . \quad . \quad . \quad (5{\cdot}68)$$

For this balanced condition, *neglecting the loss in the capacitor,*

$$\alpha = \frac{E_M}{E_S} \simeq \frac{V_M}{V_S} = \cot \phi \qquad . \quad . \quad . \quad (5{\cdot}69)$$

for V_M and V_S are in quadrature.

Also
$$FA \cos \phi = V_M = V \qquad . \quad . \quad . \quad (5{\cdot}70)$$

$$I_S X_c \cos \phi = V$$

$$I_M X_c \alpha \cos \phi = V \qquad . \quad . \quad . \quad (5{\cdot}71)$$

$$\therefore \qquad X_c = \frac{V}{\alpha I_M \cos \phi} \qquad . \quad . \quad . \quad (5{\cdot}72)$$

but
$$X_c = \frac{1}{C\omega} \qquad . \quad . \quad . \quad . \quad (5{\cdot}73)$$

$$\therefore \qquad \frac{1}{C\omega} = \frac{V}{\alpha I_M \cos \phi} \qquad . \quad . \quad . \quad (5{\cdot}74)$$

$$\therefore \qquad \underset{\text{farads}}{C} = \frac{\alpha I_M \cos \phi}{\omega V} = \frac{\alpha I_M \cos \phi}{2\pi f V} \qquad . \quad . \quad (5{\cdot}75)$$

Thus, the capacitance in farads, for an effective turns ratio α, is given by equation (5·75).

The current I_M is obtained on the basis of a symmetrical two-phase system,

i.e.
$$I_M = \frac{\text{b.h.p.} \times 746}{2 \times V \times \cos \phi \times \text{efficiency}}$$

It is clear, from Fig. 5.7, that this ideal condition exists only for one load, preferably full load, and that any change in current, due to load changes, will upset the relation between V_S and V_M, for the line *FA* will alter in length.

Analysis of Operation of Capacitor Motor under Unbalanced Conditions

We have seen earlier that the m.m.f. of a single coil on the stator, carrying a sinusoidal current is rectangular in shape. We may replace this rectangular wave by its fundamental sine wave and its various harmonics. Confining attention to the fundamental wave, we saw that the flux density at any part of the pole pitch can be represented by

$$B_1 = a_1\sqrt{2}I_1 \sin \omega t \sin x \quad . \quad . \quad . \quad (5.1)$$

This is a pulsating wave of flux, and

a_1 = the factor which converts current to flux density

I_1 = the r.m.s. current in the coil

$\omega = 2\pi \times$ frequency

and x = distance measured from the coil side to the point, where the density is B

Now equation (5.1) is equivalent to

$$B_1 = \frac{a_1 I_1}{\sqrt{2}}[\cos(x - \omega t) - \cos(x + \omega t)] \quad . \quad . \quad (5.3)$$

Equation (5.3) represents two rotating waves, one equal to the other, and each has half the amplitude of the pulsating wave. Let there be another set of coils, displaced in space by the angle β from the first set, and carrying a current differing in time phase by the angle λ.

Then keeping our origin the same, we have

$$B_2 = a_2\sqrt{2}I_2 \sin(\omega t - \lambda)\sin(x - \beta) \quad . \quad . \quad (5.76)$$

where B_2 is the density due to I_2 in the second set of coils at the point x.

Now

$$B_2 = \frac{a_2 I_2}{\sqrt{2}}[\cos(x - \beta - \omega t + \lambda) - \cos(x - \beta + \omega t - \lambda)] \quad (5.77)$$

The second set of coils also produces two opposite rotating waves as given by (5.77).

Now make $\lambda = \pi - \beta$, then equation (5.77) becomes

$$B_2 = \frac{a_2 I_2}{\sqrt{2}}[\cos(x - 2\beta - \omega t + \pi) - \cos(x + \omega t - \pi)]$$

$$= \frac{a_2 I_2}{\sqrt{2}}[-\cos(x - 2\beta - \omega t) + \cos(x + \omega t)] \quad . \quad . \quad (5.78)$$

If $a_1\bar{I}_1 = a_2\bar{I}_2$, that is if the ampere-turns of the two sets of coils are equal, then it is clear that, when $\lambda = \pi - \beta$, the reverse fields cancel, and we have two positively rotating waves only of constant amplitude. Again, if $\lambda - \beta = \pi$, we have

$$B_2 = \frac{a_2\bar{I}_2}{\sqrt{2}}[-\cos(x - \omega t) + \cos(x + \omega t - 2\beta)]$$

In this case the two coils produce a single rotating field, but rotating in the reverse direction.

In the first case, when $\lambda = \pi - \beta$, we have

$$B_1 + B_2 = \frac{a_1\bar{I}_1}{\sqrt{2}}[\cos(x - \omega t) - \cos(x - 2\beta - \omega t)]$$

$$B = \frac{2a_1\bar{I}_1}{\sqrt{2}}\sin(x - \omega t - \beta)\sin\beta \quad . \quad . \quad (5{\cdot}79)$$

since we assumed $a_1\bar{I}_1 = a_2\bar{I}_2$.

Now our object in the analysis is to resolve our unbalanced currents and voltages into two symmetrical systems; one system of currents produce a field of constant amplitude and constant speed in one direction, called the "positive-sequence" system, and the other set, called the "negative-sequence" system, produces a field of constant amplitude and speed, revolving in the opposite direction.

Let T_1 = number of turns in the main winding
T_2 = number of turns in the starting winding
f_1 = breadth factor × coil-span factor for main winding
f_2 = breadth factor × coil-span factor for starting winding

Then
$$\frac{T_1 \times f_1}{T_2 \times f_2} = \alpha$$

Let x_c = capacitive reactance of the capacitor in series with the starting winding
V = line volts
$\bar{I}_M$ = current in main winding
$\bar{I}_S$ = current in auxiliary winding

Then, since we assume that the ampere-turns of the two systems of currents in the two windings are equal, we have

$$\bar{I}_M \times T_1 \times f_1 = \bar{I}_S \times T_2 \times f_2$$

i.e.
$$\bar{I}_M\alpha = \bar{I}_S \quad . \quad . \quad . \quad . \quad (5{\cdot}80)$$

If $\bar{I}_{M_1}$ and $\bar{I}_{M_2}$ represent the positive- and negative-sequence components of current in the main winding and $\bar{I}_{S_1}$ and $\bar{I}_{S_2}$ represent the positive- and negative-sequence components of current in the auxiliary winding, we have

$$\boldsymbol{I}_M = \boldsymbol{I}_{M_1} + \boldsymbol{I}_{M_2} \qquad (5.81)$$

$$\boldsymbol{I}_S = \boldsymbol{I}_{S_1} + \boldsymbol{I}_{S_2} \qquad (5.82)$$

$$= +jI_{M_1}\alpha - jI_{M_2}\alpha \qquad (5.83)$$

from which we have

$$\boldsymbol{I}_{M_1} = \frac{\boldsymbol{I}_M - j\dfrac{\boldsymbol{I}_S}{\alpha}}{2} \qquad (5.84)$$

$$\boldsymbol{I}_{M_2} = \frac{\boldsymbol{I}_M + j\dfrac{\boldsymbol{I}_S}{\alpha}}{2} \qquad (5.85)$$

Similarly

$$\boldsymbol{I}_{S_1} = \frac{\boldsymbol{I}_S + j\boldsymbol{I}_M\alpha}{2} \qquad (5.86)$$

$$\boldsymbol{I}_{S_2} = \frac{\boldsymbol{I}_S - j\boldsymbol{I}_M\alpha}{2} \qquad (5.87)$$

Let $\boldsymbol{E}_S$ and $\boldsymbol{E}_M$ be the actual generated e.m.f.s in the auxiliary and main windings respectively; these voltages may be resolved into symmetrical components also.

Thus $$\boldsymbol{E}_S = \boldsymbol{E}_{S_1} + \boldsymbol{E}_{S_2}$$

E_{S_1} = positive-sequence component

E_{S_2} = negative-sequence component

and $$\boldsymbol{E}_M = \boldsymbol{E}_{M_1} + \boldsymbol{E}_{M_2}$$

$$= -j\boldsymbol{E}_{S_1}\alpha + j\boldsymbol{E}_{S_2}\alpha \qquad (5.88)$$

$$\therefore \quad \boldsymbol{E}_{S_1} = \frac{\boldsymbol{E}_S + j\dfrac{\boldsymbol{E}_M}{\alpha}}{2} \qquad (5.89)$$

and $$\boldsymbol{E}_{S_2} = \frac{\boldsymbol{E}_S - j\dfrac{\boldsymbol{E}_M}{\alpha}}{2} \qquad (5.90)$$

Also let $$\boldsymbol{Z}_S = \boldsymbol{Z}_M \times \frac{1}{\alpha^2} \qquad (5.91)$$

where $\boldsymbol{Z}_S$ = impedance of the auxiliary phase.

Now the positive-sequence components of the voltage, supplied to the auxiliary phase—which has the capacitor in series with it—are given by the following equations—

$$V_{S_1} = I_{S_1}\left(-j\frac{x_c}{2} + Z_{S_1}\right) - j\frac{x_c}{2} I_{S_2} \quad . \quad . \quad (5{\cdot}92)$$

and

$$V_{S_2} = I_{S_2}\left(-j\frac{x_c}{2} + Z_{S_2}\right) - j\frac{x_c}{2} I_{S_1} \quad . \quad . \quad (5{\cdot}93)$$

The zero-order and second-order components of the capacitive reactance are both $-j\frac{x_c}{2}$

Also

$$V_{S_1} = \frac{V}{2}\left(1 + \frac{j}{\alpha}\right) \quad . \quad . \quad . \quad (5{\cdot}94)$$

and

$$V_{S_2} = \frac{V}{2}\left(1 - \frac{j}{\alpha}\right) \quad . \quad . \quad . \quad (5{\cdot}95)$$

$$I_{S_1} = \frac{\frac{V}{2}\left\{\left(1 + \frac{j}{\alpha}\right) Z_{S_2} + \frac{x_c}{\alpha}\right\}}{Z_{S_1}Z_{S_2} - j\frac{x_c}{2}\left(Z_{S_1} + Z_{S_2}\right)} \quad . \quad . \quad (5{\cdot}96)$$

and

$$I_{S_2} = \frac{\frac{V}{2}\left\{\left(1 - \frac{j}{\alpha}\right) Z_{S_1} - \frac{x_c}{\alpha}\right\}}{Z_{S_1}Z_{S_2} - j\frac{x_c}{2}\left(Z_{S_1} + Z_{S_2}\right)} \quad . \quad . \quad (5{\cdot}97)$$

Also

$$I_{M_1} = \frac{\frac{V}{2}\left\{\left(\frac{1}{\alpha^2} - \frac{j}{\alpha}\right) Z_{M_2} - jx_c\right\}}{\frac{1}{\alpha^2} Z_{m_1}Z_{m_2} - j\frac{x_c}{2}\left(Z_{m_1} + Z_{m_2}\right)} \quad . \quad . \quad (5{\cdot}98)$$

$$I_{M_2} = \frac{\frac{V}{2}\left\{\left(\frac{1}{\alpha^2} + \frac{j}{\alpha}\right) Z_{m_2} - jx_c\right\}}{\frac{1}{\alpha^2} Z_{m_1}Z_{m_2} - j\frac{x_c}{2}\left(Z_{m_1} + Z_{m_2}\right)} \quad . \quad . \quad (5{\cdot}99)$$

where Z_{M_1} and Z_{M_2} represent the impedances of the main winding to positive- and negative-sequence currents. The starting torque is determined by finding the rotor currents corresponding to I_{M_1} and I_{M_2}, and referring both to the main winding of the stator. From the positive- and negative-sequence diagrams, we have the ratio—

$$\frac{I' \text{ rotor}}{I \text{ stator}} = \frac{\text{rotor current referred to stator}}{\text{stator current}} = \frac{+jx_m}{\frac{r_2'}{s} + j(x_m + x_2')}$$

Corresponding to I_{M_1}, we have the rotor current, referred to the stator—

$$I_{M_2}' = + I_{M_1} \times \frac{jx_m}{\frac{r_2'}{s} + j(x_m + x_2')} \quad . \quad . \quad . \quad (5\cdot100)$$

I_{M_2} is the current in the rotor corresponding to I_{M_1}.

Likewise for the component I_{M_2} in the stator, we have

$$I_{MR_2}' = + I_{M_2} \frac{jx_m}{\frac{r_2'}{2 - s} + j(x_m + x_2')} \quad . \quad . \quad . \quad (5\cdot101)$$

I_{MR_2}' is the current in the rotor corresponding to I_{M_2}, which is due to the reverse field.

r_2' is the rotor resistance, *referred to the main winding.*

x_m is the magnetizing reactance.

x_2' is the rotor reactance, *referred to the main winding.*

The torques due to the two fields are opposing, so we have the torques, given by the usual expressions, viz.

$$T_1 = \frac{2 \times 112\cdot7}{\text{r.p.m.}_{\text{syn}}} \times (I_{m_2}')^2 \times \frac{r_2'}{s} \text{ oz-ft} \quad . \quad (5\cdot102)$$

and $$T_2 = 2 \times \frac{112\cdot7}{\text{r.p.m.}_{\text{syn}}} \times (I_{MR_2}')^2 \times \frac{r_2'}{2 - s} \text{ oz-ft} \quad . \quad (5\cdot103)$$

and the total torque $= T_1 - T_2$

The starting torque is obtained by putting $s = 1$ in the above expression.

The starting torque will have a maximum value when the reactance of the capacitor equals $\frac{1}{\alpha^2}$ times the starting impedance of the main winding.

The maximum starting torque

$$= \frac{p}{\omega} \frac{V^2 r_2'}{R^2 + X^2} \times \alpha \times \frac{X}{R} \quad . \quad . \quad . \quad (5\cdot104)$$

where $R + jX$ is the locked impedance of the motor referred to the main winding. The maximum torque varies as α, which is the ratio of effective turns in the main winding to those of the starting winding.*

There is still another method of running a motor from a single-phase supply, which is extensively used on very small motors, such as servo motors and those used for gyro work. In effect this method uses a three-phase winding on the stator, connected in the following manner. It is known as the capacitor split method.

---

* See *Application of the Method of Symmetrical Components*, Waldo V. Lyon. McGraw-Hill (New York: 1927)

The three phases of the motor are shown as A, B, and C in Fig. 5.8, and the capacitor *C* is connected between *P* and *Q*.

It is possible to get, from the single-phase supply, a three-phase system of voltages, but the system is only symmetrical provided certain conditions are met. Let the three phases A, B, and C have

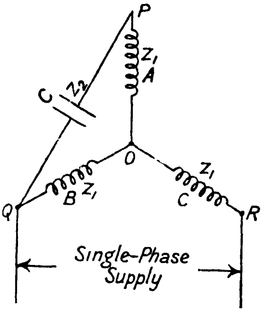


FIG. 5.8

total impedances per phase $\mathbf{Z}_1$ ohms and let $\mathbf{Z}_2$ = impedance of the capacitance. It will be assumed to have no loss, so $\mathbf{Z}_c = -jX_c$, where $X_c = \frac{1}{C\omega}$

The combined impedance, between *Q* and *O*

$$= \frac{\mathbf{Z}_1(\mathbf{Z}_1 + \mathbf{Z}_2)}{2\mathbf{Z}_1 + \mathbf{Z}_2} \quad . \quad . \quad . \quad (5{\cdot}105)$$

and the impedance between *Q* and *R*

$$= \frac{\mathbf{Z}_1(\mathbf{Z}_1 + \mathbf{Z}_2)}{2\mathbf{Z}_1 + \mathbf{Z}_2} + \mathbf{Z}_1 \quad . \quad . \quad . \quad (5{\cdot}106)$$

$$= \frac{3\mathbf{Z}_1^2 + 2\mathbf{Z}_1\mathbf{Z}_2}{2\mathbf{Z}_1 + \mathbf{Z}_2} \quad . \quad . \quad . \quad . \quad (5{\cdot}107)$$

Let $\hat{I}_T$ be the current in the line when a sinusoidal voltage $\hat{E}\sin\omega t$ is applied between *Q* and *R*.

Then $$\hat{I}_T = \frac{\hat{E} \times (2\mathbf{Z}_1 + \mathbf{Z}_2)}{3\mathbf{Z}_1^2 + 2\mathbf{Z}_1\mathbf{Z}_2} \quad . \quad . \quad . \quad (5{\cdot}108)$$

The current in the leg $OQ = \hat{I}_T \times \frac{\mathbf{Z}_1 + \mathbf{Z}_2}{2\mathbf{Z}_1 + \mathbf{Z}_2}$. (5·109)

The current in the leg $OR = \hat{I}_T$ . . . (5·110)

The current in the leg $OP = \hat{I}_T \times \frac{\mathbf{Z}_1}{2\mathbf{Z}_1 + \mathbf{Z}_2}$. (5·111)

If the three currents in the three phases are to be equal, then

$$\left|\frac{\mathbf{Z}_1}{2\mathbf{Z}_1+\mathbf{Z}_2}\right| = \left|\frac{\mathbf{Z}_1+\mathbf{Z}_2}{2\mathbf{Z}_1+\mathbf{Z}_2}\right| = 1 \qquad . \quad . \quad . \ (5.112)$$

$$\therefore \qquad |\mathbf{Z}_1+\mathbf{Z}_2| = |2\mathbf{Z}_1+\mathbf{Z}_2| \ . \quad . \quad . \ (5.113)$$

i.e.

$$|R+jX-jX_c| = |2R+2jX-jX_c| \quad . \ (5.114)$$

or

$$R^2+X^2+X_c^2-2XX_c = 4R^2+4X^2-4XX_c+X_c^2 \quad (5.115)$$

i.e.

$$X_c = \frac{3(R^2+X^2)}{2X} \quad . \quad . \quad . \ (5.116)$$

The currents will be equal in magnitude, provided

$$X_c = \frac{3(R^2+X^2)}{2X} \quad . \quad . \quad . \ (5.117)$$

They must also differ in phase by 120°.

In the branch OPQ, $\tan\phi_1 = \dfrac{X-X_c}{R}$. (5.118)

In the branch OQ, $\tan\phi_2 = \dfrac{X}{R}$

$$\tan(\phi_2-\phi_1) = \frac{\tan\phi_2-\tan\phi_1}{1+\tan\phi_2\tan\phi_1} = \frac{\dfrac{X}{R}-\left(\dfrac{X-X_c}{R}\right)}{1+\dfrac{X^2}{R^2}-\dfrac{XX_c}{R^2}} \quad (5.119)$$

$$= \frac{\dfrac{X_c}{R}}{1+\dfrac{X^2}{R^2}-\dfrac{XX_c}{R^2}} \quad . \quad . \quad . \quad . \ (5.120)$$

Now if $\phi_2-\phi_1 = 120°$,

$$\tan 120 = -\sqrt{3}$$

$$\therefore \qquad \frac{X_c}{R} = -\sqrt{3}\left(1+\frac{X^2}{R^2}-\frac{XX_c}{R^2}\right) \quad . \quad . \ (5.121)$$

$$\therefore \qquad X_c = \frac{\sqrt{3}\left(R+\dfrac{X^2}{R}\right)}{\sqrt{3}\dfrac{X}{R}-1} \quad . \quad . \quad . \ (5.122)$$

The two values of X_c, given by equations (5.122) and (5.116), must be equal

i.e.
$$\frac{\sqrt{3}\left(R+\frac{X^2}{R}\right)}{\sqrt{3}\,\frac{X}{R}-1}=\frac{3}{2}\left(\frac{R^2}{X}+X\right) \qquad . \quad . \quad . \ (5.123)$$

Let $\frac{X}{R}=K$, then from above $K=\sqrt{3}$

i.e.
$$\frac{X}{R}=\sqrt{3} \text{ and } X_c=2X \ . \qquad . \quad . \ (5.124)$$

There are thus two conditions to be satisfied for a true three-phase system of voltages to be produced—

(*a*) $$\frac{X}{R}=\sqrt{3}$$

and (*b*) $$X_c=2X$$

The X is the *total* inductive reactance of each phase. R is the resistance of each phase. The capacitive reactance must be twice the total inductive reactance of each phase.

[*Note.* Total inductive reactance, *not leakage reactance.*]

Assuming these conditions are satisfied for a symmetrical three-phase system of voltages, we may enquire as to what the power factor may be. We have

$$I_T=\bar{E}\,\frac{(2Z_1+Z_2)}{3Z_1^2+2Z_1Z_2}$$

$$Z_1=R+jX \qquad . \quad . \quad . \ (5.125)$$

$$Z_2=-jX_c=-2jX \ . \qquad . \quad . \ (5.126)$$

Then
$$\frac{2Z_1+Z_2}{3Z_1^2+2Z_1Z_2}=\frac{2R}{3R^2+2jRX+X^2} \ . \qquad . \ (5.127)$$

and
$$I_T=\frac{\mathbf{E}2R[(3R^2+X^2)-2jRX]}{(3R^2+X^2)^2+4R^2X^2} \ . \qquad . \ (5.128)$$

It is assumed than $E=\hat{E}\sin\omega t$

and
$$\tan\alpha=\frac{2RX}{3R^2+X^2} \ . \qquad . \quad . \ (5.129)$$

When $\frac{X}{R}=\sqrt{3}$,

$$\tan\alpha=\frac{2\,\frac{X^2}{\sqrt{3}}}{3\,\frac{X^2}{3}+X^2}=\frac{\frac{2}{\sqrt{3}}X^2}{2X^2}=\frac{1}{\sqrt{3}} \qquad . \ (5.130)$$

$\therefore$
$$\alpha=30^\circ$$

and
$$\cos\alpha=0{\cdot}866 \qquad . \quad . \quad . \ (5.131)$$

Thus, the power factor is 0·866 for the line current.

It is clear that, under these conditions, the performance can be estimated in the same manner as that used for a three-phase motor under a balanced system of voltages. This method of starting and running is common in small induction motors for gyro work and also for small servo motors. It is also used for hysteresis motors. The condition for a balanced system of voltages is not met in practice, and one finds both inequality of voltages and phase angles differing from 120°. For the analysis of the performance, one can resort to the system of symmetrical components.*

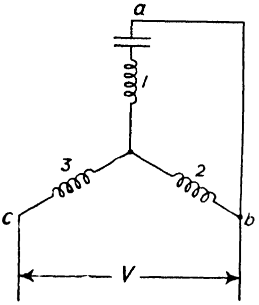


FIG. 5.9

As before, we obtain the positive- and negative-system of voltages and currents.

Let the capacitor reactance be $-jx$. Then the zero, positive- and negative-sequence components of this reactance $= -j\frac{x}{3}$

The applied potentials --

$$V_{ab} = 0;\quad V_{bc} = V;\quad V_{ca} = -V$$

$$V_{ab_1} = \tfrac{1}{3}[0 + aV - a^2V] \quad . \quad . \quad . \quad (5{\cdot}132)$$

$$= \frac{V}{3}[a - a^2] = \frac{V}{3}\sqrt{3}j - j\frac{V}{\sqrt{3}}$$

$$a = -0{\cdot}5 + j0{\cdot}866 \quad . \quad . \quad . \quad (5{\cdot}133)$$

$$a^2 = -0{\cdot}5 - j0{\cdot}866$$

$$V_{ab_2} = \tfrac{1}{3}[0 + a^2V - aV] \quad . \quad . \quad (5{\cdot}134)$$

$$= \frac{V}{3}(-j)\sqrt{3} = -j\frac{V}{\sqrt{3}} \quad . \quad . \quad (5{\cdot}135)$$

The positive-sequence component of the star voltage

$$V_{a_1} = \frac{V_{ab_1}}{\sqrt{3}}\angle -30 = \frac{V}{3}\angle 60 \quad . \quad . \quad (5{\cdot}136)$$

$$V_{a_2} = \frac{V_{ab_2}}{\sqrt{3}}\angle 30 = \frac{V}{3}\angle -60 \quad . \quad . \quad (5{\cdot}137)$$

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* Op. Cit.

The positive-sequence voltage in phase 1, i.e. the phase containing the capacitor,

$$= V_{a_1} = \frac{V}{3} \angle 60$$

The negative-sequence voltage in phase 1

$$= V_{a_2} = \frac{V}{3} \angle - 60$$

$\frac{V}{3} \angle 60$ means that the magnitude $= \frac{V}{3}$ and leads V by 60°

$\frac{V}{3} \angle - 60$ means that the magnitude $= \frac{V}{3}$ and lags V by 60°

Now for the voltage equations for the positive and negative sequences for phase 1. We have

$$\mathbf{V}_{a_1} = \mathbf{I}_{a_1}\left(\mathbf{Z}_1 - j\frac{x}{3}\right) - j\frac{x}{3} I_{a_2} \quad . \quad . \quad (5.138)$$

$$\mathbf{V}_{a_2} = \mathbf{I}_{a_2}\left(\mathbf{Z}_2 - j\frac{x}{3}\right) - j\frac{x}{3} I_{a_1} \quad . \quad . \quad (5.139)$$

The impedance of phase 1 to the positive-sequence currents is $\mathbf{Z}_1$, and to negative-sequence currents is $\mathbf{Z}_2$.

Substituting the values of $\mathbf{V}_{a_1}$ and $\mathbf{V}_{a_2}$ already found, we have

$$\frac{V}{3} \angle 60 = \mathbf{I}_{a_1}\left(\mathbf{Z}_1 - j\frac{x}{3}\right) - j\frac{x}{3}\mathbf{I}_{a_2} \quad . \quad . \quad (5.140)$$

$$\frac{V}{3} \angle - 60 = \mathbf{I}_{a_2}\left(\mathbf{Z}_2 - j\frac{x}{3}\right) - j\frac{x}{3}\mathbf{I}_{a_1} \quad . \quad . \quad (5.141)$$

or

$$\frac{V}{3} \angle - 60 = \mathbf{I}_{a_1}\left(-j\frac{x}{3}\right) + \mathbf{I}_{a_2}\left(\mathbf{Z}_2 - j\frac{x}{3}\right) \quad . \quad (5.142)$$

$$\therefore \qquad \mathbf{I}_{a_1} = \frac{\begin{vmatrix} \frac{V}{3} \angle 60 & -j\frac{x}{3} \\ \frac{V}{3} \angle - 60 & \mathbf{Z}_2 - j\frac{x}{3} \end{vmatrix}}{\begin{vmatrix} Z_1 - j\frac{x}{3} & -j\frac{x}{3} \\ -j\frac{x}{3} & \mathbf{Z}_2 - j\frac{x}{3} \end{vmatrix}} \quad . \quad . \quad (5.143)$$

$$\mathbf{I}_{a_1} = \frac{\frac{V}{3}\mathbf{Z}_2 \angle 60 - \frac{V}{3} \angle 60\, j\frac{x}{3} + \frac{V}{3} j\frac{x}{3} \angle - 60}{Z_1 Z_2 - j\frac{x}{3}(Z_1 + Z_2)} \quad . \quad (5.144)$$

$$\therefore \qquad \boldsymbol{I}_{a1} = \frac{\dfrac{V}{3}\left[\mathbf{Z}_2 \angle 60 + \dfrac{x}{\sqrt{3}}\right]}{\mathbf{Z}_1\mathbf{Z}_2 - j\dfrac{x}{3}(\mathbf{Z}_1 + \mathbf{Z}_2)} \qquad . \qquad . \quad (5{\cdot}145)$$

Note $\dfrac{V}{3}j \angle -60\dfrac{x}{3} = \dfrac{V}{3}\angle 30\dfrac{x}{3}$

and $-\dfrac{V}{3}\angle 60 j\dfrac{x}{3} = -\dfrac{V}{3}\angle 150\dfrac{x}{3}$

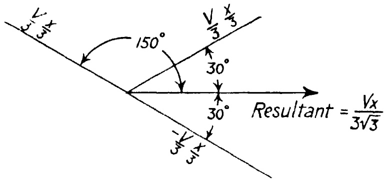


FIG. 5.10

$$\boldsymbol{I}_{a_2} = \frac{\dfrac{V}{3}\left(\mathbf{Z}_1 \angle -60 - \dfrac{x}{\sqrt{3}}\right)}{\mathbf{Z}_1\mathbf{Z}_2 - j\dfrac{x}{3}[\mathbf{Z}_1 + \mathbf{Z}_2]} \qquad . \qquad . \quad (5{\cdot}146)$$

The current in phase 3 is the line current

$$\boldsymbol{I}_c = \boldsymbol{I}_{c_1} + \boldsymbol{I}_{c_2} \qquad . \qquad . \qquad . \qquad . \quad (5{\cdot}147)$$

$$= a\boldsymbol{I}_{a_1} + a^2 I_{a_2} \qquad . \qquad . \qquad . \quad (5{\cdot}148)$$

a is an operator $= e^{j120} = \cos 120 + j \sin 120$, it turns the vector on which it operates through 120°.

$$\therefore \qquad I_c = I_{a_1}\angle 120 + I_{a_2}\angle 240$$

$$= I_a \angle 120 + I_{a_2}\angle -120$$

$$\therefore \qquad \boldsymbol{I}_c = \frac{-V[\mathbf{Z}_1 + \mathbf{Z}_2 - jx]}{3[\mathbf{Z}_1\mathbf{Z}_2] - jx[(\mathbf{Z}_1 + \mathbf{Z}_2)]} \qquad . \qquad . \quad (5{\cdot}149)$$

To make this step clear, we have

$$I_c = \frac{\dfrac{V}{3}\left[\mathbf{Z}_2 \angle 180 + \dfrac{x}{\sqrt{3}}\angle 120\right]}{\mathbf{Z}_1\mathbf{Z}_2 - j\dfrac{x}{3}(\mathbf{Z}_1 + \mathbf{Z}_2)} + \frac{\dfrac{V}{3}\left[\mathbf{Z}_1 \angle -180 - \dfrac{x}{\sqrt{3}}\angle -120\right]}{Z_1 Z_2 - j\dfrac{x}{3}(\mathbf{Z}_1 + \mathbf{Z}_2)} \qquad . \qquad . \quad (5{\cdot}150)$$

$$= V\left[\frac{\mathbf{Z}_2 \angle 180 + \mathbf{Z}_1 \angle -180 + \dfrac{x}{\sqrt{3}}\angle 120 - \dfrac{x}{\sqrt{3}}\angle -120}{3\mathbf{Z}_1\mathbf{Z}_2 - jx(\mathbf{Z}_1 + \mathbf{Z}_2)}\right] \qquad . \qquad . \quad (5{\cdot}151)$$

$$= \frac{V[-Z_2 - Z_1 + jx]}{3\mathbf{Z}_1\mathbf{Z}_2 - jx(\mathbf{Z}_1 + Z_2)} = -\frac{V[\mathbf{Z}_2 + \mathbf{Z}_1 - jx]}{3\mathbf{Z}_1\mathbf{Z}_2 - jx(\mathbf{Z}_1 + \mathbf{Z}_2)} \quad \cdot \ (5.152)$$

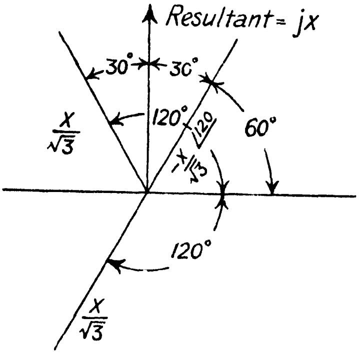


Fig. 5.11

Note. $\frac{x}{\sqrt{3}} \angle 120 - \frac{x}{\sqrt{3}} \angle -120$ has a resultant $\frac{2x}{\sqrt{3}} \cos 30j$ as shown in Fig. 5.11

$$= \frac{2x}{\sqrt{3}} \cdot \frac{\sqrt{3}j}{2} = xj$$

Our positive-sequence diagram then becomes as in Fig. 5.12.

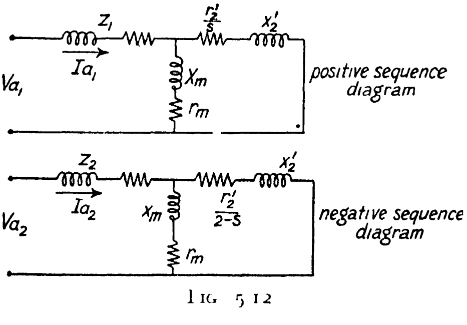


Fig. 5.12

The positive-sequence current in the rotor

$$= -I_{a_1} \times \frac{Z_m}{Z_m + \frac{r_2'}{s} + jx_2'} \simeq -I_{a_1} \times \frac{jx_m}{\frac{r_2'}{s} + j(x_m + x_2')} = I_{a_{1R}}' \quad \cdot \ (5.153)$$

The negative-sequence current in the rotor

$$= -I_{a_2} \times \frac{jx_m}{\frac{r_2'}{2-s} + j(x_2' + x_m)} = +I_{a_{2R}}' \quad \cdot \ (5.154)$$

where r_2' = rotor resistance, referred to the stator

x_2' = rotor reactance, at full frequency, referred to the stator

m_2 = number of rotor phases

The torque of the motor, in synchronous watts,

$$= m_2 \left[(I_{a_{1R}}')^2 \frac{r_2'}{s} - (I_{a_{2R}}')^2 \frac{r_2'}{2 - s} \right] \quad . \quad . \quad (5{\cdot}155)$$

The starting torque, in synchronous watts,

$$= m_2[(I_{a_{1R}}')^2 r_2' - (I_{a_{2R}}')^2 r_2'] \quad . \quad . \quad (5{\cdot}156)$$

$$\simeq m_2[(I_{a_1}')^2 r_2' - (I_{a_2}')^2 r_2'] \quad . \quad . \quad . \quad (5{\cdot}157)$$

$$\simeq m_2 \frac{V^2}{9} \left[\frac{\left(\mathbf{Z} \angle 60 + \frac{x}{\sqrt{3}}\right)^2 - \left(\mathbf{Z} \angle -60 - \frac{x}{\sqrt{3}}\right)^2}{\left(\mathbf{Z}^2 - j\frac{x}{3} 2\mathbf{Z}\right)^2} \right] r_2' \quad (5{\cdot}158)$$

Since, when $s = 1$, $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathcal{Z}$.

The starting torque then becomes

$$\simeq m_2 \frac{V^2}{9} \frac{\frac{2}{\sqrt{3}} Rxr_2'}{(R^2 + X^2)\left(R^2 + \left\{X - \frac{2x}{3}\right\}^2\right)} \quad . \quad (5{\cdot}159)$$

where R = resistance per phase

and X = the reactance per phase at standstill

Differentiating equation (5.159) with respect to x, we have

$$\frac{R^2 + \left(X - \frac{2x}{3}\right)^2 + 2\left(X - \frac{2x}{3}\right) \times \frac{2}{3} x}{\left\{R^2 + \left(X - \frac{2x}{3}\right)^2\right\}^2}$$

for a maximum value

$$R^2 + X^2 - \frac{4xX}{3} + \frac{4x^2}{9} + \frac{4Xx}{3} - \frac{8}{9} x^2 = 0 \quad . \quad (5{\cdot}160)$$

i.e.
$$R^2 + X^2 - \frac{4}{9} x^2 = 0$$

i.e.
$$x^2 = \frac{9}{4}(R^2 + X^2) \quad . \quad . \quad . \quad (5{\cdot}161)$$

and
$$x = 1{\cdot}5\sqrt{R^2 + X^2}$$

That is, the starting torque is a maximum when the reactance of the capacitor = 1·5 × impedance per phase.

Substituting we have, maximum torque in synchronous watts

$$= \frac{m_2 V^2}{6\sqrt{3}} \frac{R r_2'}{(R^2 + X^2)(\sqrt{R^2 + X^2} - X)} \quad . \quad (5\cdot162)$$

The maximum torque at the start in lb-ft

$$= \frac{7\cdot04}{\underset{\text{syn}}{\text{r.p.m.}}} \times \frac{m_2 V^2 R r_2'}{(R^2 + X^2)(\sqrt{R^2 \times X^2} - X) \times 6\sqrt{3}} . \quad (5\cdot163)$$

If $X \gg R$

$$\sqrt{X^2 + R^2} - X - X\sqrt{1 + \frac{R^2}{X^2}} - X$$

$$\bumpeq X\left(1 + \frac{1}{2}\frac{R^2}{X^2}\right) - X \bumpeq \frac{R^2}{2X}$$

and the maximum starting torque

$$= \frac{7\cdot04}{\underset{\text{syn}}{\text{r.p.m.}}} \times \frac{m_2}{6\sqrt{3}} \frac{V^2 r_2' \times 2X}{(R^2 + X^2)R} \quad . \quad . \quad . \quad (5\cdot164)$$

$$= \frac{7\cdot04}{\underset{\text{syn}}{\text{r.p.m.}}} \times \left(\frac{V}{\sqrt{3}}\right)^2 \times \frac{m_2 r_2'}{R^2 + X^2} \times \frac{X}{\sqrt{3}R} \quad . \quad (5\cdot165)$$

Thus, the maximum torque developed at the start $= \frac{X}{\sqrt{3}R}$ times that for the normally balanced motor.

CHAPTER VI

Electric Braking of Induction Motors

In certain classes of machines, such as hoists, cranes, rolling mills, etc., it is necessary to stop quickly, and hence braking is necessary. An induction motor may be braked in several ways. Alternating current may be used for excitation, but it is necessary to reverse the field of the motor by reversing two of the primary leads in the three-phase machine, and by reversing two primary leads of one phase in a two-phase machine. The rotor is then running in the opposite direction to the field. The power generated in the rotor is dissipated in I^2R loss. Remembering that the torque, in synchronous watts, is equal to the rotor input, i.e. the torque, in synchronous watts

$$= (I_{2t}')^2 \frac{r_2'}{s} \qquad . \quad . \quad . \quad . \quad (6.1)$$

Note. I_{2t}' in this case = $\sqrt{3}$ × rotor current per phase referred to the stator

and

$$\text{torque in lb-ft} = \frac{(I_{2t}')^2 r_2'}{\text{r.p.m.}_{\text{syn}} \times s} \times 7{\cdot}04 \qquad . \quad . \quad (6.2)$$

we see that, if the torque is to remain constant, during the braking period, and I_{2t}' is constant, then $\frac{r_2'}{s}$ must be constant. As the slip decreases from $s = 2$, which corresponds to full speed, with the field rotating in the opposite direction, to $s = 1$, i.e. to standstill, if the rotor current is to remain constant, with constant braking torque, *then r_2' must decrease in the same ratio as the slip decreases.* If a liquid resistance is used, it would be possible to change the resistance in such a manner that the torque and current would remain constant during the braking period. At the instant when the motor is running near synchronous speed, and the field is reversed, the frequency of the rotor currents is nearly twice that at standstill, and the secondary voltage will be nearly twice that at standstill. In large induction motors, this may impose excessive stress on the insulation, so in these machines it is desirable to apply half normal voltage to the primary, or use direct current for braking. We will now develop approximate expressions for the speed-torque and ampere-torque curves.

We will use our approximate equivalent circuit, shown in Fig. 6.1.

Let $\bar{V}_1$ = primary terminal volts; the machine is assumed to be connected in star, so that the phase voltage $= \frac{\bar{V}_1}{\sqrt{3}}$

$\bar{V}_2$ = secondary volts between rings at standstill

$\bar{I}_1 = \sqrt{3}$ × current per phase, for three-phase

= 2 × current per line, for two-phase

$\bar{I}_{0t} = \sqrt{3}$ × no-load current

$\bar{I}_{2t} = \sqrt{3}$ × secondary current per phase

Note. Since we are using the terminal volts in our equivalent circuit, the current to be used in calculating power $= \sqrt{3}$ × current per phase.

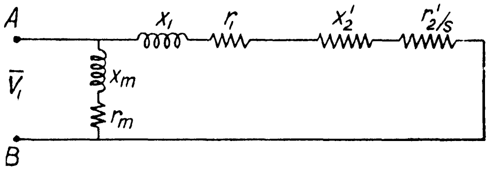


FIG. 6.1

r_1 = primary resistance per phase

r_2' = secondary resistance, referred to the primary

x_1 = primary leakage reactance per phase

x_2' = the secondary leakage reactance per phase, referred to the primary

ρ = total watts with the motor locked

τ = torque in lb-ft

s = slip

n_0 = synchronous revolutions per minute

Since the power factor at no load and at short circuit is nearly the same,

$$x_1 + x_2' \doteq \frac{\sqrt{\{(\bar{I}_{st} - \bar{I}_{0t})\bar{V}_1\}^2 - \rho^2}}{(\bar{I}_{st} - \bar{I}_{0t})^2} \quad . \quad . \quad (6.3)$$

where $\bar{I}_{st} = \sqrt{3}$ × short-circuit current per phase

$$\bar{I}_{2t}' = \frac{\bar{V}_1}{\sqrt{\left(r_1 + \frac{r_2'}{s}\right)^2 + (x_1 + x_2')^2}} \quad . \quad . \quad (6.4)$$

$$\tau = \text{torque in lb-ft} = \frac{(I_{2t}')^2 \times \frac{r_2'}{s} \times 7{\cdot}04}{n_0} \tag{6.5}$$

$$\therefore \quad \text{slip } s = \frac{(I_{2t}')^2 r_2' \times 7{\cdot}04}{n_0 \times \tau} \tag{6.6}$$

By substituting for $\bar{I}_{2t}'$ the value

$$= \frac{\bar{V}_1}{\sqrt{\left(r_1 + \frac{r_2'}{s}\right)^2 + (x_1 + x_2')^2}}$$

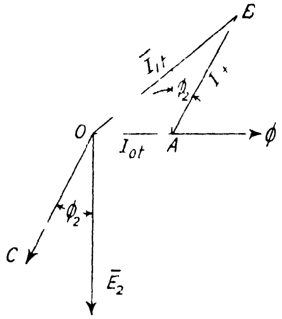


Fig. 6.2

in equation (6.6), and differentiating, we find the torque is a maximum when

$$r_1 + \frac{r_2'}{s} = x_1 + x_2'$$

Neglecting the no-load watt-component of the primary current, our vector diagram, Fig. 6.2, shows the primary current $(\times \sqrt{3}) = OB$,

and $$\bar{I}_{1t} = \sqrt{(\bar{I}_{2t}' \sin \phi_2 + \bar{I}_{0t})^2 + (\bar{I}_{2t}' \cos \phi_2)^2} \tag{6.7}$$

and since $$\sin \phi_2 = \frac{x_1 + x_2'}{\sqrt{\left(r_1 + \frac{r_2'}{s}\right)^2 + (x_1 + x_2')^2}} \tag{6.8}$$

and $$\cos \phi_2 = \frac{r_1 + \frac{r_2'}{s}}{\sqrt{\left(r_1 + \frac{r_2'}{s}\right)^2 + (x_1 + x_2')^2}} \tag{6.9}$$

we have

$$\bar{I}_{1t} = \sqrt{\left\{\frac{\bar{I}_{2t}'(x_1 + x_2')}{\sqrt{\left(r_1 + \frac{r_2'}{s}\right)^2 + (x_1 + x_2')^2}} + \bar{I}_{0t}\right\}^2 + \left\{\frac{\bar{I}_{2t}'\left(r_1 + \frac{r_2'}{s}\right)}{\sqrt{\left(r_1 + \frac{r_2'}{s}\right)^2 + (x_1 + x_2')^2}}\right\}^2} \tag{6.10}$$

From these equations the torque-slip and torque-current curves can be drawn for various values of the rotor resistance.

Note. The torque in synchronous watts = input to the rotor circuit

$$= \frac{m_2 (I_2')^2 r_2'}{s} \tag{6.11}$$

For a three-phase rotor $m_2 = 3$, but

$$I_{2t}' = \sqrt{3I_2'} \quad . \quad . \quad . \quad . \quad (6.12)$$

$$\therefore \qquad \frac{m_2(I_2')^2 r_2'}{s} = \frac{(I_{2t}')^2 \times r_2'}{s} \quad . \quad . \quad . \quad (6.13)$$

The following example is given by Specht, to whose article on braking, the reader is referred (*Journal of The American Institute of Electrical Engineers*).

A motor of 2000 b.h.p., three-phase, 6600 V line, 25 c/s, 6 poles, 500 r.p.m. (syn.), with both stator and rotor connected in star, is braked with a.c. current.

$\bar{I}_{0t} = 58\text{ A} =$ line amperes $\times \sqrt{3}$

No-load power $= 32$ kW

$\bar{I}_{st} = 1550$ A with locked rotor $= \sqrt{3} \times \bar{I}_s$

$\rho = 1950$ kW with locked rotor

$r_1 = 0{\cdot}38\ \Omega$

$\bar{E}_2 = 1700$ V between rings

The resistance per phase of the secondary at 40° $= 0{\cdot}026\ \Omega$,

$$\therefore \qquad r_2' = 0{\cdot}026 \times \left(\frac{6600}{1700}\right)^2 = 0{\cdot}39\ \Omega \quad . \quad . \quad (6.14)$$

In Fig. 6.3 nine speed-torque curves are given for the following rotor resistances, in ohms, referred to the primary—

Curve 1. $r_2' = 0{\cdot}39$ without external resistance
,, 2. $r_2' = 1{\cdot}2$ including external resistance
,, 3. $r_2' = 2{\cdot}4$,, ,, ,,
,, 4. $r_2' = 5{\cdot}0$,, ,, ,,
,, 5. $r_2' = 8{\cdot}0$,, ,, ,,
,, 6. $r_2' = 13{\cdot}0$,, ,, ,,
,, 7. $r_2' = 20{\cdot}0$,, ,, ,,
,, 8. $r_2' = 34{\cdot}0$,, ,, ,,
,, 9. $r_2' = 60{\cdot}0$,, ,, ,,

For $r_2' = 0{\cdot}39$ and $s = 2$, we have

$$\bar{I}_{2t} = \frac{6600}{\sqrt{\left(0{\cdot}38 + \frac{0{\cdot}39}{2}\right)^2 + 4{\cdot}37^2}} = 1496\text{ A} \quad . \quad (6.15)$$

$$x_1 + x_2 = \frac{\sqrt{[(1550 - 58) \times 6600]^2 - (1\,950\,000)^2}}{(1550 - 58)^2} = 4{\cdot}37\ \Omega \quad (6.16)$$

and

$$\tau = \frac{1496^2 \times \frac{0{\cdot}39}{2}}{500} \times 7{\cdot}04 = 6100\text{ lb-ft} \quad . \quad . \quad (6.17)$$

For various values of the slip from 2 to 0, the torque curve 1 and secondary ampere curve are determined.

The results are shown in Fig. 6.3.

It will be noticed that if the motor should be braked with a torque equal to the average full-load torque, and with a current not exceeding greatly the full-load current, the resistance should be decreased, step by step, until the motor stops. This is shown by the heavy zigzag line. This corresponds to the use of a metallic resistance. If a reduced voltage is applied to the stator for braking, the currents vary in the same ratio as the voltage, and the torque varies as the square of the voltage ratio. Therefore, the same curves can be used for reduced voltage as for full voltage, by changing the current scale in the ratio of the voltage and the torque scale in the square of the voltage ratio. In order to brake the motor at half-voltage with full-load current and half the full-load torque, the secondary resistance must be half that at full voltage. This torque can only be obtained when the maximum torque is at least twice full-load torque at full voltage, because the maximum torque at half-voltage will be only one-quarter of that at full voltage.

In the case of a squirrel-cage motor, one speed-torque curve only can be obtained, and in order to obtain a good torque for braking, without excessive current, a high-resistance rotor is necessary. Such a motor is very inefficient when running normally, and should only be used for very intermittent service for elevators, cranes, and hoists.

Braking by Direct Current

When using direct current for braking, the d.c. supply may be connected to the primary in the following ways, shown in Fig. 6.4. The connections for three-phase machines are given in Figs. (*a*) to (*f*), and those for two-phase machines in (*g*) to (*h*).

In the case where one terminal is connected to the negative supply and the remaining two terminals to the positive supply, the current in phase 1 (when at the maximum value) will have double the value of the currents in phases 2 and 3, and will flow in opposite direction. The current in phases 2 and 3 flows from the terminals to the star point (negative direction) and, in phase 1, it will flow from the star point to the terminal (positive direction).

This is just what happens when, in a three-phase system, the current in phase 1 is at its crest value. Let us call the r.m.s. value of the current I, then the momentary current in phase 1 is $\sqrt{2}I$ and, in phases 2 and 3, $\frac{1}{\sqrt{2}} I$

It is clear that the direct current, in this case, which replaces the three-phase excitation, will have the value $\sqrt{2}I$

Let us consider a star-wound motor and connect only two

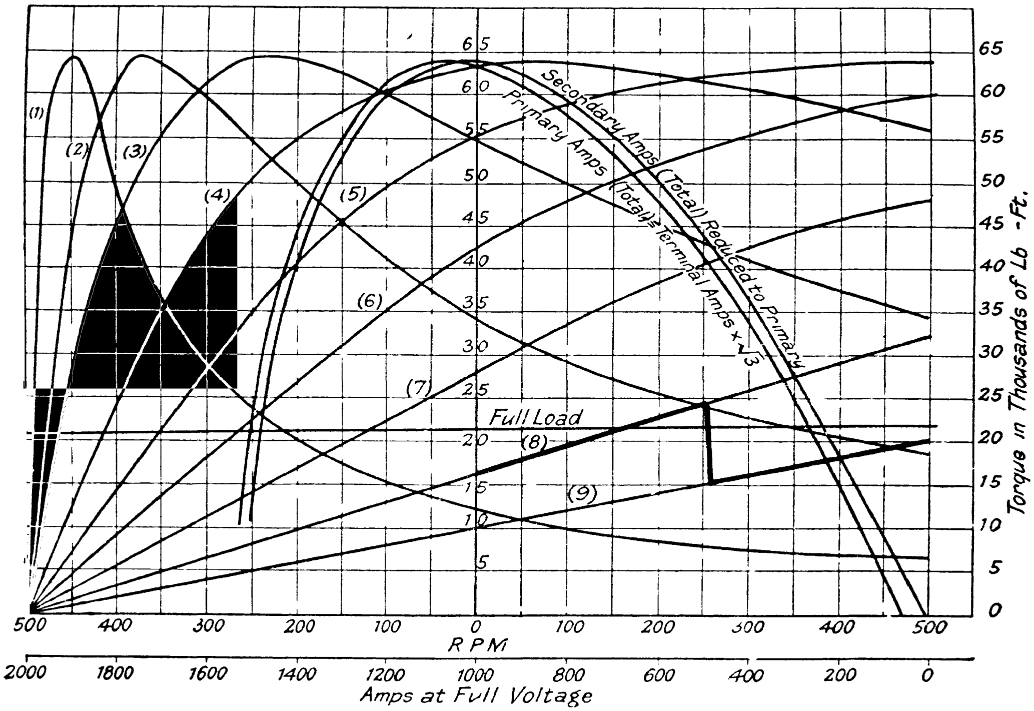


Fig 6 3 Ampere Speed/Torque Curve for A C Excitation terminals to the supply. Then the current in the third phase is zero, the currents in phases 1 and 2 have the same value, but opposite in direction. This corresponds to the moment in which, in the three-phase machine, the current in phase 3 is zero, whereas the currents in the other two phases are

$$+\sqrt{\frac{3}{2}}\,I \text{ and } -\sqrt{\frac{3}{2}}\,I = 1{\cdot}23I \text{ and } -1{\cdot}23I$$

In this case the equivalent direct current will be 23 per cent higher than the measured value of the alternating current.

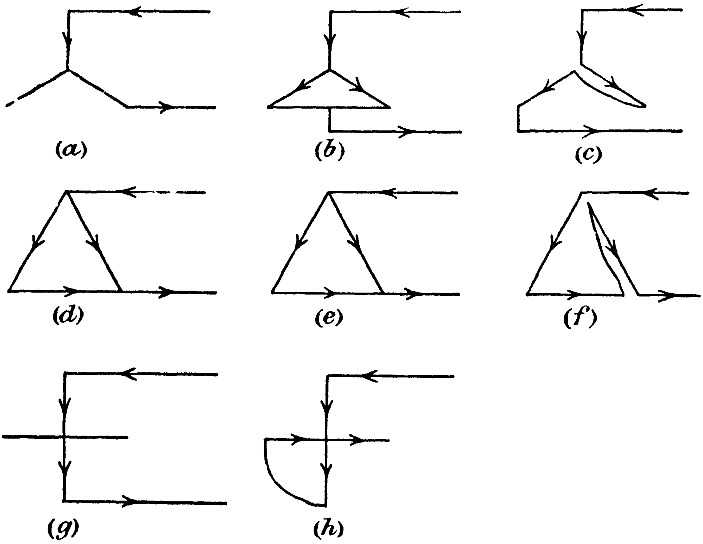


FIG. 6.4

The watt losses in the first case are, if R represents the resistance of one phase,

$$(1{\cdot}414I)^2R + 2 \times (0{\cdot}707I)^2R = 3I^2R \quad . \quad . \quad (6.18)$$

and, in the second case,

$$2(1{\cdot}23I)^2R = 3I^2R \quad . \quad (6.19)$$

Therefore, the d.c. voltages, applied in cases 1 and 2, are in inverse proportion to the currents.

Exactly the same relation holds good for delta connection. In this case, if the d.c. supply is connected to two terminals only, the direct current has to be 23 per cent greater than the measured alternating current; whereas if one terminal is connected to the positive and the two others to the negative supply, the value of the current is $\sqrt{2}I$

Let r_2 = secondary ohmic resistance measured between terminals and divided by 2
x = inductive reactance at synchronous speed
s = slip = 0 at synchronous speed of motor
= 1 at standstill
n_0 = synchronous revolutions per minute
n = revolutions per minute of motor when running
e_2 = secondary voltage between terminals at no-load speed
i_2 = *total* secondary current equivalent to single phase
[for three-phase i_2 = terminal amperes × $\sqrt{3}$]
[for two-phase i_2 = terminal amperes × 2]
i_s = total secondary short-circuit current for inductive reactance purely
i_1 = total primary amperes
T = torque in lb at 1 ft radius
i_0 = direct current for exciting
t_1 = number of turns per phase in primary
t_2 = number of turns per phase in secondary

For the d.c. excitation, an equivalent alternating current $= \frac{1}{\sqrt{2}}$ times the direct current can be substituted, and then, for synchronous speed, the voltage and short-circuit current in the secondary can be determined by transformation.

The short-circuit current is equal to the equivalent alternating magnetizing current reduced to the secondary turns. The secondary open-circuit voltage is the same as that obtained by the equivalent d.c. excitation when running at synchronous speed. In determining the corresponding alternating short-circuit current, the distribution and the amount of winding excited by direct current must be considered, i.e. the current must be multiplied by a factor C besides $\frac{1}{\sqrt{2}}$

The factor C for a three-phase winding, of which two phases are excited by direct current, is 1·15.

The short-circuit current in a three-phase secondary at synchronous speed, which would be obtained if the rotor had inductive reactance only is

$$i_s = \frac{i_0\sqrt{3}}{\sqrt{2}} \times \frac{t_1}{t_2} \times C \qquad . \quad . \quad . \ (6.20)$$

and the equivalent primary alternating current is

$$i_1 = i_0 \times \frac{\sqrt{3}}{\sqrt{2}} \times C \text{ for three-phase primary} \qquad . \ (6.21)$$

$$i_1 = i_0 \frac{2}{\sqrt{2}} \times C \text{ for two-phase primary} \qquad . \ (6.22)$$

The corresponding secondary volts e_2 can be read off on the open a.c. saturation curve.

The inductive reactance $$x = \frac{e_2}{i_s} \quad \text{. . . . . (6.23)}$$

also $$i_2 = \frac{e_2}{\sqrt{\left(\frac{r_2}{s}\right)^2 + x^2}} \quad \text{. . . (6.24)}$$

$$T = \frac{i_2{}^2 r_2 \times 7{\cdot}04}{n} \quad \text{. . . . (6.25)}$$

$$s = \frac{n_0 T}{i_2{}^2 r_2} \times 7{\cdot}04 \quad \text{. . (6.26)}$$

Maximum torque occurs when $\frac{r_2}{s} = x$.

We will compare d.c. braking with a.c. braking.

The 2000 h.p. motor, to which we have already referred, will be selected for example. Assuming the primary is excited by direct current across two terminals of the star winding, and that this current is 100 A, then

$$i_s = 100 \times \frac{\sqrt{3}}{\sqrt{2}} \times \frac{6600}{1700} \times 1{\cdot}15 = 550 \text{ A total} \quad \text{. (6.27)}$$

$$i_1 = 100 \times \frac{\sqrt{3}}{\sqrt{2}} \times 1{\cdot}15 = 141{\cdot}7 \text{ A total} \quad \text{. (6.28)}$$

From the saturation curve, the corresponding secondary voltage e_2 for $i_1 = 141{\cdot}7$ A is found to be 2420 V.

The inductive reactance $= \frac{2420}{550} = 4{\cdot}4\ \Omega$.

For various secondary resistances r_2, the following values are selected—

Curve 1. $r_2 = 0{\cdot}026\ \Omega$ without external resistance
,, 2. $r_2 = 0{\cdot}5\ \Omega$ including external resistance
,, 3. $r_2 = 1{\cdot}0\ \Omega$,, ,, ,,
,, 4. $r_2 = 2{\cdot}0\ \Omega$,, ,, ,,
,, 5. $r_2 = 3{\cdot}0\ \Omega$,, ,, ,,

For $s = 1$ and $r_2 = 0{\cdot}5\ \Omega$, the secondary current i_2 and torque T are

$$i_2 = \frac{2420}{\sqrt{\left(\frac{0{\cdot}5}{1{\cdot}0}\right)^2 + 4{\cdot}4^2}} = 548 \text{ A} \quad \text{. . (6.29)}$$

$$T = \frac{548^2 \times 0{\cdot}5}{500} \times 7{\cdot}04 = 2120 \text{ lb-ft} \quad \text{. . (6.30)}$$

In this way the currents and torques for other slips and resistances may be found. The results are shown in the following figure.

Examination of the curves shows that, without external resistance, the torque at 500 r.p.m. is nearly zero, and increases very slowly with decreasing speed, except that below 50 r.p.m. the torque increases much faster. At about 3 r.p.m. the torque reaches the maximum value (9400 lb-ft) and then drops very rapidly to zero value. It is clear that this speed-torque curve is of no practical value; and in order to obtain good braking torque over a wide speed range, it is necessary to insert a fairly large resistance.

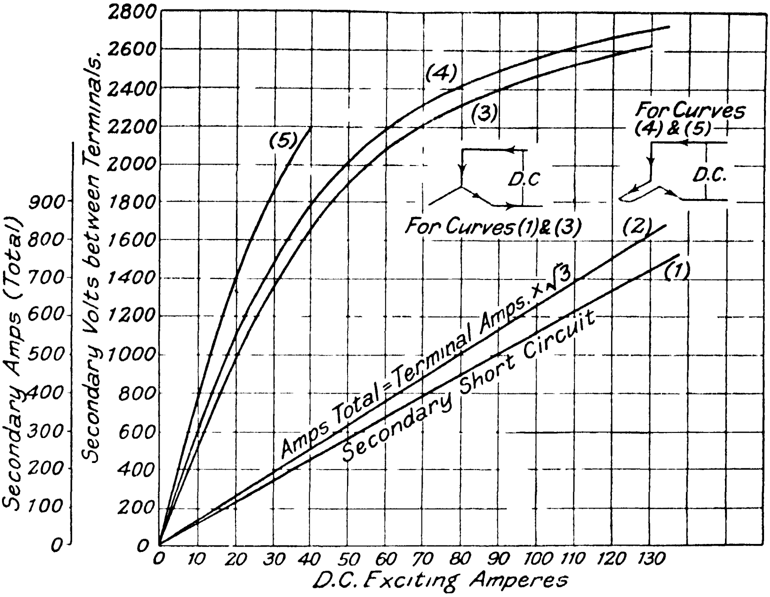


FIG. 6.5. SATURATION CURVE FOR A.C. EXCITATION

Further, the curves show that the maximum torque obtainable with an exciting current of 100 A is not even quite half full-load torque, and that the open-circuit secondary voltage at synchronous speed is 42 per cent greater than the voltage at standstill with 6600 V alternating current on the primary. It would be possible to obtain a greater torque by increasing the exciting current. This would give, however, a higher secondary voltage and stronger field, and would increase the unbalanced pull and the danger o greater potential rise in case any of the circuits should break. It is pointed out by Specht that the conditions for d.c. braking of this motor are very poor, due to the low value of the no-load current as compared to the full-load current. Motors having a larger ratio in this respect would give more favourable results.

Nevertheless, the best torque which can be obtained by braking with direct current is not greater than full-load torque. The same motor was driven by another motor at synchronous speed, and the primary was excited by direct current and the open-circuit secondary voltage measured. Then the secondary circuit was closed and the short-circuit current was measured. The results are shown in Fig. 6.5.

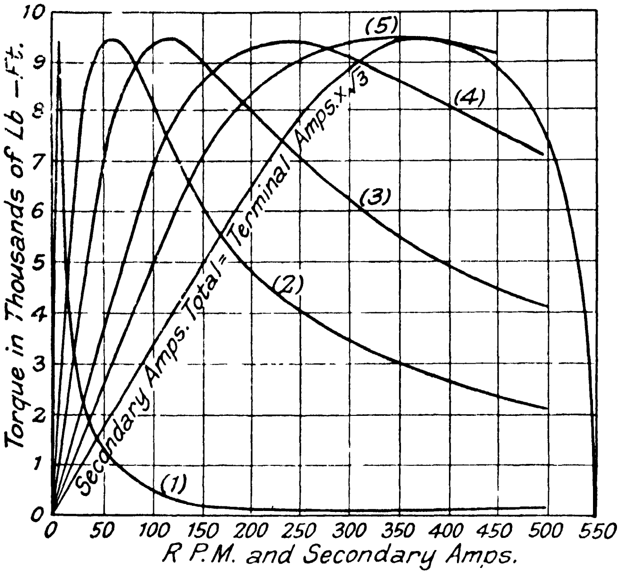


FIG. 6.6. AMPERE SPEED/TORQUE CURVE FOR D.C. EXCITATION

Curves 1 and 3 correspond to an excitation of two phases; and curves 2, 4, and 5 correspond to an excitation of three phases.

Analysis of the results shows—

(1) The braking torque obtainable by alternating current, even with only half the primary voltage, is, as a rule, considerably greater than with direct current.

(2) In braking with alternating current, the line circuit has to be taken off the motor as soon as the motor comes to rest, otherwise the motor will reverse; whereas in braking with direct current, the motor comes to rest only and will not reverse.

(3) With alternating current, it is an easy matter to obtain a strong and practically constant braking torque during the whole retardation period; whereas with d.c. excitation, it is difficult to obtain good braking torque near standstill, due to the rapid decrease in torque from maximum to zero.

(4) If it is desired to brake the motor with full-load torque by means of direct current, the secondary voltage at synchronous speed will be not far from double voltage; and, further, since the magnetic field has to be much stronger with d.c. current, there is a danger of serious voltage rises due to breaking of any of the circuits.

The only advantage of braking with direct current is the small energy which is needed. Only the I^2R losses of the primary have to be supplied with d.c. excitation; while with a.c. excitation, the full power has to be supplied to the motor which it would require for developing an equal torque at normal operating condition. This section with the curves and examples are taken from the paper by Specht.

CHAPTER VII

Speed Control of Induction Motors

THE induction motor is practically a constant-speed motor, resembling, in this respect, the d.c. shunt motor, and is admirably adapted to constant-speed work, but there are many, and varied, applications where variable speed is necessary. This variable-speed field is extensive and includes rolling-mill motors, cranes and hoists, pumps and compressors, etc. There is a trend towards the use of alternating current on ships, and variable-speed motors are necessary for winches, capstans, etc. There are numerous methods by which speed control can be obtained with the induction motor. They are as follows—

(*a*) Rheostatic control by the introduction of resistance in the rotor circuits.

(*b*) Pole-changing.

(*c*) Cascade connection.

(*d*) Change of supply frequency.

(*e*) By the use of a synchronous converter in circuit with the rotor.

(*f*) By concatenation with the three-phase series and shunt commutator motors.

(*g*) By the use of resistance and reactance in parallel in the rotor circuits.

Speed Control by Resistance in Rotor Circuit

The simplest method of varying the speed of an induction motor is to insert resistance in the rotor circuit. By this means any speed required, *below* synchronous speed is obtainable, but only at the cost of efficiency. It has been shown that the percentage efficiency is always less than the speed as a percentage of synchronous speed. It is chiefly used in the following cases—

(1) During starting a motor, where the energy wasted in resistance bears a small ratio to the total energy.

(2) For the speed control of haulage motors of medium size, where the wind is a long one.

(3) For rolling-mill motors and winding-gear equalizers.

The Continuous Slip Regulator

For rolling-mill work, where a mill has to roll a great variety of sections, it is clear that speed regulation is necessary. Smaller sections must be finished at a higher speed than the larger ones, otherwise the metal would cool too rapidly and could only be formed by the expenditure of a great deal of power, which increases the liability of the breakdown of the mill, and the accuracy of the sections and quality of the product may be affected. To obtain a reasonable production, the smaller sections must be rolled at as high a speed as possible. Within certain limits, where the speed regulation does not exceed 10 to 15 per cent, the rheostatic method of control is the simplest and most satisfactory.

In rolling-mill applications, it is very important to equalize the load on the generating plant as much as possible, to ensure maximum efficiency in operation. To that end, a flywheel is coupled to the induction motor, which is caused to drop its speed by the introduction of resistance in the rotor circuits. This allows the flywheel to give up some of its energy when peak loads have to be met. Thus, during the passes, a sudden, heavy demand is made on the motor and generating system. This demand on the generating plant is reduced by the equalizing effect of the flywheel, and, furthermore, the size of motor is also reduced. The resistance is introduced into the rotor circuit in two ways. In one the resistance is *permanently* in circuit, and the method is known as the "continuous slip regulator." It is a matter of interest to analyse the behaviour of the motor, and to show how the speed, torque, and output of the motor, and input from the line vary under load conditions. We will consider, in the first place, a rolling-mill of the continuous type, and deduce the torque and speed equations when permanent resistance is adopted.

Let I = moment of inertia of the flywheel in lb-ft^2

ω = angular velocity in radians per second at any time t seconds from the commencement of the pass

T = torque exerted by the induction motor at any time t seconds, from the commencement of the pass, in lb-ft

T_f = full-load torque in lb-ft

T_2 = maximum torque exerted by the induction motor in the pass in lb-ft

T_3 = total torque in the pass, assumed constant, in lb-ft

ω_0 = synchronous speed in radians per second

ω_f = speed of the motor, in radians per second, at full-load, with permanent resistance in circuit

E_2 = voltage per phase in the rotor, at standstill

I_2 = rotor current per phase in amperes

m_2 = number of rotor phases—usually three

R = resistance per phase in rotor circuit at working temperature

$L_2\omega$ = rotor reactance per phase at standstill

s slip $= \dfrac{\text{synchronous speed} - \text{actual speed}}{\text{synchronous speed}}$

Then

$$I_2 = \frac{sE_2}{\sqrt{R^2 + s^2L_2^2\omega^2}} \quad . \quad . \quad (7.1)$$

$$T = \frac{sE_2^2Rm_2}{R^2 + s^2L_2^2\omega^2} \text{ in synchronous watts} \quad . \quad . \quad (7.2)$$

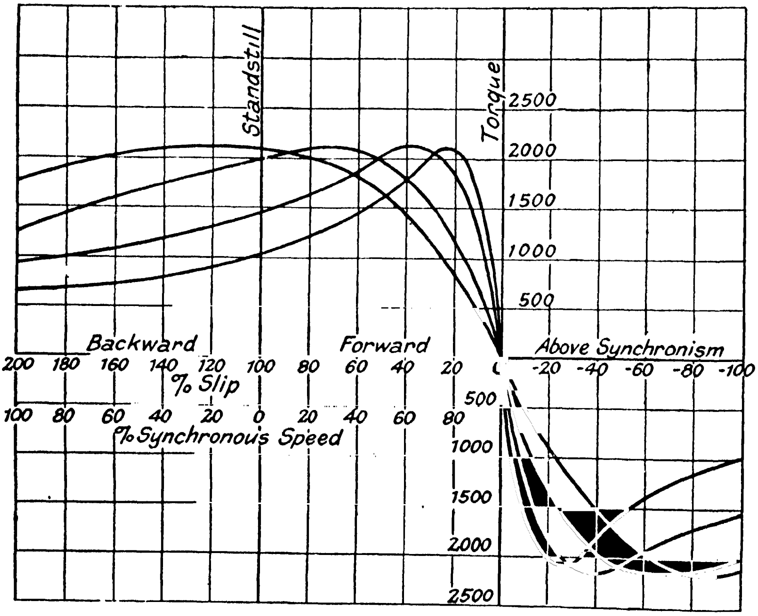


FIG 7.1. TYPICAL SPEED/TORQUE CURVES WITH DIFFERENT VALUES OF ROTOR RESISTANCE

The torque slip curves of an induction motor are given in Fig. 7.1. The slip, usually taken, is about 20 per cent in the passes. Now for *small* values of the slip

$$T \simeq \frac{m_2 s E_2^2}{R} \propto \text{slip} \quad . \quad . \quad . \quad . \quad (7.3)$$

It will be seen, from Fig. 7.1, that the assumption that the torque is proportional to slip is sufficiently accurate for practical purposes.

$$\therefore \quad \frac{T}{T_f} = \frac{\omega_0 - \omega}{\omega_0} \div \frac{\omega_0 - \omega_f}{\omega_0} = \frac{\omega_0 - \omega}{\omega_0 - \omega_f} \quad . \quad . \quad (7.4)$$

Now $$T - I\frac{d\omega}{dt} = T_3 \quad . \quad . \quad . \quad . \qquad (7.5)$$

since $\frac{d\omega}{dt}$ is negative.

$$\therefore \quad \frac{T_F w_0}{\omega_0 - \omega_f} - \frac{T_F \omega}{\omega_0 - \omega_f} - I\frac{d\omega}{dt} = T_3 \quad . \quad . \qquad (7.6)$$

Let $$\frac{T_F}{\omega_0 - \omega_f} = a \quad . \quad . \quad . \quad . \quad . \qquad (7.7)$$

Then $$a\omega_0 - a\omega - I\frac{d\omega}{dt} = T_3 \quad . \quad . \quad . \quad . \qquad (7.8)$$

i.e. $$\frac{d\omega}{dt} + \frac{a\omega}{I} = \frac{a\omega_0 - T_3}{I} = \frac{c}{I} \quad . \quad . \quad . \qquad (7.9)$$

where $$c = a\omega_0 - T_3 \quad . \quad . \quad . \quad . \quad . \qquad (7.10)$$

Multiplying by $e^{\int \frac{a}{I} dt}$ we have

$$\frac{d}{dt}\left(\omega e^{\frac{a}{I}t}\right) = \frac{c}{I}e^{\frac{a}{I}t} \quad . \quad . \quad . \quad . \qquad (7.11)$$

$$\therefore \quad \omega e^{\frac{a}{I}t} = \frac{c}{I}\int e^{\frac{a}{I}t}\,dt + E \quad . \quad . \quad . \quad . \qquad (7.12)$$

where E = constant of integration

$$\therefore \quad \omega = \frac{c}{a} + Ee^{-\frac{a}{I}t} \quad . \quad . \quad . \quad . \quad . \qquad (7.13)$$

When $t = 0$, $\omega = \omega_1$,

$$\therefore \quad \omega_1 = \frac{c}{a} + E \quad . \quad . \quad . \quad . \quad . \qquad (7.14)$$

$$\therefore \quad E = \omega_1 - \frac{c}{a} \quad . \quad . \quad . \quad . \quad . \qquad (7.15)$$

$$\therefore \quad \omega = \frac{c}{a} + \left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t} \quad . \quad . \quad . \quad . \qquad (7.16)$$

$$\therefore \omega = \omega_0 - \frac{T_3}{T_f}(\omega_0 - \omega_f) + \left\{\omega_1 - \omega_0 + \frac{T_3}{T_f}(\omega_0 - \omega_f)\right\}e^{-\frac{a}{I}t} \quad (7.17)$$

This is our speed equation and from it the speed, at any time during the pass, can be determined.

At the commencement of the first pass, ω_1 will be the speed, corresponding to the friction load.

At the beginning of the second pass, it may happen that the speed has not fallen to the frictional load speed. In that case the speed, at the end of the first interval between passes, can be found, and this speed must be substituted in the equation for the speed at the beginning of the second pass.

The average speed during the pass

$$= \frac{1}{t}\int_0^t \omega dt$$

where t = time of pass

$$= \frac{1}{t}\int_0^t \left\{\frac{c}{a} + \left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t}\right\} dt \quad . \quad . \quad . \quad (7.18)$$

$$= \frac{c}{a} + \left[-\frac{I}{a}\left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t}\right]_0^t \times \frac{1}{t} \quad . \quad . \quad (7.19)$$

$$= \frac{c}{a} - \frac{I}{a}\frac{\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}}{t} + \frac{I}{at}\left(\omega_1 - \frac{c}{a}\right) \quad . \quad . \quad (7.20)$$

Therefore, average speed, during the pass

$$= \omega_0 - \frac{T_3}{T_f}(\omega_0 - \omega_f)$$

$$+ \frac{I}{tT_f}\left[\{\omega_0 - \omega_f\}\left\{\omega_1 - \omega_0 + \frac{T_3}{T_f}(\omega_0 - \omega_f)\right\}\right]\left[1 - e^{-\frac{a}{I}t}\right] \quad . \quad (7.21)$$

During the interval between passes, we have

$$T = T_0 + I\frac{d\omega}{dt} \quad . \quad . \quad . \quad . \quad (7.22)$$

Here $\frac{d\omega}{dt}$ is positive, and T_0 = frictional torque.

$$T = \frac{T_f\omega_0}{\omega_0 - \omega_f} - \frac{T_f\omega}{\omega_0 - \omega_f}$$

$$\therefore \quad I\frac{d\omega}{dt} + a\omega = a\omega_0 - T_0 \quad . \quad . \quad . \quad (7.23)$$

$$\text{i.e.} \quad \frac{d\omega}{dt} + \frac{a\omega}{I} = \frac{a\omega_0 - T_0}{I} = \frac{f}{I} \quad . \quad . \quad (7.24)$$

where $f = a\omega_0 - T_0$

$$\therefore \quad \omega e^{\frac{a}{I}t} = \frac{f}{I}\int e^{\frac{a}{I}t}\, dt + F \quad . \quad . \quad . \quad (7.25)$$

When $t = t_2$ = time to the end of the first pass, $\omega = \omega_2$,

$$\therefore \quad \omega_2 = \frac{f}{a} + Fe^{-\frac{a}{I}t_2} \quad . \quad . \quad . \quad (7.26)$$

$$\therefore \quad F = \left(\omega_2 - \frac{f}{a}\right) e^{\frac{a}{I} t_2} \quad . \quad . \quad . \quad (7.27)$$

$$\therefore \quad \omega = \frac{f}{a} + \left(\omega_2 - \frac{f}{a}\right) e^{\frac{a}{I}(t_2 - t)} \quad . \quad . \quad . \quad (7.28)$$

$$\therefore \quad \omega = \omega_0 - \frac{T_0}{T_f}(\omega_0 - \omega_f) + \left\{\omega_2 - \omega_0 + \frac{T_0}{T_f}(\omega_0 - \omega_f)\right\} e^{\frac{a}{I}(t_2 - t)} \quad . \quad . \quad . \quad (7.29)$$

The average speed, during the interval,

$$= \frac{1}{t_3 - t_2}\int_{t_2}^{t_3} \omega dt = \frac{f}{a} + \frac{I}{a(t_3 - t_2)}\left(\omega_2 - \frac{f}{a}\right)\left\{1 - e^{\frac{a}{I}(t_2 - t_3)}\right\} \quad (7.30)$$

Coming now to the question of torque, we find during the pass,

$$T = T_3 + I\frac{d\omega}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.31)$$

$$T = T_3 + I\left\{-\frac{a}{I}\left(\omega_1 - \frac{c}{a}\right)\right\} e^{-\frac{a}{I}t} \quad . \quad . \quad . \quad (7.32)$$

$$= T_3 - a\left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.33)$$

$$= T_3 - \frac{T_f}{\omega_0 - \omega_f}\left\{\omega_1 - \omega_0 + \frac{T_3}{T_f}(\omega_0 - \omega_f)\right\} e^{-\frac{a}{I}t} \quad (7.34)$$

From this the torque at any instant during the pass can be determined.

Let T_1 = torque exerted by the motor at the beginning of the pass, i.e. $t = 0$, then we have

$$T_1 = T_3 - a\left(\omega_1 - \frac{c}{a}\right) \quad . \quad . \quad . \quad (7.35)$$

and T_2 the torque at the end of the pass $t = t_2$.

Then
$$T_2 = T_3 - a\left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t_2} \quad . \quad . \quad . \quad (7.36)$$

$$= T_3 + \frac{T_1 - T_3}{e^{\frac{a}{I}t_2}} \quad . \quad . \quad . \quad . \quad . \quad (7.37)$$

Now
$$\frac{at_2}{I} = \frac{T_f}{\omega_0 - \omega_f} \times \frac{t_2}{I} = \frac{T_f t_2}{\omega_0\left(\frac{\omega_0 - \omega_f}{\omega_0}\right) I} = \frac{T_f t_2}{\omega_0 s I} \quad . \quad (7.38)$$

$$\therefore \quad T_2 = T_3 + \frac{T_1 - T_3}{e^{\frac{T_f t_2}{\omega_0 s I}}} = T_3 + \frac{T_1 - T_3}{A^{t_2}} \quad . \quad . \quad (7.39)$$

where
$$A = e^{\frac{T_f}{\omega_0 s I}} \quad . \quad . \quad . \quad . \quad . \quad (7.40)$$

The torque exerted by the motor, during the interval

$$T = T_0 + I\frac{d\omega}{dt} \quad \text{. . . . . . .} \quad (7{\cdot}41)$$

$$= T_0 + I\frac{d}{dt}\left\{\frac{f}{a} + \left(\omega_2 - \frac{f}{a}\right)e^{\frac{a}{I}(t_2-t)}\right\} \quad \text{. .} \quad (7{\cdot}42)$$

$$= T_0 - a\left(\omega_2 - \frac{f}{a}\right)e^{\frac{a}{I}(t_2-t)} \quad \text{. . . .} \quad (7{\cdot}43)$$

where $t = t_2$, $T = T_2$

$$\therefore \qquad T_2 = T_0 - a\left(\omega_2 - \frac{f}{a}\right) \quad \text{. . . .} \quad (7{\cdot}44)$$

and when $t = t_3$ the time of the beginning of the second pass $T = T_3'$

$$\therefore \qquad T_3' = T_0 - a\left(\omega_2 - \frac{f}{a}\right)e^{\frac{a}{I}(t_2-t_3)} \quad \text{. . .} \quad (7{\cdot}45)$$

It is not our purpose to go into the considerations which determine the relative proportions of flywheel and motor, but it may be remarked that, where the power is generated at the works, it is important to use relatively heavy flywheels to keep the load on the generating plant as constant as possible. In those cases where power is supplied from an outside source, the method of charging affects the proportions; a heavy flywheel being necessary when the maximum demand system is in vogue, and a small flywheel and relatively large motor, when power is charged for on the flat-rate

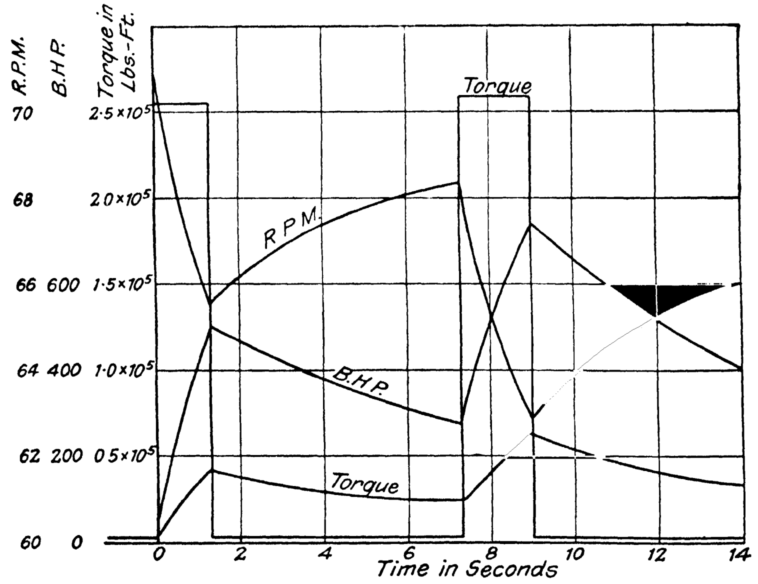


FIG. 7.2. CURVES SHOWING INDUCTION MOTORS' PERFORMANCE FOR PERMANENT RESISTANCE

system, it being important, in the latter case to reduce the friction losses as much as possible.

Application of our Formulae to a Cogging Mill

The following example will be instructive: a 20 in. to 3 ft high cogging mill for rolling ingots from 10 in. × 10 in. to 6 in. × 6 in. has six passes. The speed of the mill is 70 to 59·5 r.p.m.

The torque diagram is given in Fig. 7.2.

The flywheel employed has a moment of inertia of $0{\cdot}54 \times 10^6$ lb-ft^2 and has a diameter of 20 ft. The motor has a full-load output of 630 h.p. at 70 r.p.m., and the speed drop, arranged for, is from 70 to 59·5 r.p.m. The maximum torque, exerted by the motor = 83 700 lb-ft.

The slip, at 59·5 r.p.m. $= \frac{71{\cdot}5 - 59{\cdot}5}{71{\cdot}5} = \frac{12}{71{\cdot}5} = 0{\cdot}168$; percentage slip = 16·8.

At full load, the torque is 47 500 lb-ft $= \frac{83\,700}{1{\cdot}765}$ and the slip, at full-load torque = 9·5 per cent.

The full-load speed is, therefore, 64·7 r.p.m. and the full-load output is $\frac{47\,500 \times 2\pi \times 64{\cdot}7}{33\,000} = 585$ h.p.

Calculation of torque and speed in the first pass and interval

In the pass $\omega = \frac{c}{a} + \left(\omega_1 - \frac{c}{a}\right) e^{\frac{a}{I}t}$

$$\frac{c}{a} = \omega_o - \frac{T_3}{T_f}(\omega_o - \omega_f)$$

$$\omega_o = 2\pi \times \frac{71{\cdot}5}{60} = 7{\cdot}48 \text{ radn/sec}$$

$$\omega_f = 2\pi \times \frac{64{\cdot}7}{60} = 6{\cdot}78 \text{ radn/sec}$$

$$T_3 = 2{\cdot}505 \times 10^5 \text{ lb-ft}$$

friction torque = 4000 lb-ft

$$\frac{T_3}{T_f} = \frac{2{\cdot}5 \times 10^5}{4{\cdot}75 \times 10^4} = 5{\cdot}28$$

$$\omega_0 - \omega_f = 0{\cdot}7$$

$$\frac{c}{a} = 3{\cdot}78$$

no-load slip = 0·8 per cent

r.p.m., no-load = 70·9

$$\frac{a}{I} = \frac{47\,500}{0{\cdot}095 \times 7{\cdot}48 \times 0{\cdot}54 \times 10^6} = 0{\cdot}1236$$

Using equation (7.16) for the speed and equation (7·33) for the torque, we will tabulate the various quantities for the first pass—

Time in Seconds	ω	R.p.m.	Torque lb-ft	$e^{-\frac{a}{I}t}$	H.p. Output of Motor
0 (beginning)	7·40	70·9	4 000	1·000	54
0·30	7·27	69·3	13 500	0·965	179
0·60	7·14	68·0	22 500	0·928	292
0·90	7·00	66·8	31 500	0·895	409
1·27 end	6·88	65·5	41 900	0·856	524

For the first interval, we use equation (7.28) for the speed and equation (7.43) for the torque

$$\frac{f}{a} = 7{\cdot}421\,; \quad \omega_2 - \frac{f}{a} = 6{\cdot}88 - 7{\cdot}421 = -0{\cdot}541$$

Tabulating the results for the interval, we have—

Time in Seconds	ω	R.p.m.	Torque lb-ft	$e^{-\frac{a}{I}(t_2-t)}$	H.p. of Motor
1·50	6·894	65·7	39 600	0·973	495
2·50	6·955	66·5	35 600	0·862	450
3·50	7·014	67·0	31 500	0·753	402
4·50	7·057	67·5	28 600	0·673	366
7·27	7·164	68·4	21 400	0·475	279

For the second pass, we put $\omega_1 = 7{\cdot}164$ the value of the speed at the end of the first pass.

Tabulated results, for the second pass, are as follows—

Time in Seconds	ω	R.p.m.	Torque lb-ft	$e^{-\frac{a}{I}t}$	H.p. of Motor
0	7·164	68·4	21 400	1·000	279
0·300	7·050	67·5	30 000	0·965	384
0·600	6·910	66·0	39 000	0·928	490
0·900	6·790	64·7	48 000	0·890	592
1·200	6·690	63·8	54 000	0·865	656
1·525	6·570	62·8	62 000	0·827	740

$$c = a\omega_0 - T_3$$

here $T_3 = 2{\cdot}58 \times 10^5$

$$\frac{c}{a} = \omega_0 - \frac{T_3}{T_f}(\omega_0 - \omega_f) = 7{\cdot}48 - \frac{2{\cdot}58 \times 10^5}{4{\cdot}75 \times 10^4} \times 0{\cdot}7 = 3{\cdot}68$$

$$\omega_1 - \frac{c}{a} = 7{\cdot}164 - 3{\cdot}68 = 3{\cdot}484$$

Also $$T = T_3 - a\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}$$

$$a = \frac{T_f}{\omega_0 - \omega_f} = \frac{47\ 500}{0{\cdot}7} = 67\ 855$$

In the interval of the second pass, we have

$$\omega = \frac{f}{a} + \left(\omega_2 - \frac{f}{a}\right)e^{\frac{a}{I}(t_2 - t)}$$

and $$f = a\omega_0 - T_0$$

$$\frac{f}{a} = \omega_0 - \frac{T_0}{T_f}(\omega_0 - \omega_f) = 7{\cdot}48 - \frac{4000}{47\ 500} \times 0{\cdot}7 = 7{\cdot}421$$

$$\omega_2 = 6{\cdot}57$$

and $$\omega = 7{\cdot}421 - 0{\cdot}85\,e^{\frac{a}{I}(t_2 - t)}$$

Tabulated results for the interval after the second pass are—

Time from Commencement of Second Pass in Seconds	ω	R.p.m.	Torque lb-ft	$e^{-\frac{a}{I}(t_2 - t)}$	H.p. of Motor
2·000	6·619	63·3	58 300	0·943	700
3·000	6·718	64·4	51 700	0·827	632
4·000	6·799	64·8	46 300	0·733	570
5·000	6·871	65·5	41 200	0·646	515
6·525	7·017	67·0	31 350	0·475	399

The average speed during the first pass (from equation (7.21)) $= \omega_{av} = 7{\cdot}07$, and average speed, in revolutions per minute, in the first pass $= 67{\cdot}5$.

The motor torque, at any instant of the pass is,

$$T = T_3 - a\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t} \qquad (7{\cdot}46)$$

and the average torque, during the pass,

$$= \frac{1}{t}\int_0^t T\,dt$$

$$= \frac{1}{t}\int_0^t \left\{T_3 - a\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}\right\}dt$$

$$= \frac{1}{t}\left[T_3 t + I\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}\right]_0^t$$

$$= T_3 + \frac{I}{t}\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t} - \frac{I}{t}\left(\omega_1 - \frac{c}{a}\right) \qquad (7{\cdot}47)$$

For the first pass

$$\text{av. } T = 2{\cdot}505 \times 10^5 + (0{\cdot}54 \times 10^6 \times 3{\cdot}62 \times 0{\cdot}855 - 0{\cdot}54 \times 10^6 \times 3{\cdot}62) \times \frac{1}{1{\cdot}27}$$

$$= 27\,500 \text{ lb-ft}$$

The results, for the two passes, are shown in Fig. 7.2.

Constant Output in the Passes

The power equation for the pass is

$$T\omega + I\frac{d\omega}{dt}\omega = W \quad . \quad . \quad . \quad (7{\cdot}48)$$

where W = constant power in the pass

As before $T = a\omega_0 - a\omega$

$$\therefore \quad a\omega\omega_0 - a\omega^2 - I\frac{d\omega}{dt}\omega = W \quad . \quad . \quad (7{\cdot}49)$$

$$\therefore \quad dt = \frac{I\omega d\omega}{a\omega\omega_0 - a\omega^2 - W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7{\cdot}50)$$

$$= -\frac{1}{2a} \times \frac{I(a\omega_0 - 2a\omega)d\omega}{a\omega\omega_0 - a\omega^2 - W} + \frac{I}{2}\frac{\omega_0 d\omega}{(a\omega\omega_0 - a\omega^2 - W)} \quad (7{\cdot}51)$$

$$\therefore \; t = -\frac{I}{2a}\log(a\omega\omega_0 - a\omega^2 - W) + \frac{I\omega_0}{2}\int\frac{d\omega}{a\omega\omega_0 - a\omega^2 - W} + C \quad . \quad . \quad (7{\cdot}52)$$

$$\therefore \; t = -\frac{I}{2a}\log(a\omega\omega_0 - a\omega^2 - W) - \frac{I\omega_0}{2a\sqrt{\frac{W}{a} - \frac{\omega_0^2}{4}}}\tan^{-1}\left(\frac{\omega - \frac{1}{2}\omega_0}{\sqrt{\frac{W}{a} - \frac{\omega_0^2}{4}}}\right) + C \quad (7{\cdot}53)$$

When $t = 0$, $\omega = \omega_1$,

$$\therefore \; 0 = -\frac{I}{2a}\log(a\omega_0\omega_1 - a\omega_1^2 - W) - \frac{I\omega_0}{2a\sqrt{\frac{W}{a} - \frac{\omega_0^2}{4}}}\tan^{-1}\left(\frac{\omega_1 - \frac{1}{2}\omega_0}{\sqrt{\frac{W}{a} - \frac{\omega_0^2}{4}}}\right) + C \quad (7{\cdot}54)$$

$$\therefore \; C = \frac{I}{2a}\log(a\omega_0\omega_1 - a\omega_1^2 - W) + \frac{I\omega_0}{2a\sqrt{\frac{W}{a} - \frac{\omega_0^2}{4}}}\tan^{-1}\left(\frac{\omega_1 - \frac{1}{2}\omega_0}{\sqrt{\frac{W}{a} - \frac{\omega_0^2}{4}}}\right) \quad (7{\cdot}55)$$

$$\therefore\ t = \frac{I}{2a}\log\left(\frac{(a\omega_0\omega_1 - a\omega_1^2 - W}{a\omega_0\omega - a\omega^2 - W}\right)$$
$$+\frac{I\omega_0}{2a\sqrt{\dfrac{W}{a}-\dfrac{\omega_0^2}{4}}}\left[\tan^{-1}\left(\frac{\omega_1 - \frac{1}{2}\omega_0}{\sqrt{\dfrac{W}{a}-\dfrac{\omega_0^2}{4}}}\right) - \tan^{-1}\left(\frac{\omega - \frac{1}{2}\omega_0}{\sqrt{\dfrac{W}{a}-\dfrac{\omega_0^2}{4}}}\right)\right] \quad (7{\cdot}56)$$

In passes, the last term is usually negligibly small and, where it is not negligible, the inverse tangent can be expanded and an approximation obtained.

$$\therefore \qquad t \simeq \frac{I}{2a}\log\left(\frac{a\omega_0\omega_1 - a\omega_1^2 - W}{a\omega_0\omega - a\omega^2 - W}\right) \quad . \quad . \quad (7{\cdot}57)$$

Put $a\omega_0\omega_1 - a\omega_1^2 - W = d$

then
$$\frac{d}{a\omega_0\omega - a\omega^2 - W} = e^{+\frac{2at}{I}} \quad . \quad . \quad . \quad (7{\cdot}58)$$

$$\therefore \qquad de^{-\frac{2at}{I}} = a\omega_0\omega - a\omega^2 - W \quad . \quad . \quad . \quad (7{\cdot}59)$$

i.e.
$$a\omega^2 - a\omega_0\omega + W + de^{-\frac{2at}{I}} = 0 \quad . \quad . \quad (7{\cdot}60)$$

$$\therefore \qquad \omega = \frac{a\omega_0 \pm \sqrt{a^2\omega_0^2 - 4(W + de^{-\frac{2at}{I}})a}}{2a} \quad . \quad . \quad (7{\cdot}61)$$

Equation (7.61) gives the speed at any time t from the commencement of the pass.

The output of the induction motor, during the pass,

$$= W + I\omega\frac{d\omega}{dt} \quad . \quad . \quad . \quad . \quad (7{\cdot}62)$$

and
$$\frac{d\omega}{dt} = \frac{2ade^{-\frac{2at}{I}}}{I\sqrt{a^2\omega_0^2 - 4\,(W + de^{-\frac{2at}{I}})a}} \quad . \quad . \quad (7{\cdot}63)$$

The motor output can thus be determined for constant total output in the passes.

In the interval between the passes, the power equation is

$$I\frac{d\omega}{dt}\,\omega + W_0 = T\omega \quad . \quad . \quad . \quad (7{\cdot}64)$$

where W_0 = frictional power, which is assumed to remain constant.

The Case of the Automatic Slip Regulator

We will now consider the case of a rolling-mill cycle, in which an automatic slip regulator is introduced at a predetermined value of the load. An induction motor, of the slip-ring type, whose stator is excited from the secondary circuit of a series transformer, the primary of which is inserted in the leads to the motor, is used for operating the electrodes. The movement of the rotor is resisted by means of a spring. As the current in the leads to the stator of the main motor increases, the current in the induction motor of the slip regulator increases. The torque varies as the square of the current, so the displacement of the rotor will vary as the square of the current. A lever, attached to the rotor of the regulator motor, moves the blades of the liquid resistance switch in and out of the liquid, thus varying the resistance in the main motor circuit. An increase in load causes the blades to leave the liquid partially, and thus to increase the resistance and slip of the main motor. We will investigate shortly the law, according to which such regulators should be designed, to meet the objects in view. Unlike the case of permanent resistance, the torque of the induction motor remains constant, or, at least, this is the condition to be aimed at, since the current to the motor remains constant, if this condition is fulfilled.

Taking the case of a continuous mill, with constant total torque in the pass, our torque equation is—

$$T - I\frac{d\omega}{dt} = T_3 \quad . \quad . \quad . \quad . \quad (7{\cdot}65)$$

T_3 = constant total torque in the pass, and the other symbols have the same significance as before.

$$\frac{d\omega}{dt} = \frac{T - T_3}{I}$$

$$\therefore \quad \omega = \left(\frac{T - T_3}{I}\right) t + G \quad . \quad . \quad . \quad (7.66)$$

When $t = t_1$, the time when the slip regulator comes into action, let $\omega = \omega_1$,

$$\therefore \quad \omega_1 = \left(\frac{T - T_3}{I}\right) t_1 + G \quad . \quad . \quad . \quad (7.67)$$

$$\therefore \quad G = \omega_1 - \left(\frac{T - T_3}{I}\right) t_1 \quad . \quad . \quad . \quad (7.68)$$

and

$$\omega = \omega_1 + \frac{T - T_3}{I}(t - t_1) \quad . \quad . \quad . \quad (7.69)$$

ω_1 is easily determined, for it is simply the angular velocity, in radians per second, when the regulator comes in.

It will be seen that the relation between angular velocity and time is linear.

In the interval between passes, our torque equation is

$$T - I\frac{d\omega}{dt} = T_0 \quad . \quad . \quad . \quad . \quad (7{\cdot}70)$$

where T_0 = friction torque

$$\therefore \qquad \omega = \left(\frac{T - T_0}{I}\right) t + F \quad . \quad . \quad . \quad (7{\cdot}71)$$

When $t = t_2$, the time to the end of the first pass, $\omega = \omega_2$,

$$\therefore \qquad \omega_2 = \left(\frac{T - T_0}{I}\right) t_2 + F \quad . \quad . \quad . \quad (7{\cdot}72)$$

$$\therefore \qquad F = \omega_2 - \left(\frac{T - T_0}{I}\right) t_2 \quad . \quad . \quad . \quad (7{\cdot}73)$$

Therefore, ω in the interval

$$= \omega_2 + \left(\frac{T - T_0}{I}\right)(t - t_2) \quad . \quad . \quad . \quad (7{\cdot}74)$$

The power of the motor, at any time during the pass, with the regulator in

$$= T\omega = T\omega_1 + T\left(\frac{T - T_3}{I}\right)(t - t_1) \quad . \quad . \quad (7{\cdot}75)$$

The average speed in the pass, during the time the regulator is in action

$$= \tfrac{1}{2}(\omega_1 + \omega_2)$$

where ω_1 = angular speed at the time of entrance of the regulator

and $\quad \omega_2$ = angular speed at the end of the pass

$$= \tfrac{1}{2}\left\{\omega_1 + \omega_1 + \frac{T - T_3}{I}(t_2 - t_1)\right\} \quad . \quad . \quad (7{\cdot}76)$$

$$= \omega_1 + \frac{T - T_3}{I}\,\frac{(t_2 - t_1)}{2} \quad . \quad . \quad . \quad (7{\cdot}77)$$

In the interval, the torque equation is—

$$T - I\frac{d\omega}{dt} = T_0 \,. \quad . \quad . \quad . \quad (7{\cdot}78)$$

where T_0 = frictional torque

It is interesting to enquire what values of rotor resistance are required in order that the torque of the motor shall remain constant. The torque, in synchronous watts,

$$= \frac{m_2 E_2^{\,2} R_2 s}{R_2^{\,2} + s^2 L_2^{\,2} \omega_0^{\,2}}$$

If T = motor torque in *kilogramme-metres*, then

$$T\omega_s 9{\cdot}81 = \frac{m_2 E_2^{\,2} R_2 s}{R_2^{\,2} + s^2 L_2^{\,2} \omega_0^{\,2}} \,. \quad . \quad . \quad (7{\cdot}79)$$

where m_2 = number of rotor phases

R_2 = rotor resistance per phase + external resistance per phase

$\bar{E}_2$ = volts per rotor phase at standstill

$$\therefore \quad T\omega_s 9{\cdot}81 R_2^2 - m_2\bar{E}_2^2 s R_2 + T\omega_s 9{\cdot}81 s^2 L_2^2 \omega_0^2 = 0 \quad . \quad (7.80)$$

$$\therefore \quad R_2 = \frac{m_2\bar{E}_2^2 s \pm s\sqrt{m_2^2\bar{E}_2^4 - 386 T^2\omega_0^2\omega_s^2 L_2^2}}{19{\cdot}62 T\omega_s} \quad . \quad (7.81)$$

where $\omega_0 = 2\pi \times$ supply frequency

ω_s — synchronous angular velocity of the rotor in radians per second

$$\therefore \quad \frac{R_2}{s} = \frac{m_2\bar{E}_2^2 \pm \sqrt{m_2^2\bar{E}_2^4 - 386 T^2\omega_0^2\omega_s^2 L_2^2}}{19{\cdot}62 T\omega_s} \quad . \quad (7.82)$$

Since $$s = \frac{\omega_s - \omega}{\omega_s}$$

$$R_2 = \frac{\omega_s - \omega}{\omega_s}\left\{\frac{m_2\bar{E}_2^2 \pm \sqrt{m_2^2\bar{E}_2^4 - 386 T^2\omega_0^2\omega_s^2 L_2^2}}{19{\cdot}62 T\omega_s}\right\} \quad (7.83)$$

$\therefore$ R_2 in the pass

$$= \frac{-\omega_1 - \left(\frac{T_1 - T_3}{J}\right)(t - t_1) + \omega_s}{\omega_s}$$

$$\times \left\{\frac{m_2\bar{E}_2^2 \pm \sqrt{m_2^2\bar{E}_2^4 - 385 T^2\omega_0^2\omega_s^2 L_2^2}}{19{\cdot}62 T\omega_s}\right\} \quad (7.84)$$

Equation (7.84) gives the value of the total resistance per phase, required for constant torque, at any time during the pass.

Note. Since

$$\frac{p}{2} \times \text{revolutions per second (synchronous)} = \text{frequency} = f$$

$$\text{revolutions per second (synchronous)} = \frac{2f}{p}$$

where p = poles

$$\omega_s = 2\pi \times \text{synchronous revolutions per second} = \frac{4\pi f}{p}$$

$$\therefore \quad \omega_s = \frac{2\omega_0}{p} \quad . \quad . \quad . \quad . \quad . \quad (7.85)$$

It is clear the resistance should be directly proportional to the slip for constant torque.

Continuous Mill with Automatic Slip Regulator and Constant Output during the Pass

The power equation is—

$$T\omega - I\frac{d\omega}{dt}\omega = W \quad . \quad . \quad . \quad (7.86)$$

W = constant output during the pass.

$$\therefore \qquad \frac{I\omega d\omega}{T\omega - W} = dt \quad . \quad . \quad . \quad . \quad (7.87)$$

$$\frac{I}{W}\left\{\frac{W}{T}\left(\frac{1}{\frac{T}{W}\omega - 1} + 1\right)\right\}d\omega = dt \quad . \quad . \quad (7.88)$$

$$\therefore \qquad C + \frac{IW}{T^2}\log\left(\frac{T}{W}\omega - 1\right) + \frac{I}{T}\omega = t \quad . \quad . \quad (7.89)$$

To determine C, put $t = t_1$, then $\omega = \omega_1$, the speed at time t_1, when the regulator comes in—

$$C + \frac{IW}{T^2}\log\left(\frac{T}{W}\omega_1 - 1\right) + \frac{I}{T}\omega_1 = t_1 \quad . \quad . \quad (7.90)$$

$$\therefore \qquad C = -\frac{IW}{T^2}\log\left(\frac{T}{W}\omega_1 - 1\right) + t_1 - \frac{I}{T}\omega_1 \quad . \quad (7.91)$$

$$\therefore \qquad t = \frac{IW}{T^2}\log\frac{\left(\frac{T}{W}\omega - 1\right)}{\left(\frac{T}{W}\omega_1 - 1\right)} + \frac{I}{T}(\omega - \omega_1) + t_1 \quad . \quad (7.92)$$

$$\therefore \quad t = \frac{IW}{T^2}\log\left\{1 + \frac{\frac{T}{W}(\omega - \omega_1)}{\frac{T}{W}\omega_1 - 1}\right\} + \frac{I}{T}(\omega - \omega_1) + t_1 \quad (7.93)$$

Expanding the log, we have—

$$\log\left\{1 + \frac{\frac{T}{W}(\omega - \omega_1)}{\frac{T}{W}\omega_1 - 1}\right\}$$

$$= \frac{\frac{T}{W}(\omega - \omega_1)}{\frac{T}{W}\omega_1 - 1} - \frac{1}{2}\left\{\frac{\frac{T}{W}(\omega - \omega_1)}{\frac{T}{W}\omega_1 - 1}\right\}^2 + \ldots \text{ etc.}$$

The second and higher powers are negligibly small.

$$\therefore \qquad t \simeq \frac{IW}{T^2}\left\{\frac{\frac{T}{W}(\omega-\omega_1)}{\frac{T}{W}\omega_1 - 1}\right\} + \frac{I}{T}(\omega-\omega_1) + t_1 \qquad . \quad (7{\cdot}94)$$

and
$$\omega = \omega_1 + \left(\frac{T}{I} - \frac{W}{I\omega_1}\right)(t - t_1) \qquad . \quad . \quad (7{\cdot}95)$$

If, as is sometimes the case, $\frac{IW}{T^2}$ is fairly large, it may be necessary to take the second, or even the third power of the expansion of the log, and these will give a quadratic and a cubic equation in ω, which can be solved by the usual methods.

Example on Steel Tube Mill

This is an admirable case of the application of the automatic slip regulator, for the passes and intervals are relatively long.

The cycle consists of one pass of 45 sec and one interval of 75 sec.

The total torque required in the pass = 26 000 lb-ft.

The friction torque = 2100 lb-ft.

The speed of the mill is 150 r.p.m.

The motor is geared to the mill shaft, and the reduction ratio is 3·23.

The total torque-time in lb-ft/sec = 26 000 × 45 + 2100 × 75
$= 1{\cdot}327 \times 10^6$

The average torque in the cycle $= \frac{1{\cdot}327 \times 10^6}{120}$
$= 11{\cdot}05 \times 10^3$ lb-ft

To find the size of the flywheel, we proceed as follows—

Let I = moment of inertia of the flywheel in lb-ft^2

ω_1 = angular velocity, in radians per second, at the beginning of the pass

ω_2 = angular velocity, in radians per second, at the end of the pass

The energy given out, by the wheel, in the pass

$$= \tfrac{1}{2}I(\omega_1^2 - \omega_2^2)$$

where $\omega_1 = 51{\cdot}8$ radn/sec = 495 r.p.m.

$$\omega_2 = 2\pi \times \frac{450}{60} = 47{\cdot}1 \text{ radn/sec}$$

The torque given out by the flywheel in the pass

$$= \frac{26\,000 - 11\,050}{3{\cdot}23} = 4640 \text{ lb-ft}$$

$$\therefore \qquad \tfrac{1}{2}I(51{\cdot}8^2 - 47{\cdot}1^2) = 4640 \times 2\pi \times \frac{475}{60} \times 45$$

$$\therefore \qquad I = 45\,000 \text{ lb-ft}^2$$

Average horse-power of the motor

$$= \frac{11\,050 \times 2\pi \times 474}{3{\cdot}23 \times 33\,000 \times 0{\cdot}95} = 326 \text{ h.p.}$$

This, of course, will not be the rating of the motor. The motor will be required to be put in for the r.m.s. brake-horse-power over the cycle. We have seen that the speed line is a straight line, after the regulator comes in. Before its introduction, however, the motor is working with permanent resistance in its circuit, viz. its own rotor resistance. Applying our equations, we have

$$\omega = \frac{c}{a} + \left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}$$

$$\frac{c}{a} = \omega_0 - \frac{T_3}{a}$$

$$\omega_0 = 500 \times \frac{2\pi}{60} = 52{\cdot}4 \text{ radn/sec}$$

$$T_3 = \frac{26\,000}{3{\cdot}23} = 8050$$

$$a = \frac{T_f}{\omega_0 - \omega_f} = \frac{11\,050 \times 60}{3{\cdot}23 \times 2\pi \times 15} = 2180$$

Natural slip at full load = 3 per cent.

$$\frac{c}{a} = 52{\cdot}4 - \frac{8050}{2180} = 52{\cdot}4 - 3{\cdot}69 = 48{\cdot}71$$

$$\frac{a}{I} = \frac{2180}{45\,000} = 0{\cdot}0485$$

The time to reach full-load torque is given by the equation—

$$50{\cdot}85 = 48{\cdot}71 + 3{\cdot}09e^{-\frac{a}{I}t}$$

$$\therefore \qquad t = 7{\cdot}6 \text{ sec}$$

for

$$50{\cdot}85 = 48{\cdot}71 + 3{\cdot}09e^{-0{\cdot}0485t}$$

The torque equation, for the beginning of the pass up to the point when the regulator comes in is—

$$T = T_3 - a\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t}$$

$$= 8050 - 2180(51{\cdot}8 - 48{\cdot}71)e^{-0\ 0485t}$$

$$= 8050 - 6745e^{-0\ 0485t}$$

The results, for the pass, are tabulated below—

Time in Seconds	ω	R p m	$e^{-0\ 0485t}$	Torque lb-ft	H p of Motor
0	51 80	495 0	1 000	1305	122 7
2 0	51 50	492 0	0 905	1950	182 5
4 0	51 25	490 0	0 822	2520	235 1
6 0	51 01	487 5	0 746	3025	281 1
7 5	50 84	485 0	0 692	3390	311 0
Reg comes in					
45	47 91	457 0	—	3390	295 0

When the regulator is in action, the speed falls in a linear fashion and

$$\omega = \omega_1 + \frac{T - T_3}{I}(t - t_1)$$

therefore, at the end of the pass

$$\omega = 51{\cdot}8 + \frac{3390 - 8050}{45\ 000} \times 37\ 5 = 47\ 91$$

In the interval, the speed rises in a straight line, the torque remaining constant, till the natural torque-slip curve of the motor is

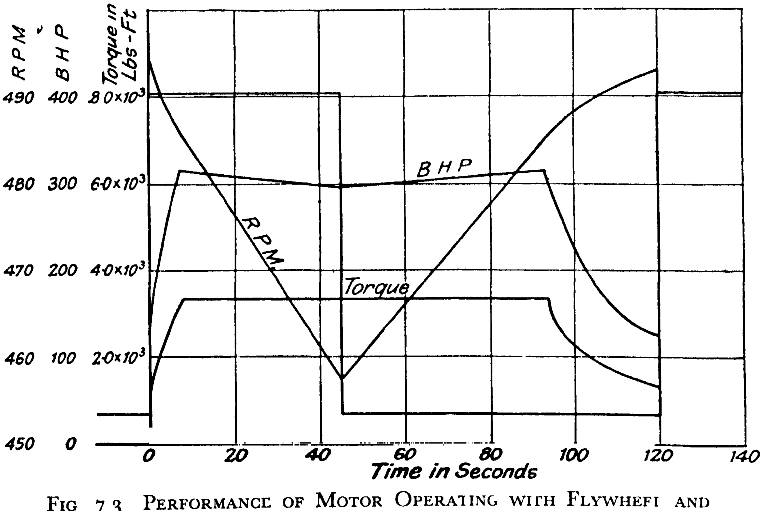


FIG 7 3 PERFORMANCE OF MOTOR OPERATING WITH FLYWHEEL AND AUTOMATIC SLIP REGULATOR reached. This is seen in Fig. 7.3. Any further increase in speed, beyond that corresponding to the natural slip of the motor, results in a reduction in torque and our speed and torque equations, from this point, are simply those for permanent resistance, i.e. the resistance of the rotor windings, viz. equation (7.28) and (7.43).

In the interval, the speed rises according to the equation

$$\omega = \omega_2 + \frac{T - T_0}{I}(t - t_2)$$

$$\therefore \qquad \frac{d\omega}{dt} = \frac{T - T_0}{I} = \frac{3390 - 650}{45\,000} = 0{\cdot}061$$

The speed of the motor, at full-load torque,

$$= 2\pi \times \frac{485}{60} = 50{\cdot}84 \text{ radn/sec}$$

Increase in speed from the *end* of the pass

$$= 50{\cdot}84 - 47{\cdot}91 = 2{\cdot}93 \text{ radn/sec}$$

Therefore, time to reach the natural torque-slip curve

$$= \frac{2{\cdot}93}{0{\cdot}061} = 48 \text{ sec}$$

The remaining 27 sec is taken up on the permanent resistance of the rotor.

The equation

$$\omega = \frac{f}{a} + \left(\omega_2 - \frac{f}{a}\right)e^{\frac{a}{I}(t_1 - t)}$$

applies to this period.

$$\frac{f}{a} = \omega_0 - \frac{T_0}{a} = 52{\cdot}25 - \frac{650}{2180} = 51{\cdot}954$$

$$\omega_2 = 50{\cdot}84$$

$$\omega_2 - \frac{f}{a} = -1{\cdot}114$$

Tabulating the results for the 27-sec interval, we have—

Time *from* 48 *sec* in interval	ω	R.p.m.	$e^{-0{\cdot}0485t}$	Torque	H.p.
2	50·950	486·0	0·905	2850	263·5
4	51·024	488·0	0·822	2650	264·0
6	51·117	489·0	0·746	2460	229·0
10	51·270	490·0	0·610	2130	198·5
14	51·380	490·3	0·510	1890	176·5
17	51·465	491·0	0·433	1700	158·5
27	51·648	493·0	0·275	1318	124·0

The corresponding torque in this period of 27 sec is

$$T = T_0 - a\left(\omega_2 - \frac{f}{a}\right) e^{\frac{a}{I}(t_2 - t)}$$

$$= 650 - 2180(-1{\cdot}114)e^{\frac{a}{I}(t_2 - t)}$$

$$= 650 + 2430e^{-0{\cdot}0485(t_2 - t)}$$

These results are plotted in Fig. 7.3.

R.M.S. H.P.

Let t_1 = time in pass, until regulator comes into action

t_2 = time till the end of the pass, from beginning of pass

t_3 = time in interval that the motor is on the regulator

t_4 = time till end of cycle

The power, in the period t_1,

$$= \left\{\frac{c}{a} + \left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t}\right\}\left\{T_3 - a\left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t}\right\} = W$$

$$= T_3\frac{c}{a} + T_3\left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t} - c\left(\omega_1 - \frac{c}{a}\right) e^{\frac{a}{I}t}$$

$$- a\left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t} = W \quad . \quad . \quad . \quad . \quad . \quad (7{\cdot}96)$$

$$W^2 = T_3^2\frac{c^2}{a^2} + 2T_3^2\frac{c}{a}\left(\omega_1 - \frac{c}{a}\right)e^{-\frac{a}{I}t} - 2T_3\frac{c^2}{a}\left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t}$$

$$- 2T_3c\left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t} + T_3^2\left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t}$$

$$- 2T_3a\left(\omega_1 - \frac{c}{a}\right)^3 e^{-\frac{3a}{I}t} - T_3c\left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t}$$

$$+ 2ac\left(\omega_1 - \frac{c}{a}\right)^3 e^{-\frac{3a}{I}t} + c^2\left(\omega_1 - \frac{c}{a}\right)^2 e^{\frac{2a}{I}t}$$

$$+ a^2\left(\omega_1 - \frac{c}{a}\right)^4 e^{-\frac{4a}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7{\cdot}97)$$

$$\begin{aligned}\int_0^{t_1} W^2 dt = {} & T_3^2 \frac{c^2}{a^2} t_1 - 2T_3^2 \frac{c}{a^2} I \left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t_1} \\ & + 2T_3^2 \frac{c}{a^2} I \left(\omega_1 - \frac{c}{a}\right) + 2T_3 \frac{c^2}{a^2} I \left(\omega_1 - \frac{c}{a}\right) e^{-\frac{a}{I}t_1} \\ & - 2T_3 \frac{c^2}{a^2} I \left(\omega_1 - \frac{c}{a}\right) + \frac{T_3 cI}{a} \left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t_1} \\ & - \frac{T_3 cI}{a} \left(\omega_1 - \frac{c}{a}\right)^2 - \frac{T_3^2 I}{2a} \left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t_1} \\ & + \frac{T_3^2 I}{2a} \left(\omega_1 - \frac{c}{a}\right)^2 + \frac{2}{3} T_3 I \left(\omega_1 - \frac{c}{a}\right)^3 e^{-\frac{3a}{I}t} \\ & - \frac{2}{3} T_3 I \left(\omega_1 - \frac{c}{a}\right)^3 + \frac{T_3 cI}{2a} \left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t_1} \\ & - \frac{T_3 cI}{2a} \left(\omega_1 - \frac{c}{a}\right)^2 - \frac{2}{3} Ic \left(\omega_1 - \frac{c}{a}\right)^3 e^{-\frac{3a}{I}t_1} \\ & + \frac{2}{3} Ic \left(\omega_1 - \frac{c}{a}\right)^3 - \frac{Ic^2}{2a} \left(\omega_1 - \frac{c}{a}\right)^2 e^{-\frac{2a}{I}t_1} \\ & + \frac{Ic^2}{2a} \left(\omega_1 - \frac{c}{a}\right)^2 - \frac{aI}{4} \left(\omega_1 - \frac{c}{a}\right)^4 e^{\frac{4a}{I}t_1} + \frac{aI}{4} \left(\omega_1 - \frac{c}{a}\right)^4 \end{aligned} \qquad (7.98)$$

The power in time

$$t_2 - t_1 = T\omega_1 + T\left(\frac{T - T_3}{I}\right)(t - t_1) = W_2 \qquad (7.99)$$

$$= T\omega_1 + \frac{T^2 - TT_3}{I}(t - t_1) = W_2 \qquad (7.100)$$

$$\begin{aligned} W_2^2 = {} & T^2\omega_1^2 + 2T\left(\frac{T^2 - TT_3}{I}\right)\omega_1(t - t_1) \\ & + \left(\frac{T^2 - TT_3}{I}\right)^2 \{t^2 - 2tt_1 + t_1^2\} \end{aligned} \qquad (7.101)$$

$$\int_{t_1}^{t_2} W_2^2 dt = T^2\omega_1^2(t_2 - t_1) + \omega_1 2T\left(\frac{T^2 - T_1T_3}{I}\right)\left[\tfrac{1}{2}t^2 - t_1 t\right]_{t_1}^{t_2}$$

$$+\left(\frac{T^2 - TT_3}{I}\right)^2\left\{\frac{t^3}{3} - t^2t_1 + t_1^2 t\right\}_{t_1}^{t_2} \quad . \quad (7.102)$$

$$= T^2\omega_1^2(t_2 - t_1) + 2T\omega_1\left(\frac{T^2 - TT_3}{I}\right)\left[\tfrac{1}{2}t_2^2 + \tfrac{1}{2}t_1^2 - t_1t_2\right]$$

$$+\left(\frac{T^2 - TT_3}{I}\right)^2\left[\frac{t_2^3}{3} - \frac{t_1^3}{3} - t_2^2t_1 + t_1^2t_2\right] \quad . \quad (7.103)$$

A corresponding expression for the interval $t_3 - t_2$ as this applies, and a similar expression to the first for the last part of the cycle. The speed in the third part of the cycle

$$\omega = \omega_2 + \left(\frac{T - T_0}{I}\right)(t - t_2) \quad . \quad . \quad (7.104)$$

and

$$W_3 = T\omega = T\left[\omega_2 + \left(\frac{T - T_0}{I}\right)(t - t_2)\right] \quad . \quad (7.105)$$

and

$$\int_{t_2}^{t_3} W_3^2 dt = T^2\omega_2^2(t_3 - t_2) + 2T^2\left(\frac{T - T_0}{I}\right)\omega_2\left[\tfrac{1}{2}t_3^2 + \tfrac{1}{2}t_2^2 - t_3t_2\right]$$

$$+\left(\frac{T^2 - TT_0}{I}\right)^2\left[\frac{t_3^3}{3} - \frac{t_2^3}{3} - t_3^2t_2 + t_2^2t_3\right] \quad . \quad (7.106)$$

Again for the last period

$$\omega = \frac{f}{a} + \left(\omega_2 - \frac{f}{a}\right)e^{-\frac{a}{I}t} \quad . \quad . \quad . \quad (7.107)$$

and the torque $= T_0 - a\left(\omega_2 - \dfrac{f}{a}\right)e^{-\frac{a}{I}t}$

and $W_4 = T\omega$

$$\int_{t_3}^{t_4} W_4^2 dt = \int_{t_3}^{t_4}\left[\left\{T_0 - a\left(\omega_2 - \frac{f}{a}\right)e^{-\frac{a}{I}t}\right\}\left\{\frac{f}{a} + \left(\omega_2 - \frac{f}{a}\right)e^{-\frac{a}{I}t}\right\}\right]^2 dt$$

$$. \quad . \quad (7.108)$$

It will be noticed that this is the same product, in form, as for the first period, and one should substitute, in the expression for the first period, T_0 for T_3, $\frac{f}{a}$ for $\frac{c}{a}$, and ω_2 for ω_1.

The r.m.s. power

$$= \sqrt{\frac{\int_0^{t_1} W_1^2 dt + \int_{t_1}^{t_2} W_2^2 dt + \int_{t_2}^{t_3} W_3^2 dt + \int_{t_3}^{t_4} W_4^2 dt}{t_4}} \quad . \quad (7.109)$$

It is simpler, in any given case, to square the ordinates of power at each point, and find the area with the planimeter. Then find the mean of the squared ordinates and take the square root.

The Ward-Leonard System

Consider next the case of the electric winder, operating on the Ward-Leonard system. The induction motor is provided with a flywheel on the shaft, and an automatic slip-regulator in the rotor circuits.

Fig. 7.4 gives the winding diagram for an electric hoist.

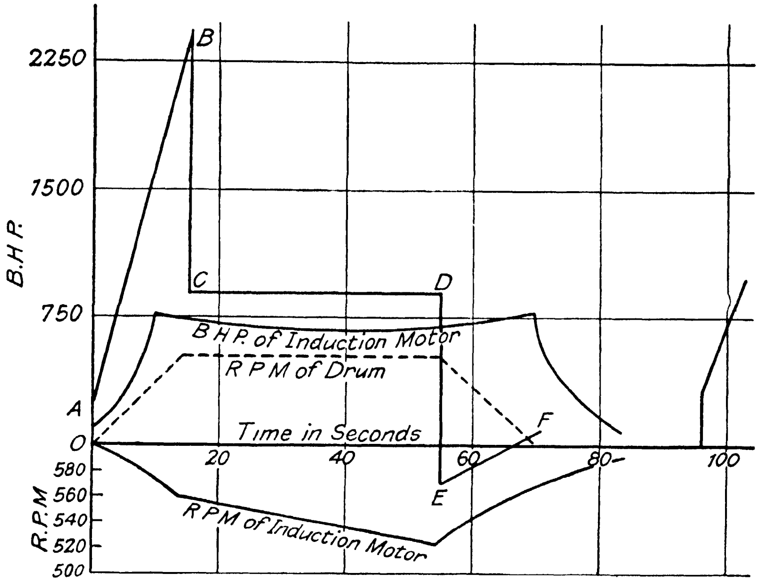


FIG. 7.4

Let W = output at B

t_1 = period of acceleration in seconds

Then
$$T\omega - I\omega\frac{d\omega}{dt} = W \times \frac{t}{t_1} \quad . \quad . \quad . \quad (7.110)$$

and
$$\frac{T}{T_f} = \frac{\omega_0 - \omega}{\omega_0 - \omega_f}; \quad \therefore \quad T = \frac{T_f\omega_0}{\omega_0 - \omega_f} - \frac{T_f\omega}{\omega_0 - \omega_f} \quad . \quad (7.111)$$

$$\therefore \qquad a_1\omega - a_2\omega^2 - \omega\frac{d\omega}{dt} = c_1 t \quad . \quad . \quad . \quad (7.112)$$

where $c_1 = \frac{W}{It_1}$

$$a_1 = \frac{T_f\omega_0}{(\omega_0 - \omega_f)I}; \quad a_2 = \frac{T_f}{(\omega_0 - \omega_f)I} \quad . \quad . \quad (7.113)$$

The solution of equation (7.112) may be obtained, approximately, thus—

Let $\omega = a$ when $t = 0$, and let $\omega = ue^{-a_2t}$, so that $u = a$ when $t = 0$.

$$\frac{d\omega}{dt} = \frac{du}{dt}e^{-a_2t} - a_2ue^{-a_2t} \quad . \quad . \quad . \quad (7.114)$$

$$\therefore \quad \omega\frac{d\omega}{dt} = u\frac{du}{dt}e^{-2a_2t} - a_2u^2e^{-2a_2t} \quad . \quad . \quad (7.115)$$

$$\therefore \quad a_1ue^{-a_2t} - a_2u^2e^{-2a_2t} - u\frac{du}{dt}e^{-2a_2t} + a_2u^2e^{-2a_2t} = c_1t \quad . \quad (7.116)$$

$$\therefore \quad a_1ue^{-a_2t} - u\frac{du}{dt}e^{-2a_2t} = c_1t \quad . \quad . \quad . \quad (7.117)$$

$$\therefore \quad a_1ue^{-a_2t} - c_1t = u\frac{du}{dt}e^{\;2a_2t} \quad . \quad . \quad . \quad (7.118)$$

and
$$\frac{du}{dt} = a_1e^{a_2t} - \frac{c_1}{u}e^{2a_2t} \times t \quad . \quad . \quad . \quad (7.119)$$

Expand u in a series of Maclaurin

$$u = f(t) = f(o) + tf'(o) + \frac{t^2}{|\underline{2}}f''(o) + \dots \quad . \quad (7.120)$$

$$f(o) = a$$

$$f'(t) = \frac{du}{dt} = a_1e^{a_2t} - \frac{c_1}{u}e^{2a_2t} \times t \quad . \quad . \quad (7.121)$$

$$\therefore \quad f'(o) = a_1$$

$$f''(t) = a_1a_2e^{a_2t} - \frac{c_1}{u}e^{2a_2t} - \frac{2a_2c_1}{u}e^{2a_2t} \times t + \frac{c_1te^{2a_2t}}{u^2}\left(a_1e^{a_2t} - \frac{c_1}{u}e^{2a_2t}\right) \quad . \quad . \quad (7.122)$$

$$f''(o) = a_1a_2 - \frac{c_1}{a} \quad . \quad . \quad . \quad (7.123)$$

$$\therefore \quad \omega = ue^{-a_2t} \quad . \quad . \quad . \quad . \quad (7.124)$$

$$= e^{-a_2t}\left\{a + a_1t - \tfrac{1}{2}\left(\frac{c_1}{a} - a_1a_2\right)t^2 + \dots\right\} \quad . \quad (7.125)$$

This expression represents the manner in which the speed of the induction motor falls, *during the accelerating stage of the winding cycle of the d.c. motor.*

We will find the speed and output of the induction motor during the remainder of the cycle.

The power equation is

$$T\omega - I\frac{d\omega}{dt}\omega = Ct \quad . \quad . \quad . \quad (7.126)$$

where C = rate of increase of power with time on the accelerating part of the curve (i.e. the part AB in Fig. 7.4.).

In this part of the cycle, the torque of the motor is constant

$$\therefore \qquad \omega\frac{d\omega}{dt} - \frac{T\omega}{I} + \frac{C}{I}t = 0 \qquad (7.127)$$

Let $\omega = vt$.

$$\frac{d\omega}{dt} = t\frac{dv}{dt} + v$$

then equation (7.127) becomes

$$vt^2\frac{dv}{dt} + v^2t - \frac{T}{I}vt + \frac{C}{I}t = 0 \qquad (7.128)$$

$$vt^2\frac{dv}{dt} + t\left(v^2 - \frac{T}{I}v + \frac{C}{I}\right) = 0$$

i.e.

$$-\frac{vdv}{v^2 - \frac{T}{I}v + \frac{C}{I}} = \frac{dt}{t} \qquad (7.129)$$

Let

$$\frac{T}{I} = b \text{ and } \frac{C}{I} = d \qquad (7.130)$$

$$\int\frac{dt}{t} = -\int\frac{vdv}{v^2 - \frac{T}{I}v + \frac{C}{I}} = -\int\frac{vdv}{v^2 - bv + d} \qquad (7.131)$$

$$= -\frac{1}{2}\int\frac{(2v - b)dv}{v^2 - bv + d} - \frac{b}{2}\int\frac{dv}{v^2 - bv + d} \qquad (7.132)$$

$$\therefore \log_e t = -\frac{1}{2}\log(v^2 - bv + d) - \frac{b}{2}\int\frac{dv}{v^2 - bv + d} + K \qquad (7.133)$$

The latter integral has different forms, depending on whether $b^2 \begin{smallmatrix}<\\=\\>\end{smallmatrix} 4d$, i.e. $\frac{T^2}{I^2} \begin{smallmatrix}>\\<\end{smallmatrix} \frac{4C}{I}$

Obviously $b^2 > 4d$, for

$$C = \frac{T\omega}{t} - \frac{I}{t}\frac{d\omega}{dt}\omega \qquad (7.134)$$

$$\frac{C}{I} = \frac{T}{I}\frac{\omega}{t} - \frac{\omega}{t}\frac{d\omega}{dt} \qquad (7.135)$$

$$\frac{T^2}{I^2} \begin{smallmatrix}>\\=\\<\end{smallmatrix} \frac{4T}{I}\frac{\omega}{t} - \frac{4\omega d\omega}{tdt} \qquad (7.136)$$

but $I\frac{d\omega}{dt}$ usually $> T$

$$\therefore \quad \frac{d\omega}{dt} > \frac{T}{I} = m\frac{T}{I} \qquad (7.137)$$

$$\therefore \quad \frac{T^2}{I^2} + 4m\frac{T\omega}{It} > \frac{4T\omega}{It} \qquad (7.138)$$

$$\therefore \quad \frac{T^2}{I^2} > \frac{4C}{I} \qquad (7.139)$$

$$\therefore \quad \int \frac{dv}{v^2 - bv + d} = \int \frac{dv}{(v - \frac{1}{2}b)^2 - \left(\frac{b^2}{4} - d\right)} \qquad (7.140)$$

$$= -\frac{1}{\sqrt{\frac{b^2}{4} - d}} \coth^{-1} \frac{(v - \frac{1}{2}b)}{\sqrt{\frac{b^2}{4} - d}} \qquad (7.141)$$

$$\therefore \log_e t = -\tfrac{1}{2}\log(v^2 - bv + d) + \frac{b}{2\sqrt{\frac{b^2}{4} - d}} \coth^{-1} \frac{(v - \frac{1}{2}b)}{\sqrt{\frac{1}{4}b^2 - d}} + K \qquad (7.142)$$

When $t = t_1$, the time when the regulator comes in, $\omega = \omega_1$,

$$\therefore \quad K = \log_e t_1 + \tfrac{1}{2}\log\left(\frac{\omega_1^2}{t_1^2} - \frac{b\omega_1}{t_1} + d\right) - \frac{b}{2\sqrt{\frac{b^2}{4} - d}} \coth^{-1} \frac{\left(\frac{\omega_1}{t_1} - \frac{b}{2}\right)}{\sqrt{\frac{1}{4}b^2 - d}} \qquad (7.143)$$

$$\therefore \quad \log_e \frac{t}{t_1} = \frac{1}{2}\log \frac{\left(\frac{\omega_1^2}{t_1^2} - \frac{b\omega_1}{t_1} + d\right)}{\left(\frac{\omega^2}{t^2} - \frac{b\omega}{t} + d\right)} + \frac{b}{2\sqrt{\frac{b^2}{4} - d}} \left[\coth^{-1} \frac{\left(\frac{\omega}{t} - \frac{b}{2}\right)}{\sqrt{\frac{b^2}{4} - d}} - \coth^{-1} \frac{\left(\frac{\omega_1}{t_1} - \frac{b}{2}\right)}{\sqrt{\frac{b^2}{4} - d}}\right] \qquad (7.144)$$

We have, approximately (neglecting the second term)

$$\omega = \frac{\frac{T}{I}t \pm t\sqrt{\frac{T^2}{I^2} - 4\left\{\left(\frac{C}{I} - \frac{\omega_1^2}{t^2} + \frac{T}{I}\frac{\omega_1 t_1}{t^2} + \frac{t_1^2}{t^2}\frac{C}{I}\right)\right\}}}{2} \qquad (7.145)$$

This equation holds up to the peak. The output is constant from C to D and our equation becomes

$$T\omega - I\frac{d\omega}{dt}\omega = E \quad . \quad . \quad . \quad (7.146)$$

where E = constant output

This equation is similar to equation (7.86), and is solved in the same way.

Speed Control by Pole-changing

Since the revolutions per minute $= \dfrac{120 \times \text{frequency}}{\text{number of poles}}$, it is clear that speed change can be obtained by changing the number of poles. This may be effected by a regrouping of the coils of a single winding, or by using two or more independent windings. For speed ranges of 2 : 1, this type of motor, using a single stator winding and a squirrel-cage rotor, is commonly used. There are a number of different ways of connecting the windings of such motors; the selection of the connection, in any case, depending on the relative maximum outputs required at the two speeds. The connection most frequently used, and in which the material is used to the best advantage, has a half-speed rating of from 60 to 70 per cent of the full-speed rating. The following table gives several different connections, which may be used, with the approximate relative outputs for the different connections—

	Speed	Connection	Approximate Maximum Output
(1)	100	2 circuit Δ	100
	50	Y-Δ	11
(2)	100	2 circuit Y	100
	50	1 circuit Y	22
(3)	100	2 circuit Y	100
	50	1 circuit Δ	66
(4)	100	1 circuit Δ	100
	50	2 circuit Y	117
(5)	100	1 circuit Y	100
	50	2 circuit Y	350
(6)	100	Y-Δ	100
	50	2 circuit Δ	700

The values are approximate only and will vary with the ratio of reactance to resistance, and also with the ratio of end connection reactance to the slot reactance. The method of pole changing is very simple. It is to reverse the current in half the winding for the smaller number of poles. The following diagram shows how it is done for four poles and two poles.

Fig. 7.5 shows a winding, for one phase, arranged for four poles. Fig. 7.6 shows the regrouping effected for two poles.

For the smaller number of poles it will be seen that one terminal is connected to the mid-point tapping *A*, and the other terminal of the winding is connected to the lead which joins the beginning and end of the winding together. If we wish to keep the same number of coils in series, in each case, the winding may be arranged as in Fig. 7.7.

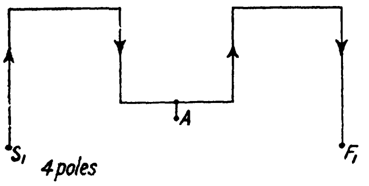


FIG. 7.5

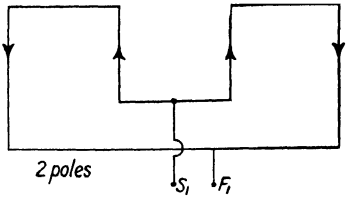


FIG. 7.6

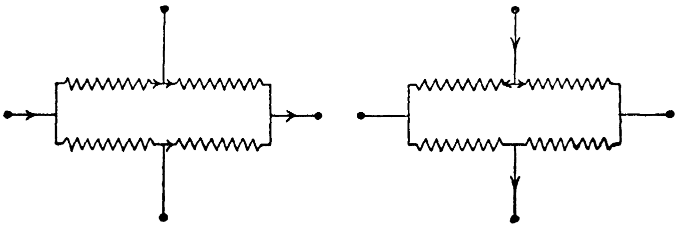
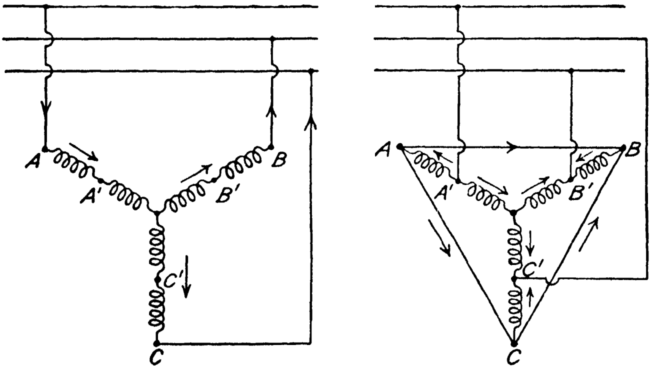


FIG. 7.7



(*a*) *8 poles* (*b*) *4 poles*

FIG. 7.8

In Fig. 7.8 (*a*) and (*b*) the winding is connected in star for the larger number of poles and in two parallel star for the smaller number of poles. In the case of a single-layer hemi-tropic winding for eight poles, each phase will consist of four groups of coils, two groups straight and two groups bent, straight and bent coils alternating. Alternate coils are connected in series, i.e. two straight coils are connected in series and two bent coils are connected in series. For the eight-pole connection, the circuit is completed in series through the two bent coils of the same phase and a tapping is brought out at the junction of the straight and bent coils. A similar method is adopted in regard to the other phases. This method of connection is necessary in order that the direction of current in alternate groups of coils may be reversed when the two halves of each phase are connected in parallel.

In order that the direction of rotation shall remain unchanged when the poles are changed, two of the phases must be reversed in relation to the line wires, with this connection. The larger number of poles is obtained by connecting the line wires to the ends of the winding; while the smaller number is obtained by connecting the lines to the tappings, and short-circuiting the ends of the winding.

The development of a three-phase winding for pole-changing, according to Fig. 7.8 (*a*) and (*b*), is shown in Fig. 7.9.

It will be seen from Fig. 7.9 that, with the smaller number of poles, certain conductors, at the centre of each pole, carry currents in the wrong direction. A part of the winding is ineffective. For this class of winding one-third of the turns per pole are ineffective.

Let T = turns per phase for the larger number of poles

p = poles

For the smaller number of poles, i.e. $\frac{p}{2}$, the effective turns per phase will be

$$\frac{1}{2}\left(1\tfrac{2}{3}\frac{T}{p}\times\frac{p}{2}\right)=\frac{5}{12}T$$

Ratio of Fluxes and Magnetizing Currents

Let E = counter e.m.f. per phase $\simeq V$

$\hat{\phi}$ = maximum flux per pole for larger number of poles

$\hat{\phi}'$ = maximum flux per pole for smaller number of poles

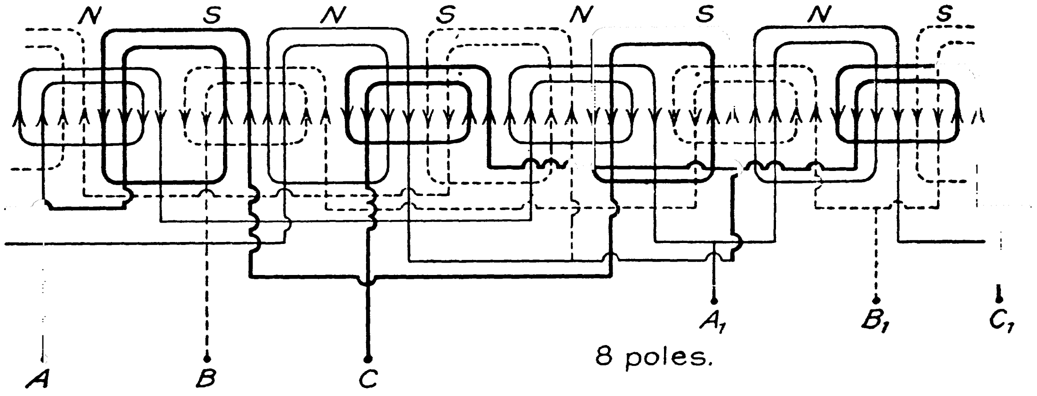
f = frequency

K_1 = breadth factor for p poles

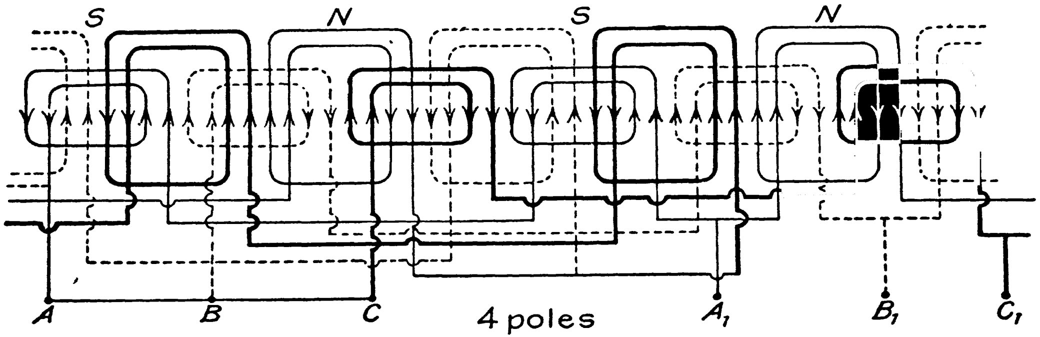
K_2 = breadth factor for $\frac{p}{2}$ poles

K_3 = coil span factor for p poles

K_4 = coil span factor for $\frac{p}{2}$ poles



8 poles.



4 poles

FIG. 7.9

Then for p poles

$$\hat{E}_1 = \sqrt{2}\pi \times K_1 \times K_3 \times \hat{\phi} \times T \times f \times 10^{-8}$$

for $\frac{p}{2}$ poles

$$\hat{E}_2 = \sqrt{2}\pi \times K_2 \times K_4 \times \hat{\phi}' \times \frac{5}{12} T \times f \times 10^{-8} \quad . \quad (7.147)$$

Let ψ = span of the coil, in electrical degrees; for the smaller number of poles $\psi = 90°$, corresponding to half-pitch

$$K_4 = \sin\frac{\psi}{2} = \sin 45° = 0{\cdot}707 \quad . \quad . \quad . \quad . \quad (7.148)$$

and since

$$\hat{E}_1 \simeq \hat{E}_2 \simeq V \quad . \quad . \quad . \quad (7.149)$$

$$\frac{\hat{\phi}'}{\hat{\phi}} = \frac{5}{12} \times 0{\cdot}707 = 3{\cdot}4 \quad . \quad . \quad . \quad (7.150)$$

The area of the pole face is doubled, so the flux density in the gap, with this connection, is 1·7 times that for the larger number of poles.

If one neglects saturation, the magnetizing current is proportional to the gap density, and inversely proportional to the turns per pole per phase.

$$\frac{i_m}{i_m'} = \frac{B_m}{\frac{T}{p}} \div \frac{B_m'}{0{\cdot}707 \times \frac{5}{12}\frac{T}{p} \times 2} \quad . \quad . \quad . \quad (7.151)$$

and

$$\frac{B_m}{B_m'} = \frac{1}{1{\cdot}7}$$

$$\therefore \quad \frac{i_m}{i_m'} = \frac{1}{1{\cdot}7} \times \frac{2 \times 5 \times 0{\cdot}707}{12} = 0{\cdot}347 \quad . \quad . \quad (7.152)$$

$$\therefore \quad i_m' = 2{\cdot}88 i_m \quad . \quad . \quad . \quad (7.153)$$

Careful attention must be paid with this method of connection, viz. star for larger number of poles and two parallel star for the smaller number, to the areas of core and teeth, if saturation is to be avoided on the smaller number of poles. This method of connection is *not* a good one and it is preferable to use—

(1) Delta connection for the larger number of poles.

(2) *Two parallel circuits per phase* in *star* for the smaller number of poles.

With (1) delta connection for p poles and (2) two parallel star for $\frac{p}{2}$ poles.

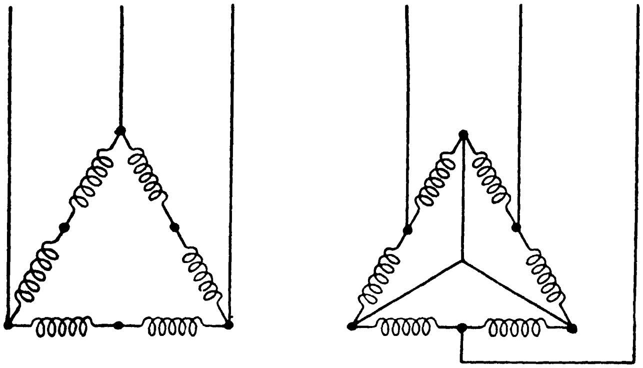
For the larger number of poles

(3) $$V \doteqdot E_1 = \sqrt{2}\pi\hat{\phi} \times T \times f \times K_1 \times K_3 \times 10^{-8} \quad . \quad (7.154)$$

(4) For $\frac{p}{2}$ poles

$$\frac{E_2}{\sqrt{3}} \doteqdot \frac{V}{\sqrt{3}} = \sqrt{2}\pi\hat{\phi}' \times \frac{5}{12} T \times 0{\cdot}707 \times f \times 10^{-8} \quad (7.155)$$

$$\therefore \qquad \frac{\hat{\phi}'}{\hat{\phi}} = \frac{12}{5 \times 0{\cdot}707 \times \sqrt{3}} = 1{\cdot}96 \quad . \quad . \quad (7.156)$$



8 poles
(a)

4 poles
(b)

FIG. 7.10

Since the pole area is doubled,

$$B_m' = 0{\cdot}98B$$

and $$\frac{i_m}{i_m'} = \frac{2 \times 0{\cdot}707 \times \frac{5}{12}}{0{\cdot}98} = 0{\cdot}6 \quad . \quad . \quad (7.157)$$

$$\therefore \qquad i_m' = 1{\cdot}66 i_m$$

This method of connection is preferable, since the performance is better and the motor can be made smaller in external diameter, and lighter than one connected as in Fig. 7.8. It may also be mentioned that, with the machine delta-connected for the larger number of poles and star-connected for the smaller number of poles, no reversal of line wires and phases is necessary. This connection is shown in Fig. 7.10 (*a*) and (*b*).

The wave form of m.m.f., on the smaller number of poles, is very irregular, giving rise to harmonics of large magnitude, especially when the coil pitch is chosen as full pitch for the larger number of poles. It will be shown, in the section on the revolving field, that in three-phase machines the fifth harmonic, the eleventh, etc., travel in the opposite direction to the fundamental, while the seventh, thirteenth, etc., revolve in the same direction as the fundamental. These harmonic fluxes produce torques, by interaction with the currents in the rotor of corresponding frequency, and they modify, profoundly, the resultant torque-slip curve of the motor. The torque-slip curve is no longer smooth, but contains saddle-backs, and it may be that the motor may fail to accelerate to full speed. Indeed, the motor may run steadily at a sub-synchronous speed, and get very hot in the process. Furthermore, noise and vibration may be caused and, of course, excessive iron and eddy-current losses.

It should also be mentioned that, where separate windings are provided for the different pole numbers, the size of the motor is increased appreciably and, of course, the power factor falls greatly on the slow speeds.

For large speed-ranges, the control arrangements for changing the coil groups become complicated.

The problem of multi-speed motor design is so to arrange the windings that the change of connections, in changing from one number of poles to another, involves the least number of switching arrangements.

Speed-control by Cascade Connection

Assume that we have two induction motors, mechanically coupled together, and let the rotor winding of the first be connected to the stator winding of the second motor, and the rotor winding of the second either short-circuited or closed through external resistance. If the fields rotate in the same direction in both machines, the set will run at a speed corresponding to that of a motor having a number of poles equal to the sum of the numbers of poles in the two machines.

Let p_1 = number of poles in the first machine
p_2 = number of poles in the second machine
f = supply frequency

The synchronous speed of the first machine, in revolutions per second $= \dfrac{2f}{p_1}$

The synchronous speed of the second machine, when supplied at full frequency $= \dfrac{2f}{p_2}$ revolutions per second.

If s = slip of the first machine, then the speed of machine No. 1 is $(1 - s)\dfrac{2f}{p_1}$ revolutions per second.

The frequency of the rotor currents of machine No. 1 is sf, and this is the frequency of the stator currents of machine No. 2.

Let s' = the slip of motor No. 2

$$= \frac{\frac{2sf}{p_2} - (1 - s)\frac{2f}{p_1}}{\frac{2sf}{p_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.158)$$

$$s' = \frac{\frac{s}{p_2} - \frac{1 - s}{p_1}}{\frac{s}{p_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.159)$$

The slip of the second motor, $s' = 0$,

when
$$\frac{s}{p_2} - \frac{1 - s}{p_1} = 0 \quad . \quad . \quad . \quad (7.160)$$

$$\therefore \quad \frac{1 - s}{s} = \frac{p_1}{p_2} \quad . \quad . \quad . \quad . \quad (7.161)$$

i.e.
$$\frac{1}{s} - 1 = \frac{p_1}{p_2} \quad . \quad . \quad . \quad . \quad (7.162)$$

i.e.
$$\frac{1}{s} = \frac{p_1 + p_2}{p_2} \quad . \quad . \quad . \quad . \quad (7.163)$$

$$\therefore \quad s = \frac{p_2}{p_1 + p_2} \quad . \quad . \quad . \quad . \quad (7.164)$$

The speed of the set, when $s' = 0$, in revolutions per second, i.e. the *cascade synchronous speed,*

$$= (1 - s)\frac{2f}{p_1}$$

$$= \left(1 - \frac{p_2}{p_1 + p_2}\right) \times \frac{2f}{p_1} \text{ revolutions per second}$$

$$= \frac{2f}{p_1 + p_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.165)$$

That is, the set runs at a cascade synchronous speed

$$\frac{2f}{p_1 + p_2} \text{ revolutions per second}$$

i.e. corresponding to a motor having a number of poles equal to the sum of the number of poles in both machines. Thus, if $p_1 = p_2$ the set will run at half speed.

If the fields are arranged to rotate in *opposite* directions in both machines, then the speed of the set is that of a motor, having a number of poles equal to the difference of the numbers of poles in the two machines,

$$\text{i.e.} \qquad \frac{2f}{p_1 - p_2} \text{ revolutions per second} \qquad (7.166)$$

The Cascade System of induction motor control is shown in Fig. 7.11.

The method of building up the circle diagram for two motors in cascade is shown in Fig. 7.12.

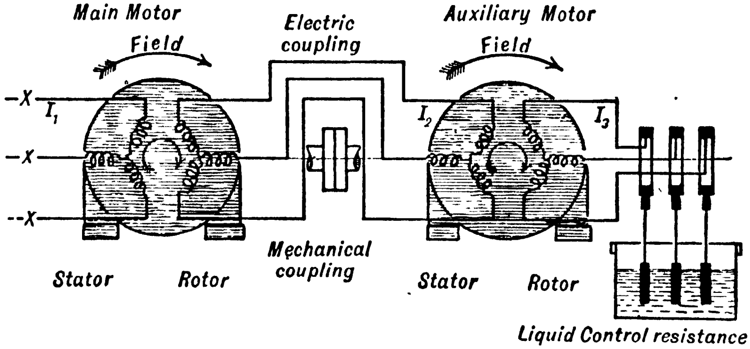


FIG. 7.11. CASCADE SYSTEM OF INDUCTION MOTOR CONTROL

The calculation of the performance of a cascade set is best effected by the aid of complex quantities. One starts with the current in the rotor of the auxiliary motor and works back to the primary of the first machine.

Let $\bar{E}$ = back e.m.f. generated in the second motor rotor by the flux linking both stator and rotor, *at full frequency*

x_4 = leakage reactance of the auxiliary rotor, at full frequency

$\bar{I}_4$ = rotor current of the auxiliary motor

s = slip of the first motor

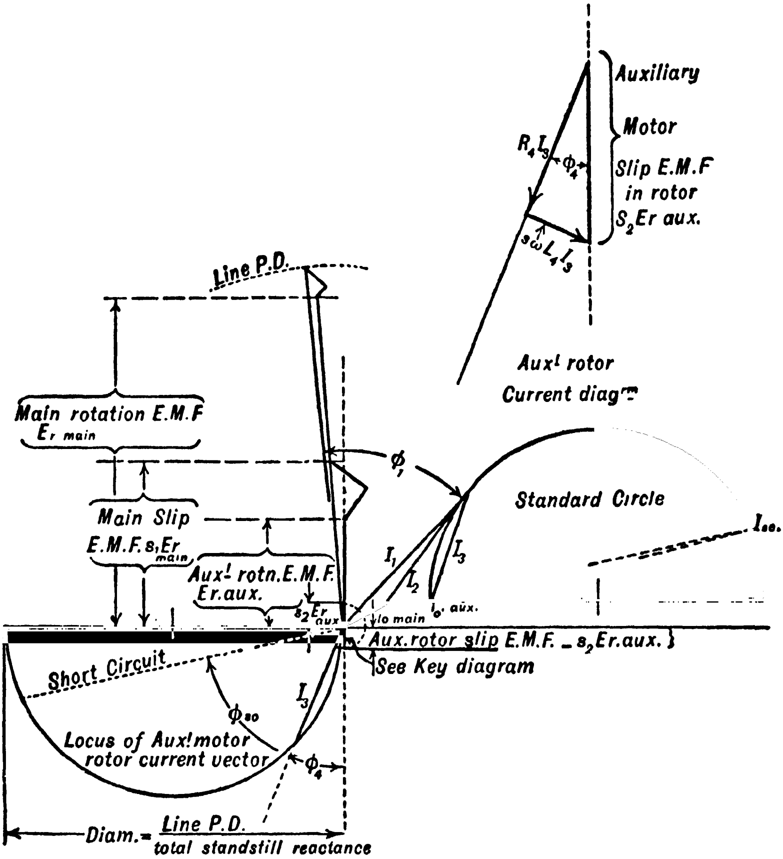
$a = \frac{p_2}{p_1}$ = ratio of number of poles

where p_2 = number of poles in motor No. 2

p_1 = number of poles in motor No. 1

The relative speed of rotation, with respect to the field in the auxiliary machine, in revolutions per second

$$= \left\{\frac{2fs}{p_2} - (1 - s)\frac{2f}{p_1}\right\}$$



Plane I—Coils 1 and 4. Plane II—Coils 2 and 5. Plane III—Coils 3 and 6
Coils numbered round stator in clockwise direction
Start of coil to innermost slot coil of coil group
Finish of coil from outermost slot coil of coil group

Fig. 7.12. Standard Circle Diagram for Two Induction Motors in Cascade

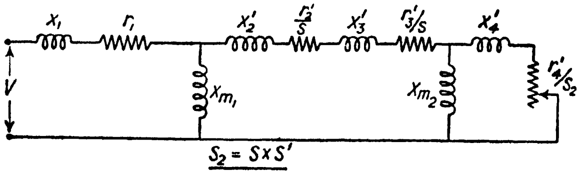


Fig. 7.13. Equivalent Circuit for Two Induction Motors in Cascade

The frequency of the rotor currents in the second machine

$$= \left\{\frac{2fs}{p_2} - (1-s)\frac{2f}{p_1}\right\} \times \frac{p_2}{2}$$

$$= f\left\{s - (1-s)\frac{p_2}{p_1}\right\}$$

$$= f\{s - (1-s)a\} = f\{s(1+a) - a\}$$

$$= s_2 f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.167)$$

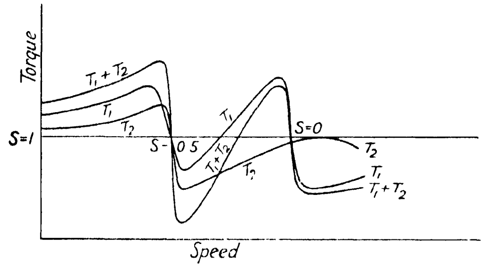


FIG. 7.14. TORQUE CURVES FOR TWO MACHINES IN CASCADE

$s_2 = s(1 + a) - a.$

$$\text{Thus, } s_2 = s \times s' = s \times \frac{\left\{\frac{s}{p_2} - \frac{1-s}{p_1}\right\}}{\frac{s}{p_2}} = s(1+a) - a \quad . \quad (7.168)$$

s = slip of machine No. 1
s' = slip of machine No. 2

$$\mathbf{I}_4 = \frac{s_2 \mathbf{E}}{r_4 + j s_2 x_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.169)$$

= *rotor* current in motor No. 2

where r_4 = resistance per phase of rotor No. 2
x_4 = leakage reactance per phase of rotor No. 2 (at full frequency)
$\mathbf{E}$ = e.m.f. generated at standstill in rotor No. 2

$$\mathbf{I}_4 = \frac{s_2 r_4 \mathbf{E}}{r_4^2 + s_2^2 x_4^2} - \frac{j s_2^2 x_4 \mathbf{E}}{r_4^2 + s_2^2 x_4^2} \quad . \quad . \quad (7.170)$$

$$= \mathbf{E}(\alpha - j\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.171)$$

where $\alpha = \dfrac{s_2 r_4}{r_4^2 + s_2^2 x_4^2}$

$\beta = \dfrac{s_2^2 x_4}{r_4^2 + s_2^2 x_4^2}$

Now if $g_2 - jb_2$ = exciting admittance of the second motor. The no-load current

$$\mathbf{I}_{\mu_2} = \mathbf{E}'(g_2 - jb_2) \qquad (7.172)$$

The primary current of motor No. 2

$$\mathbf{I}_3 = \frac{\mathbf{I}_4}{\alpha_2} + \mathbf{I}_{\mu_2} \qquad (7.173)$$

where α_2 = ratio of transformation of motor No. 2

and $\mathbf{E}' = \mathbf{E} \times \alpha_2 \qquad (7.174)$

Therefore, primary current of motor No. 2

$$\mathbf{I}_3 = \frac{\mathbf{E}(\alpha - j\beta)}{\alpha_2} + \mathbf{E}'(g_2 - jb_2) \qquad (7.175)$$

$$= \frac{\mathbf{E}'}{\alpha_2^2}(\alpha - j\beta) + \mathbf{E}'(g_2 - jb_2) \qquad (7.176)$$

$\mathbf{E}'$ = e.m.f. generated, in stator No. 2, per phase at standstill,

$$\therefore \quad \mathbf{I}_3 = \mathbf{E}'(x - jy) \qquad (7.177)$$

$$x = g_2 + \frac{\alpha}{\alpha_2^2} \text{ and } y = b_2 + \frac{\beta}{\alpha_2^2} \qquad (7.178)$$

The voltage induced in the rotor of motor No. 1, at slip s

$$= s\mathbf{E}_2 = s\mathbf{E}' + \mathbf{I}_3\mathbf{Z} \qquad (7.179)$$

where $\mathbf{E}_2$ = voltage generated in rotor No. 1, at standstill

and $\mathbf{Z} = (r_2 + r_3) + js(x_2 + x_3) \qquad (7.180)$

where r_2 = resistance of rotor No. 1, per phase
r_3 = resistance of stator No. 2, per phase
x_3 = leakage reactance of stator No. 2, per phase, at full frequency
x_2 = leakage reactance of rotor No. 1, per phase, at full frequency

$$\therefore \quad \mathbf{E}_2 = \mathbf{E}' + \frac{\mathbf{I}_3 Z}{s} \qquad (7.181)$$

$$= \mathbf{E}' + \frac{\mathbf{E}'(x - jy)\{(r_2 + r_3) + js(x_2 + x_3)\}}{s} \qquad (7.182)$$

$$= \mathbf{E}'(v + jw) \qquad (7.183)$$

$$v = 1 + \frac{x(r_2 + r_3) + sy(x_2 + x_3)}{s} \qquad (7.184)$$

$$w = x(x_2 + x_3) - \frac{y(r_2 + r_3)}{s}$$

Let
$$\mathbf{E}_2' = \mathbf{E}_2 \times \alpha_1 \tag{7.185}$$
where α_1 = ratio of transformation of motor No. 1

The no-load current of the first machine
$$= \mathbf{E}_2'(g_1 - jb_1) \tag{7.186}$$
$$= \mathbf{E}'\{v + jw\}\{(g_1 - jb_1)\}\alpha_1$$
$$= \mathbf{E}'(p - jq) \tag{7.187}$$

Now the primary current of the first machine
$$\mathbf{I}_1 = \frac{\mathbf{I}_3}{\alpha_1} + \mathbf{E}'(p - jq) \tag{7.188}$$
$$= \frac{\mathbf{E}'}{\alpha_1}(x - jy) + \mathbf{E}'(p - jq) \tag{7.189}$$
$$= \mathbf{E}'(m - jn) \tag{7.190}$$
where
$$m = \frac{x}{\alpha_1} + p \tag{7.191}$$
$$n = \frac{y}{\alpha_1} + q \tag{7.192}$$

The applied volts per phase of motor No. 1
$$\mathbf{V} = \mathbf{E}_2' + \mathbf{Z}\mathbf{I}_1 \tag{7.193}$$
$$= \alpha_1\mathbf{E}_2 + (r_1 + jx_1)\mathbf{E}'(m - jn) \tag{7.194}$$
$$= \alpha_1\mathbf{E}'(v + jw) + \mathbf{E}'(r_1 + jx_1)(m - jn) \tag{7.195}$$
$$= \mathbf{E}'(r + js) \tag{7.196}$$
$$\therefore \qquad \mathbf{V} = \mathbf{E}'(r + js) \tag{7.197}$$
$$\therefore \qquad E' = \frac{V}{\sqrt{r^2 + s^2}} \tag{7.198}$$
and
$$I_1 = \frac{V\sqrt{m^2 + n^2}}{\sqrt{r^2 + s^2}} \tag{7.199}$$

The power input to the set
$$= \text{real part of } \mathbf{V}\mathbf{I}_1$$
$$= \mathbf{E}'\{(r + js)\}\mathbf{E}'(m + jn) \tag{7.200}$$
$$= (E')^2\{(rm - sn)\} \tag{7.201}$$
power input to the set
$$= \frac{V^2}{r^2 + s^2}(rm - sn) \tag{7.202}$$

Volt-ampere input
$$= \mathbf{V}\mathbf{I}_1 \tag{7.203}$$

Torque of the auxiliary motor
$$= E^2\alpha \tag{7.204}$$

Torque of the main motor, real component

$$= \mathbf{E}_2\mathbf{I}_3 \qquad (7.205)$$

$$= \mathbf{E}'(v + jw)\mathbf{E}'(x + jy) \qquad (7.206)$$

$$= (\bar{E}')^2(vx - jw) \qquad (7.207)$$

With two similar motors in cascade, the torque becomes zero at cascade synchronous speed. Above this speed, the torque of the second machine becomes a generator torque. Due to the reversal of the current in the secondary of the first motor, its torque becomes negative also. Above cascade synchronous speed the set acts as an induction generator. When approaching full synchronous speed, the generator torque of the auxiliary motor becomes very small and the second motor acts merely as an impedance in the secondary circuit of the first machine, which again exerts a motor torque. Thus, somewhere between half synchronous speed and full speed, the torque of the first motor becomes zero, while the second motor still gives a small generator torque. A little above this speed, the torque of the set becomes zero, and above this speed, the set gives a motor torque, although the auxiliary motor still gives a small negative torque. Above full synchronous speed, the set acts as generator but practically only the first machine contributes to the generator torque above, and the motor torque below full synchronous speed.

With high resistance in the rotor circuit of the auxiliary motor, the set gives a generator torque above half synchronous speed and remains a generator torque at all higher speeds. Instead of connecting the rotor windings of the first machine to the stator of the second we may connect the two rotor windings in series, and insert the resistance for starting or speed control, in the stator of the second machine, and avoid the use of slip-rings. The phases of the second rotor must be so connected to give a direction of rotation of the auxiliary field opposite to the rotation of the set. With two machines in cascade, the first motor carries the magnetizing current of both machines. The total self-inductive impedance of the two machines is the sum of the separate impedances. It is clear, therefore, that the dispersion coefficient of the set is practically four times that of one machine (assuming two similar machines in cascade), and, therefore, the power factor is rather low, even when each machine has a small number of poles. The output of the set, when running in cascade, is equal approximately to that of one machine at full speed.

Cascade connection is used on the Italian railways.

Combined with pole-changing it gives a variety of speeds and, of course, regenerative braking can be used, which is a very useful feature. By the combination of pole-changing and cascade connection, one can get a range of speeds efficiently, but there are wide gaps

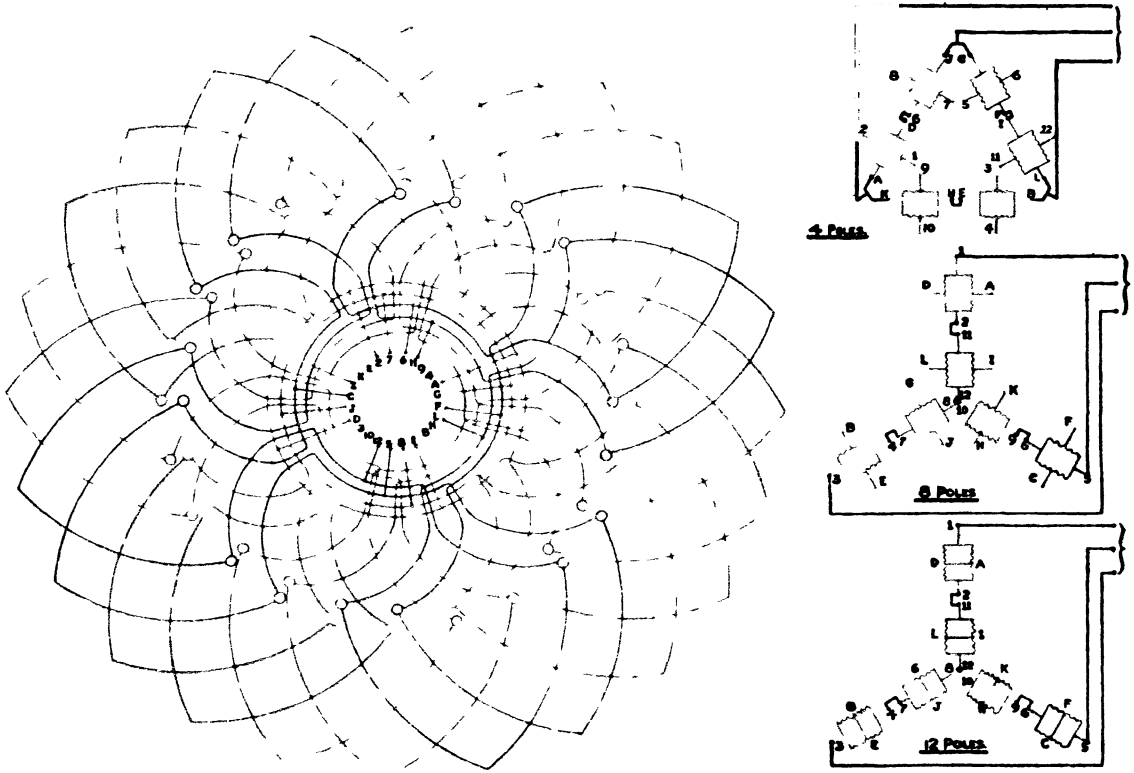


Fig. 7.15. Stator Winding suitable for Four, Eight or Twelve Poles

in the speed-range and continuous speed changes, such as are obtainable in the shunt d.c. motor by field control, are only obtainable by other means. Attempts were made to combine the two cascade machines into one and Lydall patented such a motor in 1902, in which the two windings were arranged in the same slots in stator and in rotor. The two sets of windings were wound for dissimilar numbers of poles, so that the stator windings were mutually non-inductive. The need to use deep slots to accommodate both windings caused a high leakage reactance and poor power factor, efficiency, and overload capacity.

Hunt went further and substituted one winding to perform the function of two separate windings. The stator winding must be suitable for the circulation of two independent currents, namely the main current of the frequency of supply and also the induced currents of slip frequency. The connections were such that these slip-frequency currents could only circulate when paths outside the winding were provided for them. It will be seen that control of starting torque and speed is effected by the use of resistance, connected to the stator circuits. Thus, the use of slip-rings was avoided in some cases. Mr. Hunt, at the time of his invention, was chief engineer of an important company, whose activities lay chiefly in the manufacture of mining machinery, and it will be appreciated that the elimination of possible sparking at slip-rings was a very desirable feature in machines operating in mines. Two sets of terminals were required, one set for connecting to the mains and the other set to the resistances. The main currents must not flow through the external secondary paths and, therefore, it is essential to connect the secondary terminals to two points, in the stator winding, between which no "main" potential difference exists. A "parallel-connected" winding is necessary for the stator.

There are two conditions to be satisfied, namely—

(1) The numbers of poles in the two fields must be so chosen that their windings are mutually non-inductive.

(2) The two fields, when superimposed, must not produce an unbalanced radial pull on the rotor.

These conditions are satisfied when the two numbers of poles are such that when divided by their greatest common factor the quotient is in one case even and in the other odd. Further, the greatest common factor must be greater than two. These rules are not perfectly general, as has been shown by Creedy. Any two numbers of poles must be even and must, therefore, have a common factor of 2. Hence their highest common factor must be a multiple of 2. This means that the two numbers of *pairs* of poles must have a common factor. The circumference of the machine is divided up into two or more identical sections, so that any values of magnetic density, which occur at any point, also occur at a point diametrically opposite, or at other equally spaced points, according as the common factor is

2, 3, 4, etc. The rule for magnetic balance is: the two pole numbers must differ by more than two to secure magnetic balance.

Mr. Creedy describes types of stator windings in which it is not necessary to have any common factor other than unity between the

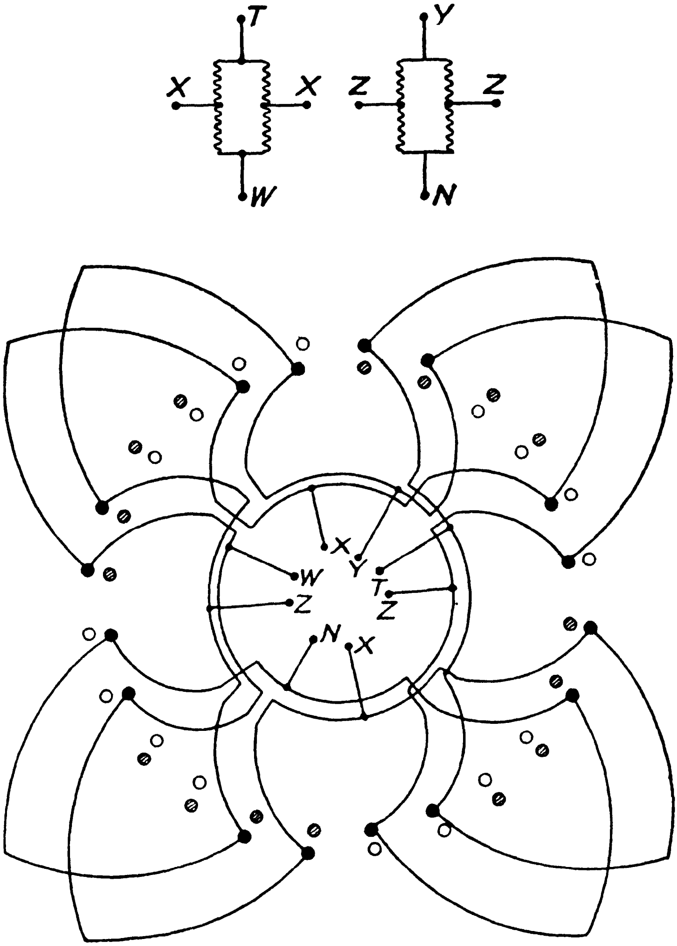


Fig. 7.16

two numbers of pairs of poles, in which case both numbers of pairs of poles must be odd. This gives machines of higher speed than is possible with the Hunt rule.

Fig. 7.15 shows the stator winding for four, eight, and twelve poles. The figures and letters, on the key diagram, correspond to those on the winding diagram.

Fig. 7.16 shows one phase of a pole-changing winding for the stator, suitable for four or eight poles.

Fig. 7.17 shows the windings of one phase of an eight-pole stator, with, for simplification, one slot per pole per phase. Each radial pair of small circles represents the two coil sections, occupying one slot, and the crosses and dots indicate the directions of the main currents. The main currents produce eight poles and the coils must be so connected that currents, induced by a four-pole field, can circulate in them. This four-pole field is shown in Fig. 7.17.

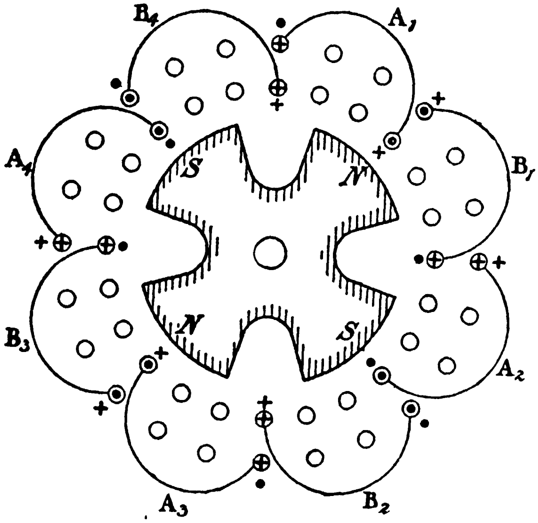


FIG. 7.17

Four of the slots are opposite the pole centres and four are situated midway between them.

The e.m.f.s induced in the conductors at the centres of the poles will be in quadrature with those induced in the conductors midway between the poles. The coils are marked *A* and *B* alternately, and all the induced currents in the *A* coils are in phase with one another, and in quadrature with those in the *B* coils. Fig. 7.18 shows the key diagram and Fig. 7.19 shows one phase connected up to comply with this diagram.

The main currents in the key diagram are shown by arrows inside the windings; the induced currents are shown by the arrows outside. All the *A* coils are included between *T* and *C*, and the *B* coils between *C* and the star point.

The rotor winding is exceedingly ingenious. The ratio of the numbers of turns in the main and auxiliary windings of the rotor is 1·73 to 1·0. This gives an auxiliary field with a flux per pole 73 per cent greater than that of the main field and a gap density of 86·5 per cent of that of the main field. The auxiliary magnetizing current is 75 per cent of the main-field magnetizing current. With a given

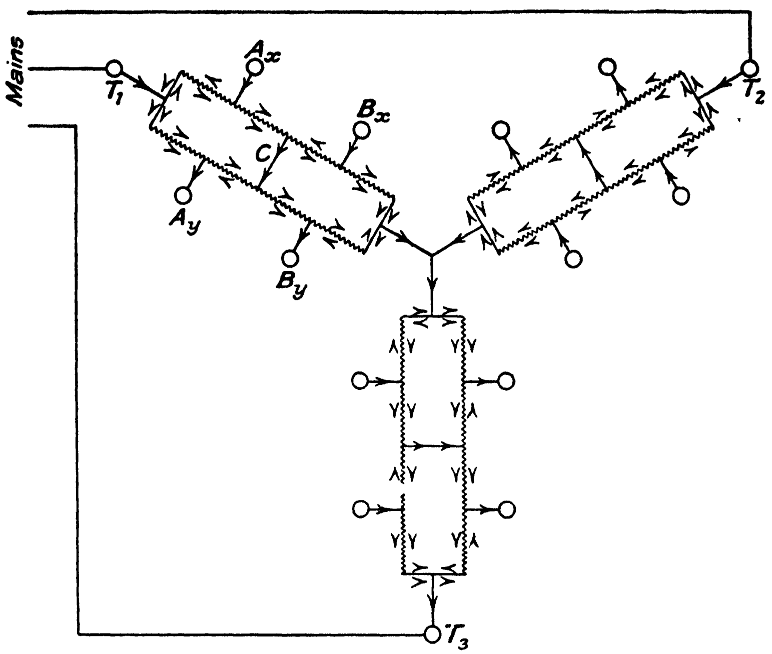


FIG. 7.18

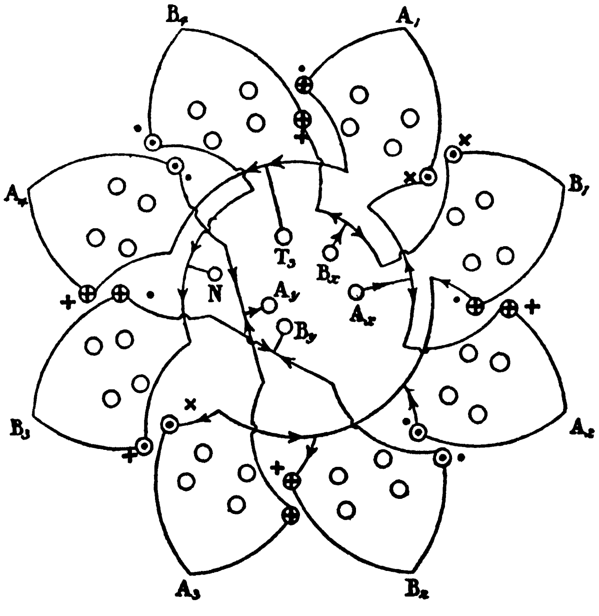


FIG. 7.19. ELEMENTS OF "HUNT" ROTOR WINDING

stator winding the magnetizing current of the auxiliary field varies inversely as the square of the number of turns in the auxiliary winding. The rotor winding consists of a main rotor winding, connected in star, and an auxiliary rotor winding connected in mesh.

Fig. 7.20 shows the elements of the two windings and the method of connection.

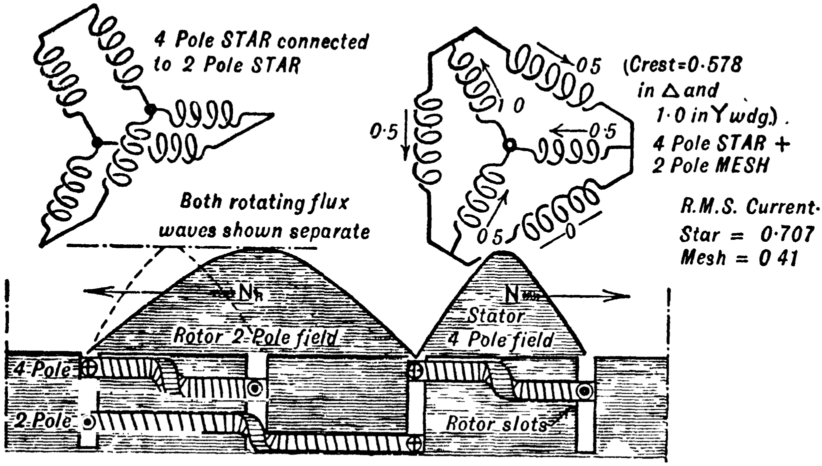


FIG. 7.20. CASCADE ROTOR WINDING

Fig. 7.21 shows a cascade rotor winding for twelve poles. The main and auxiliary rotor windings are superimposed to form a single winding.

In order to show how this superposition and combination is effected, a four-pole and two-pole combination is shown in Fig. 7.22. This is merely for the purpose of illustration and could not be used in practice, but it serves as a unit from which to build up other pole combinations.

The upper diagram of Fig. 7.22 shows the component windings for a four-pole and two-pole rotor, wound in twelve slots. It will be noticed that, in some slots, the currents are equal and flow in opposite directions and hence neutralize each other. These conductors are omitted and the remaining conductors are then connected up to form a resultant winding, which produces the four-pole and two-pole fields. Six slots have three bars and six have one bar per slot. The three bars are combined into two, so that one carries a double current and the other a single current. In one set of bars the currents are $\sqrt{3}$ times the currents in the other bars. These bars, carrying the $\sqrt{3}$ times the current in the other bars, are then connected in star and supply the single current bars, which are connected in delta. This arrangement saves 25 per cent of copper and reduces the inductance.

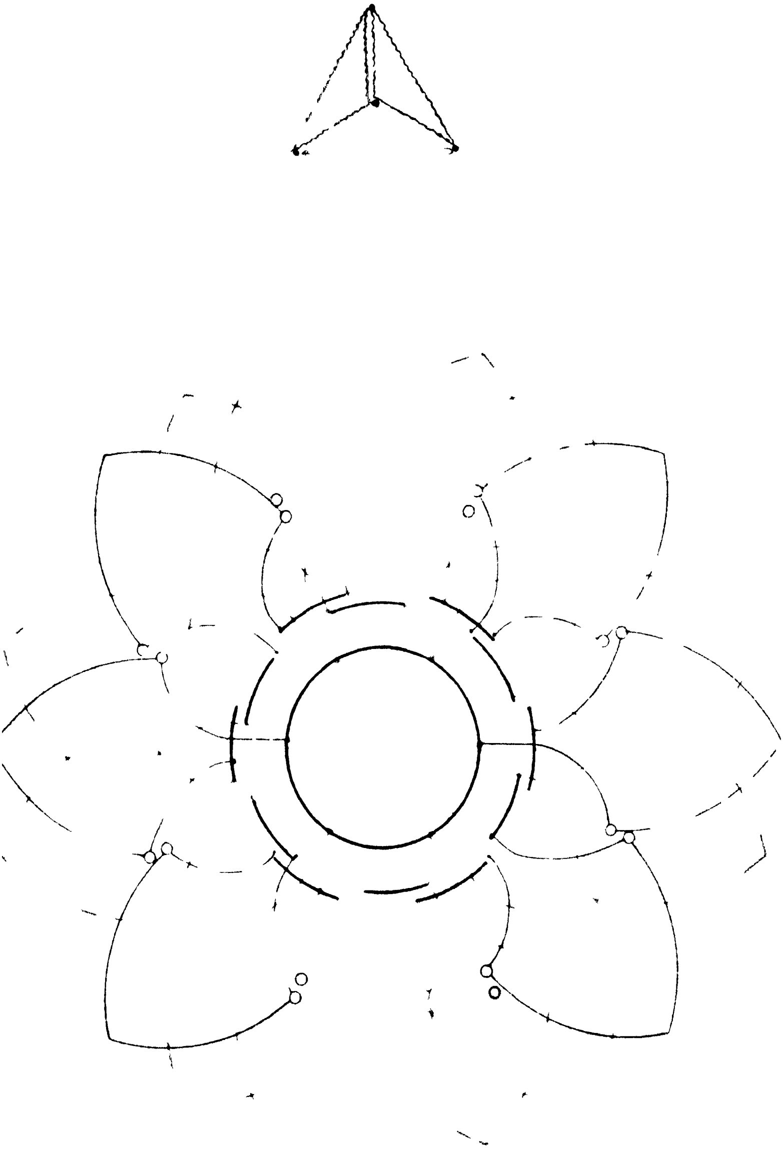


Fig 7 21

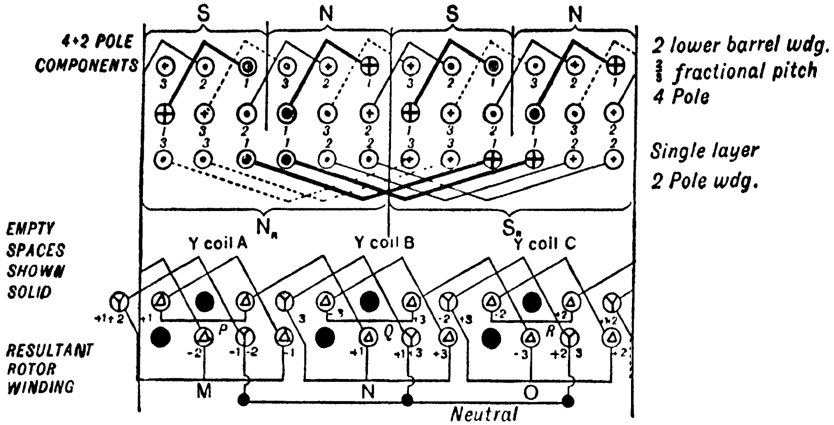


FIG. 7.22. DIAGRAMS OF *Y*. Δ ROTOR RESULTANT WINDING

Fig. 7.23 shows the resultant diagram.

The e.m.f.s in the mesh-connected winding, due to the main field, are balanced by the e.m.f.s generated by the auxiliary field, and current can circulate only by way of the star.

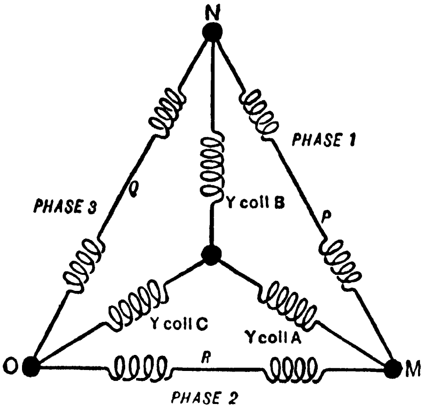


FIG. 7.23

Speeds, other than cascade speed, are obtained by bringing the ends of the mesh to slip-rings, which can be short-circuited. If the mesh windings are short-circuited, the motor will run as an induction motor having the main number of poles. The two-speed motor can be operated at a third speed, if the stator windings are so arranged that they can produce two different numbers of poles.

Fig. 7.24 shows an eight- + four-pole stator winding, with two groups of coils in each phase, divided and the neutral point opened.

To connect this winding for eight poles, two groups of coils in adjacent sections are placed in series and then, by means of a suitable switch, the whole winding is connected in star. (*See* Fig. 7.25.) Fig. 7.26 shows the same winding reconnected to give four poles.

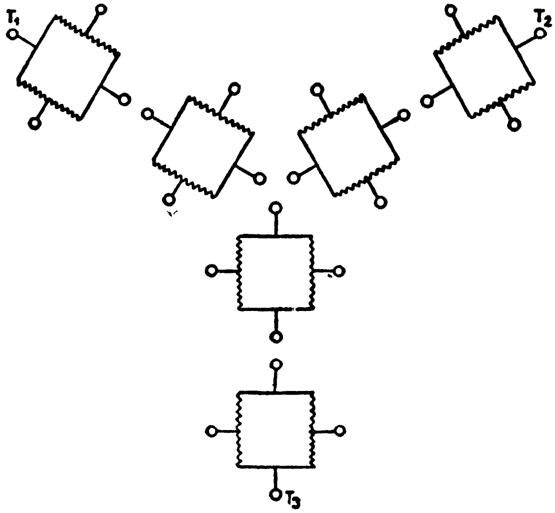


Fig. 7.24

Fig. 7.27 shows a cascade rotor winding suitable for four, eight, or twelve poles (twelve poles cascade).

It should be noted that the cascade machine is essentially a low-speed machine and, as pointed out by Mr. Hunt, shows to best advantage when designed for speeds which are abnormal for single-field motors.

There is a trend to-day, in marine work, to use alternating current on board ship, and the question of variable-speed motors for winches, etc., is a pressing one. The solution for many of these problems may be found in the use of a Hunt motor, combined with resistance in the secondary circuits. Another solution is the use of an induction motor driving McFarlane converters, each converter supplying two winches.

Change of Speed by Varying the Supply Frequency

Since r.p.m. $= \dfrac{120 \times f}{\text{poles}}$, it is clear that the speed may be varied by changing the supply frequency. The generators, driven by diesel engines or steam turbines, may have their speeds adjusted over a fair range, and the induction motors, supplied from the generators, will vary their speed as the frequency is varied. This is done usually on ships having an electric drive. On warships, where cruising and fighting speeds are required, a low-frequency generator is provided for the low-speed, and separate and more powerful generators of higher frequency, for the high speed. If it is desired to use all the

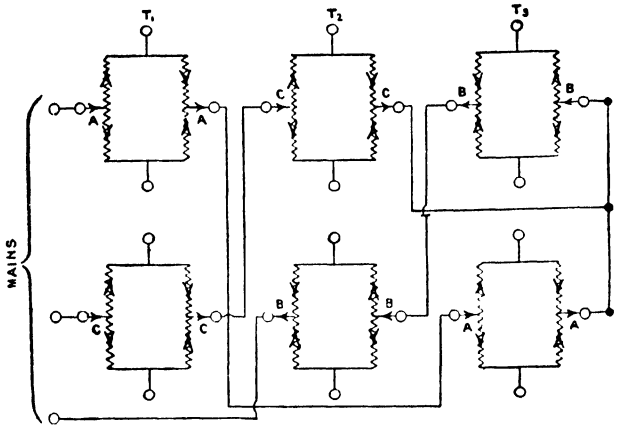


FIG. 7.25

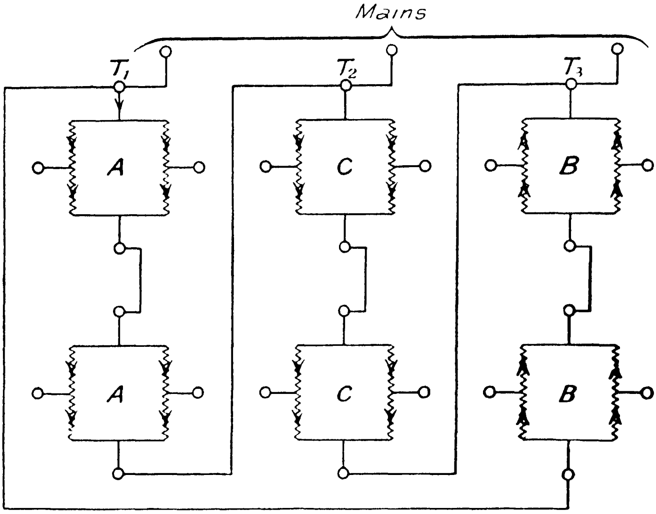


FIG. 7.26. CASCADE STATOR WINDING SUITABLE FOR FOUR POLES

generators at the high speed, then all that is necessary is to use motors giving the higher speed at the low frequency.

The Krämer System of Speed Control

In this system the rotor winding of the induction motor is connected to the slip-rings of a rotary converter. The slip energy is converted

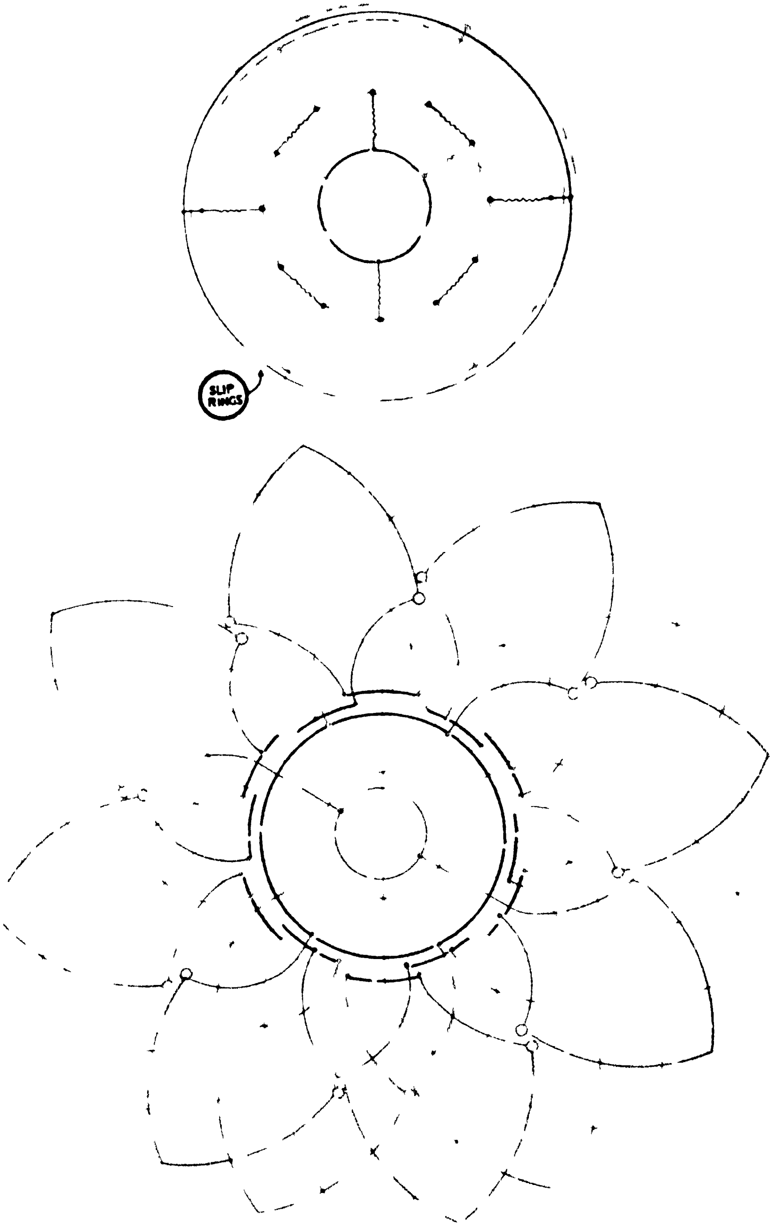


FIG 7 27

into continuous-current energy, which can be given to a continuous-current network, or to a continuous-current auxiliary motor, coupled to the shaft of the induction motor. As the slip increases the power given out by the auxiliary motor rises by the same amount as that given out by the main motor falls. Thus, the sum of the outputs of the two motors remains constant over the whole-speed range, i.e. the torque varies inversely as the speed. This is a property frequently required in rolling-mill work. When working in conjunction with a flywheel, the requisite drop in speed can be obtained by a compound winding on the auxiliary d.c. motor. The auxiliary motor determines the speed of the set. To lower the speed, the field of the d.c. motor must be increased.

With the connections shown in Fig. 7.28 no special synchronizing gear is necessary. On switching on the excitation of the auxiliary motor and converter, the latter runs up to speed and falls into step. The maximum speed of the set is determined by the lowest frequency necessary for the stable running of the rotary converter. This is about 2 or 3 c/s, so that with a frequency of 50 c/s the highest speed will be about 4 to 6 per cent below synchronous speed of the main motor, or, if artificially raised above synchronism, 4 to 6 per cent above this will be the lowest stable speed. If the voltage on the slip-rings becomes small, as compared with the ohmic drop, the converter becomes unstable. The power factor of the set can be made unity over nearly the whole range by adjusting the excitation of the converter. If the speed of the induction motor is very low, the cost of a d.c. motor, on the same shaft, may be excessive. It is preferable, in this case, to return the slip energy to the supply system, by means of a high-speed motor-generator set. The generator may be of the asynchronous type. For work which requires constant torque the latter method is suitable.

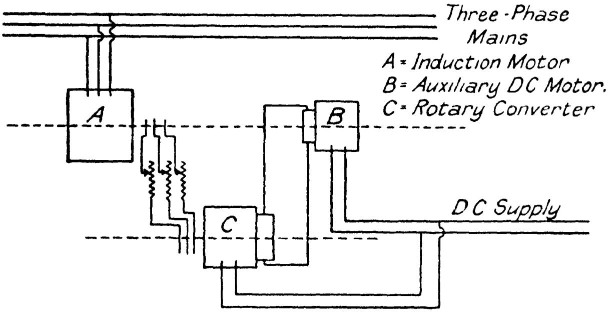


FIG. 7.28

The Krämer system is shown above in Fig. 7.28.

It will be seen that this method of controlling speed is expensive, in that costly auxiliary machines are required.

VARIATION OF SPEED AND POWER FACTOR BY THE USE OF THREE-PHASE COMMUTATOR MOTORS

Instead of converting to continuous current the slip-energy may be supplied to a three-phase commutator motor. This three-phase auxiliary motor is then coupled direct or by means of a belt-drive to the main motor. The diagram of connections is given in Fig. 7.29.

According to the characteristic required, a series motor, with brush-shifting device, or a shunt or compound motor with voltage regulation is used as auxiliary motor. With this arrangement the output is constant for a given input, so that the torque varies inversely as the speed.

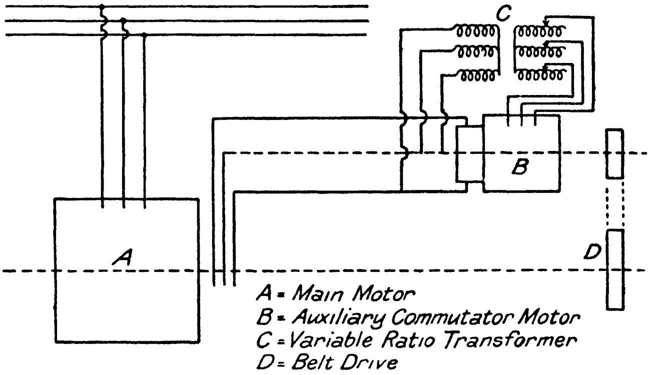

C
B
A
D
A = Main Motor
B = Auxiliary Commutator Motor
C = Variable Ratio Transformer
D = Belt Drive

FIG. 7.29. NEUTRALIZED SERIES-EXCITED THREE-PHASE A.C. COMMUTATOR MOTOR AND CONNECTIONS

The Scherbius System

In this system, used by Brown-Boveri, the rotor current is led to a three-phase commutator motor. The latter is not coupled to the main motor but to a three-phase generator, which returns the slip energy to the line. The generator is of the induction type, running above synchronous speed. The speed characteristic obtained depends on the type of commutator motor, i.e. whether series, shunt, or compound.

This system is chiefly used for drives where the power falls with the speed, as in centrifugal machines. It has the advantage that the main motor can be run independently of the auxiliary set. The most common arrangements of regulating sets are (1) a series commutator machine for single-range regulation, i.e. for speed ranges below synchronism; (2) a shunt-wound commutator machine for single-range regulation; and (3) a shunt-wound commutator machine with special exciter, for double-range speed regulation above and below synchronism.

The commutator machine is mechanically coupled to an induction machine and, in the case of single-range regulation below synchronism, the induction machine acts as generator. In double-range sets, the induction machine will act as a generator below synchronism, and as a motor above synchronism. The speed of the regulating set will change but little with change of speed of the main motor, since a very small percentage change of speed is sufficient to change from full-load motor to full-load generator action. It may be regarded, therefore, as having an approximately constant value. In all these cases an e.m.f. is injected into the rotor circuit of each phase. If this injected e.m.f. has a component, in phase opposition to the actual e.m.f. producing the rotor current, i.e. the e.m.f. overcoming the resistance drop, and tending to reduce the rotor current, it is clear the slip of the induction motor must increase to such a value that the slip e.m.f. will be sufficient to overcome the injected e.m.f. plus the resistance drop, due to the current required for the load torque.

Likewise, by adjusting the phase of the injected e.m.f. we can control the power factor.

Speed and power-factor control is, therefore, obtained by injecting an e.m.f. into the rotor circuit of the induction motor. If this e.m.f. opposes the slip e.m.f. the speed must fall; if it assists the speed must rise, because now a smaller slip is required to produce the rotor current, since the injected e.m.f. assists the slip e.m.f.

Clearly, also, if the injected e.m.f. has a component in quadrature with the slip e.m.f. of the main motor, the current will either lag or lead this e.m.f., depending on the phase of the injected e.m.f.

We shall be making a study of the three-phase commutator motor later, but it can be seen that the injected e.m.f., in the series machine, is proportional (neglecting saturation) to the rotor current of the main motor, since the speed of the commutator motor is practically constant and its phase is controlled by brush position.

Speed Regulation Below Synchronism

The diagram of connections is shown in Fig. 7.30. The form of circle diagram used, in this analysis, is one used by Behrend and Blondel, and considers only resistance drops and *not* leakage reactance voltages. These leakage reactance voltages are represented by corresponding *leakage* fluxes. The fluxes produced by primary and secondary currents are divided into fluxes which (*a*) link one winding (total flux); (*b*) mutual flux; and (*c*) leakage flux.

Fig. 7.31 gives a very clear picture of the performance of an induction motor working in conjunction with commutator sets.

Dr. Meyer-Delius and Mr. John I. Hull gave this analysis.

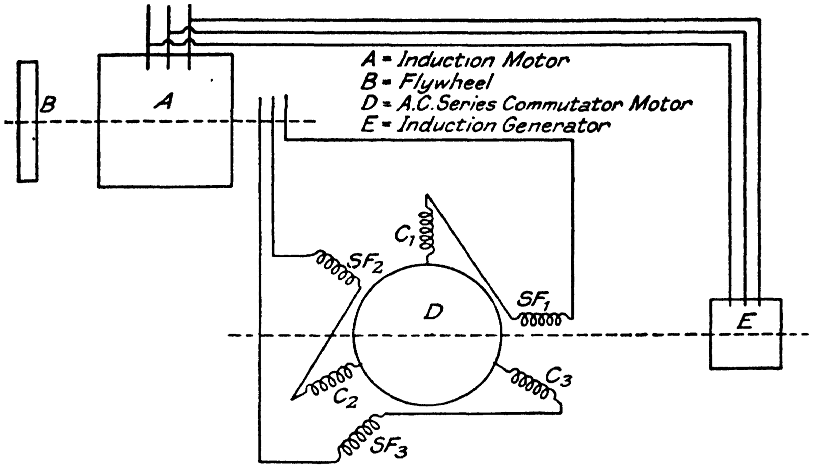


FIG. 7·30

For automatic single-range regulation of an induction motor equipped with a flywheel, to reduce peaks on line SF_1, SF_2, and SF_3 are the series coils, C_1, C_2 and C_3 are neutralizing windings, neutralizing the armature ampere-turns

In Fig. 7·31

AH — I_1, proportional to the primary current, and represents the flux linking *the primary and secondary*, which would be produced by the primary current alone, neglecting saturation

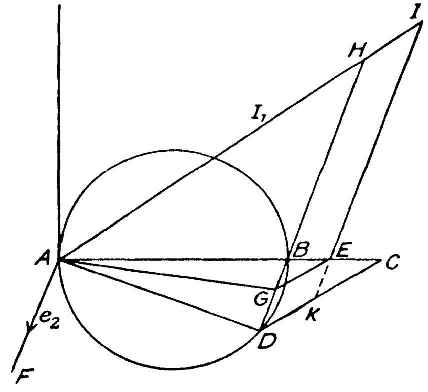


FIG. 7·31

$HI = C_1I_1$ and represents primary leakage flux

$HG = IE = I_2$, proportional to the secondary current, and represents the flux linking both primary and secondary, which would be produced by the secondary current alone, neglecting saturation

$GD = C_2I_2$, represents secondary leakage flux

AE represents the total flux linking the primary, and represents the vector sum of the fictitious total primary flux and the fictitious flux, produced by the secondary current, which links the primary.

Neglecting resistance drop, *E* is a fixed point for constant line volts and frequency, since *AE* is the flux which generates the counter e.m.f. to balance the applied volts. *HG* intersects *AE* at *B*, and it is clear that

$$\frac{AB}{AE} = \frac{AH}{AI} = \frac{I_1}{I_1(1 + C_1)} = \frac{1}{1 + C_1} \quad . \quad . \quad (7.208)$$

Therefore, $AB = \dfrac{AE}{1 + C_1}$ and, since *AE* is constant, *B* is a fixed point.

AD is the resultant of *AH* and *DH*, i.e. of I_1 and $I_2 + C_2 I_2$, and generates all secondary e.m.f.s, except the resistance drop, which is in phase opposition to e_2, set up in the secondary, in quadrature with the flux *AD*.

AF is, therefore, parallel to *HD* and to *IE*, and the angle *ADH* is a right angle. A line, parallel to *AI* from *D*, intersects *AE* produced in *C*.

$$BD = HD - HB = I_2(1 + C_2) - \frac{I_2}{1 + C_1} \quad . \quad (7.209)$$

since

$$\frac{HB}{I_2} = \frac{I_1}{I_1(1 + C_1)} \quad . \quad . \quad . \quad (7.210)$$

$$\therefore \qquad BD = \frac{I_2}{1 + C_1}\{C_1 + C_2(1 + C_1)\} \quad . \quad . \quad (7.211)$$

$$\frac{CD}{AI} = \frac{BD}{IE} \quad . \quad . \quad . \quad (7.212)$$

$$CD = I_1(1 + C_1) \times \frac{I_2\{C_1 + C_2(1 + C_1)\}}{I_2(1 + C_1)}$$

$$= I_1[C_1 + C_2(1 + C_1)] \; . \quad . \quad . \quad . \quad (7.213)$$

Now *EA* represents the flux, whose counter e.m.f. balances the applied voltage. Let *EA* be represented by I_m.

Then

$$BA = I_0 = \frac{I_m}{1 + C_1} \quad . \quad . \quad . \quad (7.214)$$

$$\frac{CB}{CD} = \frac{AE}{AI} = \frac{I_m}{I_1(1 + C_1)} \quad . \quad . \quad . \quad (7.215)$$

$$\therefore \qquad CB = CD \times \frac{I_m}{I_1(1 + C_1)} \quad . \quad . \quad . \quad (7.216)$$

$$= \frac{I_m[C_1 + C_2(1 + C_1)]}{1 + C_1} \quad . \quad . \quad (7.217)$$

$$= I_0[C_1 + C_2(1 + C_1)]. \quad . \quad . \quad (7.218)$$

$$CA = EA + CE = I_m(1 + C_2) \quad . \quad . \quad (7.219)$$

since

$$\frac{CE}{EK} = \frac{I_m}{I_2}$$

$$\therefore \qquad CE = I_m \times \frac{C_2 I_2}{I_2} = C_2 I_m \;. \quad . \quad . \quad (7.220)$$

$$CA = I_m(1 + C_2) = \frac{I_m(1 + C_2)}{C_1 + C_2(1 + C_1)} \times [C_1 + C_2(1 + C_1)] \quad (7.221)$$

CA is, therefore, constant for constant I_m, and C is a fixed point. By selecting a suitable scale, I_m could be made to represent the magnetizing current for the whole flux, which is the usually calculated quantity. I_0 could be made to represent the running-light current. By changing the scale of the diagram by the factor $C_1 + C_2(1 + C_1)$, we may say that $CB = \frac{I_m}{1 + C_1} = I_0$; the primary current $I_1 = CD$

and

$$\frac{I_2}{1 + C_1} = BD \quad . \quad . \quad . \quad (7.222)$$

At standstill, with zero secondary resistance, AD, the resultant secondary flux must be zero, which means that D coincides with A and $CD = CA$, i.e. the ideal short-circuit current at standstill $= CA$. Since $C_1 I_1$ is defined as the primary leakage flux, the primary leakage reactance drop, with current I_m is $C_1 E_1$, since I_m produces the flux, which generates E_1.

If X_1 = primary leakage reactance

$$C_1 E_1 = I_m X_1 \quad . \quad . \quad . \quad (7.223)$$

$$\therefore \qquad C_1 = \frac{I_m X_1}{E_1} \text{ and } C_2 = \frac{I_m X_2}{E_1} \quad . \quad . \quad (7.244)$$

In order to draw the diagram, we need to know the primary and secondary leakage reactances X_1 and X_2, and the magnetizing current I_m.

The diagram is shown in Fig. 7.32.

In Fig. 7.32

$$CB = I_0 \quad . \quad . \quad . \quad . \quad (7.225)$$

$$CA = \frac{I_m(1 + C_2)}{C_1 + C_2(1 + C_1)} \quad . \quad . \quad . \quad (7.226)$$

CM = watt component of the input current

$$CD = I_1$$

The secondary current is $BD(1 + C_1) = \bar{I}_2$.

To find the second voltage $\bar{E}_2 = AF$, one can first find its value at full frequency. Since AE (Fig. 7.31) is proportional to $\bar{E}_1$ at full frequency

$$AB = \frac{\bar{E}_1}{1 + C_1}$$

also

$$AD = \frac{AD}{AB} \times \frac{\bar{E}_1}{(1 + C_1)} \quad . \quad . \quad . \quad (7.227)$$

at full frequency. The actual value of the secondary volts $= \bar{I}_2 r_2$, so the percentage slip

$$= \frac{\bar{I}_2 r_2}{AD} \quad . \quad . \quad . \quad . \quad (7.228)$$

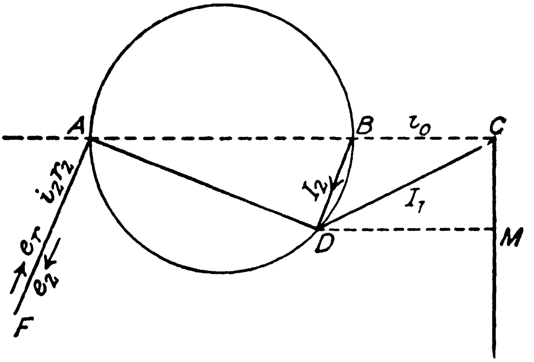


FIG. 7.32

The torque per phase, in synchronous watts,

$$= BD \times AD \quad . \quad . \quad . \quad (7.229)$$

The output per phase

$$= (1 - s)BD \times AD \quad . \quad . \quad . \quad (7.230)$$

The efficiency

$$= \frac{(1 - s)BD \times AD}{\bar{E}_1 \times MC} \quad . \quad . \quad . \quad (7.231)$$

and

$$\cos \phi = \frac{MC}{CD} \quad . \quad . \quad . \quad . \quad (7.232)$$

Now it is clear, from this simple diagram, that if an injected $\bar{E}_r'$, due to the commutator motor, exists in the rotor and if this e.m.f. is in phase opposition to the secondary current, the slip of the motor must increase to make the generated e.m.f. $\bar{E}_2 = \bar{E}_r + \bar{I}_2 r_2$. In this case the slip is increased, for the effect of the injected e.m.f. is equivalent to an addition of resistance in the rotor circuit. If $\bar{E}_r$ is in phase with $\bar{E}_2$, and assists $\bar{E}_2$, the slip will decrease.

Now it is possible to inject, by means of the commutator motors, e.m.f.s having any phase angle with the rotor current. If the injected e.m.f. has a component, which is 90° ahead of *BD*, the effect will be to make *BD* lead. In other words, speed control and power factor correction are possible.

With the neutralized, series, commutator motor, shown in Fig. 7.30, the e.m.f.s at the terminals of the commutator motor are the leakage reactance drop, the resistance drop, and the rotational voltage of the armature. This rotational e.m.f. is proportional to the flux and speed of rotation, and the flux is proportional to the secondary currents. Since the speed of the commutator motor is practically constant, the injected e.m.f. is proportional to the rotor current of the induction motor. Further, the phase angle between the rotor current and injected e.m.f., i.e. between the resistance drop and rotational voltage is constant, and can only be changed by altering the construction of the machine. The diagram for this case is shown in Fig. 7.33.

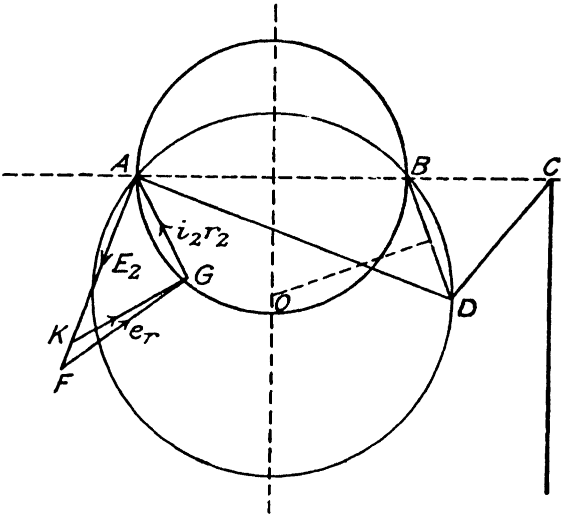


FIG. 7.33

Since $FG \propto AG$, and the angle FGA is constant, angle FAG is constant.

Also AF is at right angles to AD.

$$\therefore \qquad \angle GAD = 90^\circ - \angle FAG = \text{constant}$$

But AG is parallel to BD, i.e. $\bar{I}_2$,

$$\therefore \qquad \angle GAD = \angle BDA = \text{constant}$$

and, since AB is of constant length, it follows that the locus of D is a circle. The centre of the circle is easily found, since the circle passes through A, B, and D. Bisect AB and BD, and draw perpendiculars to each through the points of bisection. Then the perpendiculars intersect at the point O, the centre.

Points A, B, and C are determined, as for Fig. 7.31, but now for X_2 we must substitute $X_2 + X_c + X_{cs}$, where X_c = leakage reactance, per phase, of the regulating motor, at primary frequency, and

$$X_{cs} = \frac{\text{volt-amperes to excite regulating motor}}{I_2^2 \times \sqrt{3}} \quad . \quad (7.233)$$

It is usually permissible to neglect saturation, but if present, it reduces X_{cs} and gives us a new and larger circle, but the ratio of injected e.m.f. to rotor current or exciting current is reduced. The two effects partially offset one another. In any case, it is simple to draw the new circle, provided we can determine the excitation, at full frequency, from the magnetizing current of the regulating motor.

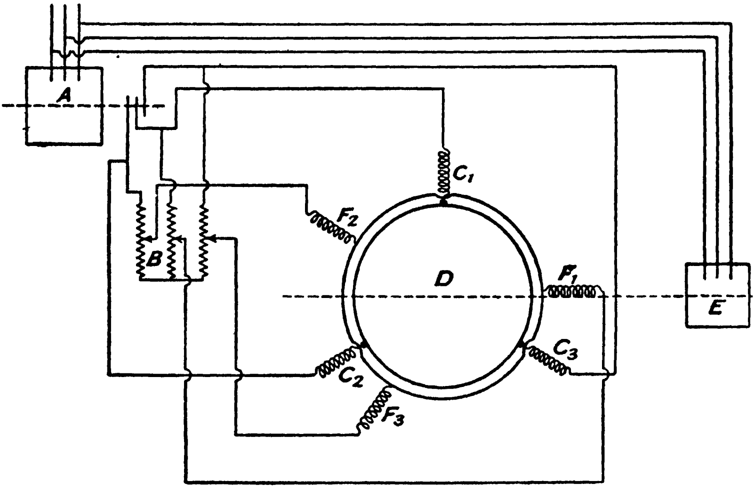


FIG. 7.34

The series regulating motor is most suitable for cases where large and rapidly fluctuating loads occur and where it is desirable to reduce the peak loads by means of a flywheel.

With a constant load, the speed may be regulated by hand, as desired, if the field windings of the commutator motor are supplied through a suitable transformer with taps.

Speed Regulation and Power-factor Control by Means of a Three-phase Shunt Commutator Motor

For some purposes speed variation is required but the speed should not vary greatly with the load. If the flux of the commutator motor can be adjusted to a constant value, the injected e.m.f. will be practically constant, since the speed of the commutator motor is practically constant. By adjusting the magnitude of the field, the rotational e.m.f. can be given any suitable value, and this e.m.f. will be independent of the load. The field coils are fed from taps on the auto-transformer *B* (*see* Fig. 7.34). When the voltage across the rings increases, the slip frequency increases at the same rate and, therefore, the flux in the auto-transformer is independent of the slip. For the same reason the flux in F_1, F_2, and F_3 is constant for a given tap position and independent of the slip. It can be changed, however, by altering the tapping point on *B*. The circle diagram for this case is shown in Fig. 7.35.

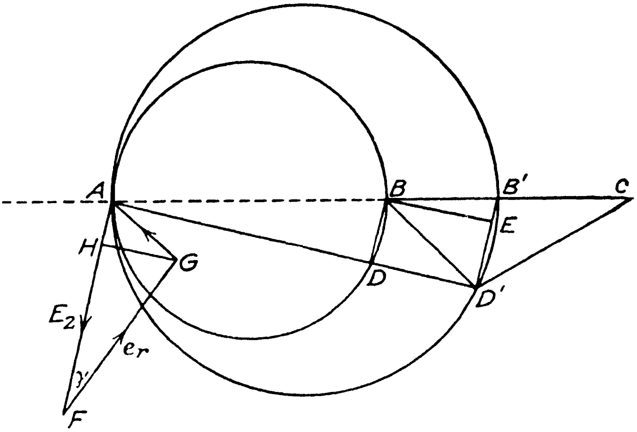


FIG. 7.35

With the commutator motor stationary, we get the circle *BDA*. In constructing this we must include the leakage reactance of the regulating motor, so that $X_{2+c} = X_2 + X_c$, and for C_2 we have C_{2+c}. With the commutator motor running, we get secondary current *BD'*, and the total induced e.m.f. of the induction motor rotor *AF* is proportional to *AD'* and the slip. *FG*, the injected e.m.f. of the commutator motor, is proportional to *AD'* and is at constant angle to *AF*. This angle is determined by the connections of the transformer and exciting windings F_1, F_2, and F_3.

Resolve the resistance drop *GA* into two components: *HA* in phase opposition to *AF* and *GH* in quadrature to *AF*; the corresponding components of secondary current being *BD* and *DD'*. *DD'* is proportional to *HG*.

$HG = FG \sin \gamma$, so that $HG \propto FG$ for constant γ; but $FG \propto AD'$, therefore, $DD' \propto AD'$ and $AD = AD' - DD'$; therefore, *AD* is proportional to *AD'*, therefore, $\frac{AB'}{AB} = \frac{AD'}{AD}$ = constant and *B* is a fixed point.

Therefore, the locus of *D'* is a circle.

When running light, i.e. zero torque, BD and AH are zero. The torque of the motor is proportional to the product of mutual flux and the component of secondary current in quadrature with it, i.e. to $BD \times AD$. It is zero, when BD is zero. BD is the torque producing component of BD'. The slip, when running light, $s_0 = \frac{FH}{AD'}$, and the additional slip, due to the load $= \frac{AH}{AD'}$.

The running-light slip is determined by the angle γ and the ratio of $\frac{FG}{AD'}$, which conditions are adjusted by the connections at the auto-transformer. The load slip s_1 is the same for all values of s_0, provided the angle γ is chosen so that $\frac{HG}{AD'}$, remains constant.

The power factor can obviously be improved and the maximum torque of the motor increased.

Characteristics, intermediate between those obtained by means of a series and shunt a.c. commutator motor, can be obtained by means of a compound-excited neutralized commutator motor. It is necessary, in this case, to change the field voltage of the commutator motor, as the load changes, in order to change the flux and hence the injected e.m.f. This is effected by a series transformer, which has an air-gap in its magnetic circuit, so that its flux is proportional to the resultant of the primary and secondary ampere-turns.

In those cases in which speed regulation is required, below and above synchronism, a special exciter or frequency changer is required. This case has the advantage that, for a given range of speed regulation, a regulating set of only half the output of that required for a full-speed range below synchronism is needed.

A second important advantage is that the synchronous speed of the main motor is in the middle of the speed range, so that many processes may be carried out running as a plain induction motor, with the set shut down.

The special exciter is simply a rotor mounted on the shaft of the main motor. It is provided with a closed-circuit winding, connected to slip-rings on one side and to a commutator on the other, just like the armature of a rotary converter. It is wound for the same number of poles as the induction motor and the commutator has three brush sets per pole pair. The magnetic circuit of the armature is completed by a ring of laminated steel, placed over the slots and rotating with the armature. The slip-rings are connected with the mains, in such a manner that the field rotates in a direction opposite to that of rotation of the rotor. Since the speed of the field relative to the rotor is that of synchronism, the brush p.d. on the commutator (neglecting the drop in the rotor windings) will be practically constant. The frequency of the brush p.d. will be equal to the slip frequency of the main motor. This is a frequency changer, changing from supply frequency to slip frequency. In passing through synchronism the phase sequence of the brush p.d.s is automatically reversed. Now this exciter is used to excite the fields of the commutator motor. Imagine that, from a suitable external source, whose frequency is always that of slip frequency, we excite the commutator motor so as to reverse the phases of the e.m.f.s, which it generates before they are reduced to zero. These e.m.f.s will be in phase with the rotor e.m.f.s, and the immediate effect is to increase the rotor currents and accelerate the rotor. As the speed increases, the rotor e.m.f.s of the induction motor decrease, and the current decreases till it reaches the value required to produce the load torque. The rotor current is now only partially maintained by the rotor e.m.f.s and is partially maintained by the commutator machine. The commutator machine is now acting as a generator and is driven by the induction machine coupled to it.

If the excitation of the commutator machine is now increased, the speed of the induction motor will rise and reach synchronism. At synchronism the induction-motor rotor e.m.f. is zero, and the rotor current is produced entirely by the e.m.f. of the commutator machine. This current, of zero frequency, is a continuous current. If this current is still further increased, the speed will rise above synchronism and the induction motor rotor will now generate e.m.f.s of opposite phase sequence and of reversed phase. Let us suppose that, as the main motor passes through synchronous speed, the phase sequence of the source, supplying the excitation of the commutator motor is reversed automatically, but without phase reversal of the e.m.f.s. The commutator motor will have the same sequence of e.m.f.s as the e.m.f.s in the rotor of the induction motor, but the latter will oppose the former. The rotor currents flow in the direction of the commutator e.m.f.s and, therefore, this machine acts as a generator, supplying the rotor copper losses of the induction motor, and also additional power, which is converted into mechanical power. The induction motor is now doubly fed, both stator and rotor receiving power from external sources. With increasing excitation of the commutator motor the speed of the induction motor will rise.

It is clear, therefore, that some source of excitation for the commutator machine is required, whose frequency is slip frequency and whose phase sequence is reversed as the motor passes through synchronous speed.

Fig. 7.36 shows the diagram of connections for double-range speed regulation. The field coils are shown as F_1, F_2, and F_3. They are fed from taps on the auto-transformer B, whose terminals are connected across the slip-rings of the induction motor. The inner ends of the field coils F are connected through the rheostats M to the commutator brushes of the frequency changes.

When it is required to change the speed, the flux is increased by altering the tappings. The reactance drop component of the impedance drop of the field circuit, being proportional to the frequency as well as to the flux, is proportional to the square of the slip, while the resistance drop is proportional to the field current and slip.

By connecting to taps of *B*, whose distance from the star point is proportional to the slip, we get a voltage proportional to the square of the slip, since the total e.m.f. of *B* is proportional to the slip. By changing the taps on resistance *M*, so that the entire resistance of the circuit is proportional to $\frac{1}{\text{slip}}$, we first permit the constant-voltage frequency changer to supply the resistance-drop balancing e.m.f., while the auto-transformer *B* furnishes the reactance drop balancing e.m.f. One set of switches can be arranged to vary *M* and *B* at once.

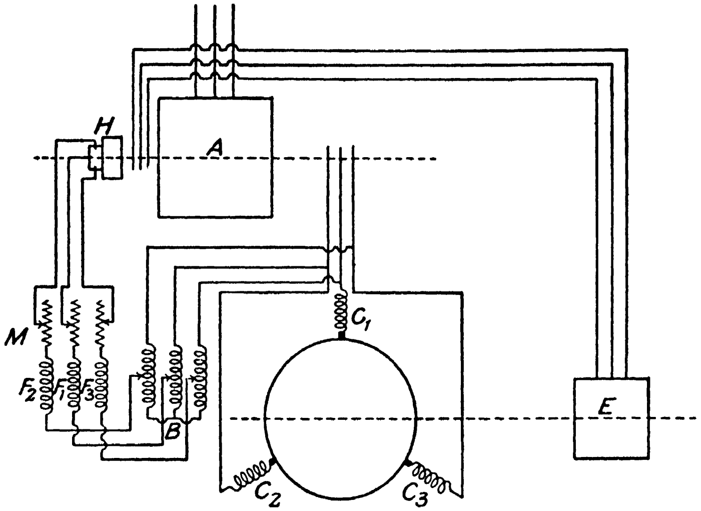


Fig. 7·36

The reader is referred to an article in the *Journal of the American Institute of Electrical Engineers* for June, 1920, and to the *E.T.Z.* for 1913, for further information.

Plate VIII (facing page 97) and Plate IX (facing page 212) show Ilgner sets for rolling mills.

CHAPTER VIII

The Three-phase Series and Shunt Commutator Motors

In Chapter VII concatenation of induction motors with the three-phase series, and three-phase shunt commutator motors was investigated. It will be of interest to give some more detailed account of these motors.

THE THREE-PHASE SERIES COMMUTATOR MOTOR

The series motor is due to the late Prof. Wilson, who took out a patent in 1888, and it also is attributed to Goerges, who took out a German patent in 1891. This machine has a stator similar, in all respects, to that of the three-phase induction motor. The rotor is similar to the armature of a d.c. machine and is provided with a commutator. The armature may be lap or wave wound. It has three sets of brushes per pair of poles; brushes of equi-potential are joined together to form a single rotor terminal. The rotor is connected in series with the stator, as shown in Fig. 8.1. The rotor is supplied through a series-type induction regulator transformer, by which means it is possible to obtain a whole range of speed-torque characteristics. Where a series transformer, with a fixed secondary is used, control of speed is effected by movement of the brushes.

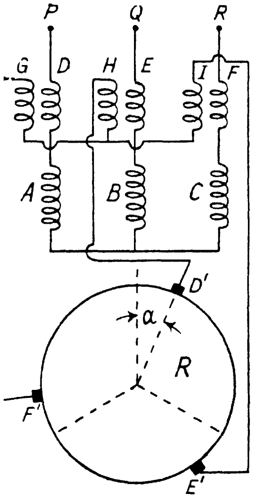


FIG. 8.1

Assume that this motor is supplied with a.c. current of frequency f. The stator current will, as in the induction motor, produce a revolving field. So also will the current in the rotor produce a revolving field. The actual revolving field in the machine is produced by the resultant of the stator and rotor ampere-turns. The magnitude of the resultant is determined, not only by the magnitude of the ampere-turns of stator and rotor, but by the position of the brushes on the commutator. That brush position, for which the ampere-turns of stator and rotor directly oppose each other, is called the "neutral position" of the brushes. It is the datum position, from which the angle of brush shift is measured.

The direction of the torque depends on the direction of movement of the brushes from the datum position. Let us call the angle of brush shift from the datum position α. When $\alpha = 0$ there is no torque, for there is no displacement of stator and rotor fields. For a clockwise movement of the brushes, from the datum or neutral position, we have clockwise rotation, and vice versa.

In Fig. 8.1, *A*, *B*, and *C* represent the three stator phases. *D*, *E*, and *F* are the primaries of the three-phase series transformers. *G*, *H*, and *I* are the secondaries of the three-phase series transformer. *P*, *Q*, and *R* are three-phase supply terminals, while *D'*, *E'*, and *F'* represent the three brush sets on the commutator of the rotor.

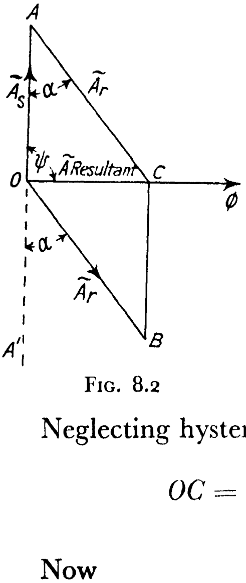


FIG. 8.2

In Fig. 8.2, *OA* represents the stator ampere-turns per pole, and *OB* the rotor ampere-turns per pole.

When $\alpha = 0$, *OB* is opposite to *OA*, and the brushes are in the neutral position. *OA* and *OB* oppose each other in this position.

Let the brushes be moved through an angle α counter-clockwise, then *OB*, the rotor ampere-turns per pole, makes an angle α with *OA'*, the neutral position.

OC = resultant ampere-turns per pole.

Neglecting hysteresis, the flux per pole is in phase with *OC*.

$$OC = \sqrt{\tilde{A}_s^2 + \tilde{A}_r^2 - 2\tilde{A}_s\tilde{A}_r \cos\alpha} = \tilde{A}_R \quad . \quad . \quad (8.1)$$

Now

$$\frac{\tilde{A}_R}{\sin\alpha} = \frac{\tilde{A}_r}{\sin\psi} \quad . \quad . \quad . \quad . \quad . \quad (8.2)$$

$\therefore$

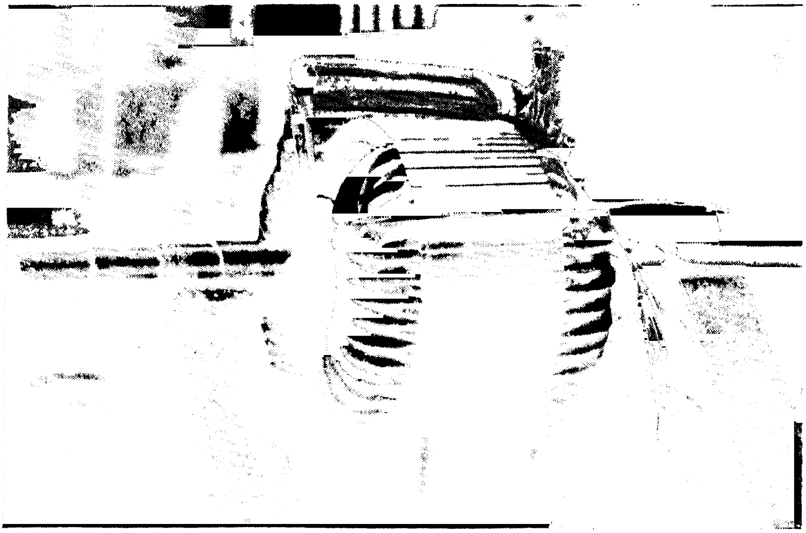
$$\sin\psi = \frac{\tilde{A}_r \sin\alpha}{\tilde{A}_R} \quad . \quad . \quad . \quad . \quad . \quad (8.3)$$

Vector Diagram for Three-phase Series Motor, Omitting Regulator

The resultant m.m.f. *OC* produces the flux ϕ, in phase with *OC* (neglecting hysteresis), and the e.m.f. which it induces in the stator winding lags the flux ϕ by 90°. Now, if the brushes are moved through an angle α, it is clear that the e.m.f.s, induced in the rotor, will be altered in time phase by the amount of the angle of brush shift, for the revolving field will cut the phase winding later with a shift of the brushes in the direction of rotation of the field.

PLATE IX. ILGNER SET WITH EXCITER SET FOR REVERSING RAIL-MILL MOTOR

(Courtesy English Electric Co., Ltd.)



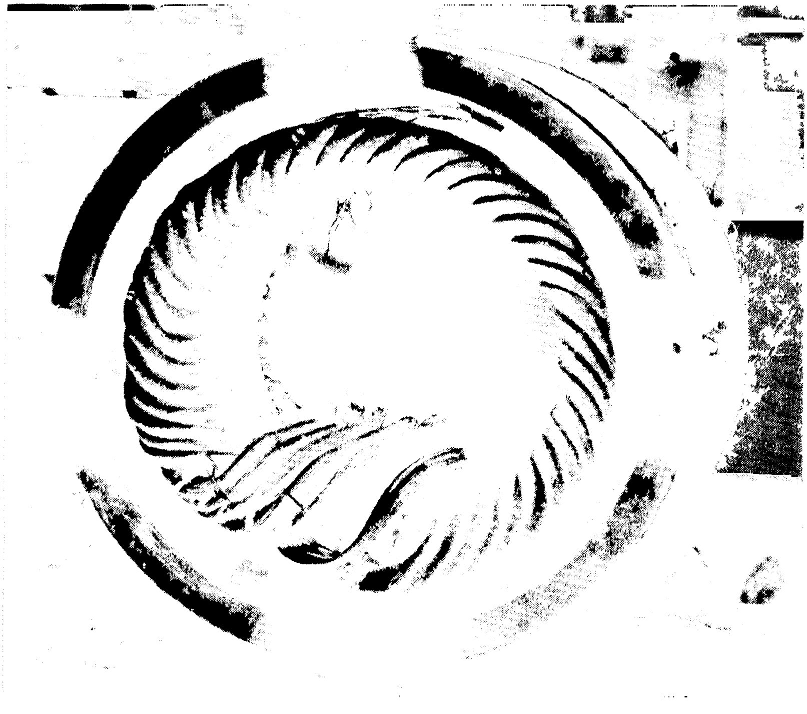


PLATE X

(*Upper*) MUSH-WOUND ROTOR IN PROCESS OF WINDING

(*Courtesy Clarke Chapman & Co Ltd*)

(*Lower*) MUSH WINDING FOR THE STATOR OF A THREE-PHASE MOTOR, IN PROCESS OF WINDING

(*Courtesy Clarke Chapman & Co, Ltd*)

In Fig. 8.3, OA = stator ampere-turns per pole and also the phase of the current

OB = rotor ampere-turns per pole; the angle $AOB = \pi - \alpha$

α = angle of brush shift from the neutral position

OC = resultant ampere-turns per pole, and phase of the flux ϕ. This is a combined space and time diagram.

$E_s = OF$ = e.m.f. induced in the stator per phase, and lags $\frac{\pi}{2}$ behind ϕ

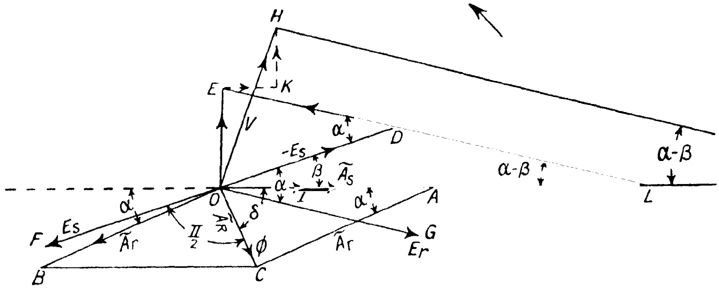


FIG. 8.3

$OD = OF = -E_s$ = component of supply volts to overcome E_s

OG = e.m.f. induced in the rotor per phase by the flux ϕ; it makes an angle $180 - \alpha$ with E_s

DE = equal and opposite to OG is the component of supply volts to overcome E_r

OE = resultant of OD and DE

EK = total resistance drop in stator and rotor, in phase with I, i.e. with OA

KH = component of supply volts to overcome the sum of the *leakage* reactance voltages in the machine

and OH = supply volts per phase

There should be a vector from the point E drawn parallel to the current vector $I(R_s + R_r')$, and a vector $I(X_s + X_r')$ leading the current by 90°. Here R_s = resistance of the stator per phase; R_r' = resistance of the rotor per phase, referred to the stator; X_s is the leakage reactance of the stator; and X_r' is the leakage reactance of the rotor per phase, referred to the stator. Since the same value of I is used for the drops, it is assumed that the delta-connected armature is replaced by an equivalent star-connected winding. Then the applied volts per phase = $OH = V$.

The e.m.f. E_s, generated in the stator by the revolving field ϕ

$$= 2{\cdot}22 \times \hat{\phi} \times Z_s \times f \times 10^{-8} \times K_1 \times K_3 = E_s \quad . \ (8.4)$$

where $\hat{\phi}$ = flux per pole (maximum value)

Z_s = number of conductors in series, per phase, on the stator

f = supply frequency

$$K_1 = \text{breadth factor for the fundamental} = \frac{\sin q_1 \frac{\lambda_1}{2}}{q_1 \sin \frac{\lambda_1}{2}}$$

where q_1 = slots per pole, per phase, in the stator

λ_1 = electrical slot pitch angle in the stator

$= \frac{180}{3q}$ for a three-phase machine

$K_3 = \cos \frac{\varepsilon}{2}$ = stator coil span factor for the fundamental

where ε = deficiency of pitch of coil from full pitch in electrical degrees

Also $$E_r = 2{\cdot}22 \times K_2 \times K_4 \times s \times f \times Z_2 \times \hat{\phi} \times 10^{-8} \quad . \ (8.5)$$

where E_r = generated volts per phase in the rotor

s = slip

K_2 = rotor breadth factor

$$= \frac{\sin q_2 \frac{\lambda_2}{2}}{q_2 \sin \frac{\lambda_2}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \ (8.6)$$

K_4 = coil span factor in the rotor

q_2 = slots per pole per phase in rotor

λ_2 = slot pitch angle in electrical degrees in rotor

It is clear that as the current varies OA and OB will vary proportionately, and since for a given brush shift α, OB is fixed in direction, the resultant vector OC, which produces the flux is fixed with respect to OA, i.e. with respect to the current. The angle DOA, in Fig. 8.3, is fixed, and so also is the angle ODE, which is equal to α, the brush shift angle. Since OE is fixed and equal to the applied voltage per phase, neglecting resistance and leakage reactance voltages, and the angle ODE is constant, it follows that the locus of the point D is a circle, i.e. as the load and speed vary the locus of D is a circle, of which OE is a chord. Produce ED to L, and the current vector to L, i.e. ED and OA produced intersect at L. Then the angle ELO is equal to $\alpha - \angle DOA = \alpha - \beta$. But the angle DOA is a fixed angle, as we have shown, therefore, the locus of L, when the load and speed vary for a given brush shift α, is another circle through the points O, E, and L, neglecting resistance and leakage reactance drops.

In Fig. 8.4 is shown the circular current locus, viz. OLE and the circle of voltages, both neglecting resistance drops and leakage reactance drops. When D coincides with E synchronous speed is reached; $E_r = 0$ and $E_s = V$. The radius of the voltage circle is given by the equation

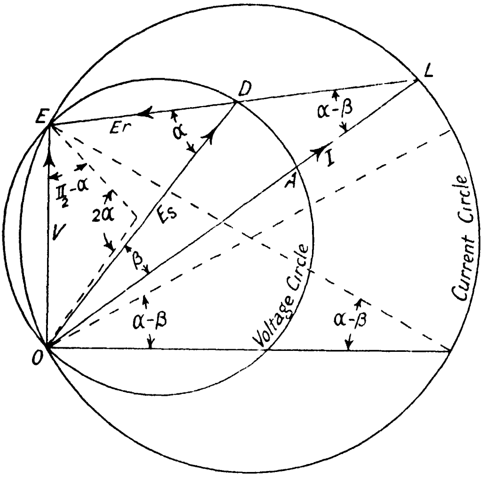


Fig. 8.4

$$2R_1 \sin \alpha = V \quad . \qquad . \; (8.7)$$

$$\therefore \qquad R_1 = \frac{V}{2 \sin \alpha}$$

At standstill $s = 1$,

$$\frac{E_r}{E_s} = \frac{K_2 \times K_4 \times Z_2}{K_1 \times K_3 \times Z_s} \quad . \qquad . \qquad . \; (8.8)$$

where Z_2 = conductors in series per phase in the rotor in the equivalent star winding.

From standstill to synchronous speed the point D, in Fig. 8.4, moves around the voltage circle, from the standstill point, in a counter-clockwise direction till it reaches the synchronous speed point E. Further movement, in a counter-clockwise sense, of the point D around the voltage circle brings the speed range above synchronism, until we reach twice synchronous speed; at which point E_r has the same value as at standstill.

The radius of the larger current circle $= R_2$,

where
$$R_2 = \frac{V}{2 \sin (\alpha - \beta)} \tag{8.9}$$

The current scale can be determined from the value of E_r at standstill ($s = 1$), or from the normal value of ϕ the flux per pole at full load and the open-circuit characteristic. It is clear that, below synchronous speed, the stator absorbs energy from the line, while the rotor gives energy to it. At synchronous speed the stator alone absorbs energy. Above synchronous speed the stator and rotor absorb energy, which is converted into mechanical energy. The torque is proportional to the product of flux per pole and the armature ampere-turns $\times$ sin α,

$$= k\bar{I}^2 \sin \alpha$$

The torque is a maximum when the rotor ampere-turns make an angle of 90° with the resultant ampere-turns OC in Fig. 8.3, i.e. with ϕ.

When $\angle OCA$ in Fig. 8.3 is a right angle, we have

$$\tilde{A}_s \cos \alpha = \tilde{A}_r$$

also
$$\frac{\tilde{A}_{\text{Resultant}}}{\tilde{A}_r} = \tan \alpha \tag{8.10}$$

Let
$$m = \frac{\tilde{A}_r}{\tilde{A}_s} = \frac{\text{ampere-turns rotor}}{\text{ampere-turns stator}} = \cos \alpha \tag{8.11}$$

Maximum torque occurs when $\tilde{A}_R$ and $\tilde{A}_r$ are at right angles

$$= K\hat{A}_R \times \tilde{A}_r$$

$$= K\tilde{A}_s \sin \alpha \times \tilde{A}_s \cos \alpha$$

Therefore, maximum torque

$$= K\tilde{A}_s^2 \sin \alpha \cos \alpha$$

$$= K\tilde{A}_s^2 \frac{\sin 2\alpha}{2} \tag{8.12}$$

i.e. maximum torque occurs when $\alpha = 45°$ or the angle $AOB = 135°$ (*see* Fig. 8.2).

Then, with this value of the brush shift

$$\frac{\tilde{A}_r}{\tilde{A}_s} = \cos 45 = \frac{1}{\sqrt{2}} \tag{8.13}$$

i.e. for maximum torque the ratio of

$$\frac{\text{rotor ampere-turns}}{\text{stator ampere-turns}} = \frac{1}{\sqrt{2}} \tag{8.14}$$

For this ratio of $\dfrac{\text{rotor ampere-turns}}{\text{stator ampere-turns}}$ the power factor, which is the cosine of the angle HOA, is low.

For this maximum torque condition, $\bar{A}_R = \bar{A}_r \tan \alpha = \bar{A}_r$ for $\alpha = 45°$.

For good power-factor conditions, usual values for

$$\frac{\bar{A}_R}{\bar{A}_r} \simeq 0{\cdot}6 \text{ to } 0{\cdot}4 \text{ and } \alpha = 22° \text{ to } 30° \quad . \quad . \quad (8.15)$$

and $$\frac{\bar{A}_r}{\bar{A}_s} \simeq 1{\cdot}17 \text{ to } 1{\cdot}08 \text{ with mean of } 1{\cdot}12$$

The angle β in Fig. 8.3 is the angle DOA

$$= \frac{\pi}{2} - \angle AOC = \frac{\pi}{2} - \delta$$

Now $$\frac{\bar{A}_r}{\sin \delta} = \frac{\bar{A}_R}{\sin \alpha} \quad . \quad . \quad . \quad . \quad . \quad (8.16)$$

$$\therefore \quad \sin \delta = \frac{\bar{A}_r \sin \alpha}{\bar{A}_R} \quad . \quad . \quad . \quad . \quad . \quad (8.17)$$

Clearly by making δ large and $> \frac{\pi}{2}$, then OD falls below OA, and it is clear that the power factor is greatly improved.

The whole series of characteristics can be plotted from Fig. 8.4 by taking various positions for D on the circle, for speeds below and above synchronous speed.

Thus, for a given value of ED, i.e. E_r, from the e.m.f. equation we can determine $s \times \phi$, and from the corresponding value for E_s, viz. OD, we can determine ϕ from the e.m.f. equation. Thus, the slip can be determined and the speed $= \omega_0(1 - s)$. Also, the torque, current, slip, input, and efficiency can be readily determined.

Characteristic Curves for the Three-phase Series Commutator Motor

Characteristic curves are shown in Fig. 8.5 for the series type of motor, from which the behaviour of the motor can be seen for speeds up to 1·5 times the synchronous speed.

One must draw a similar diagram for every brush position to that of Fig. 8.4. The angle α of brush shift is a variable parameter, which is held fixed for one set of curves; by varying α various sets of such curves are obtained. The part BC of the torque curve (Fig. 8.5) represents unstable conditions. Thus, the torque falls with the speed near standstill. This part of the curve can be modified by suitable design.

This motor has admirable characteristics for applications such as hoists, winches, etc. It resembles, in many ways, the d.c. series motor, but instead of one torque-speed curve being obtained, as in the series d.c. motor, with the three-phase series type, a whole series of such curves are obtainable by brush shift. Instead of shifting the brushes, it is usual to use a series induction regulator, which is a rotary current transformer with movable secondary. Thus, the phase of the applied voltages to the rotor can be varied by moving the secondary of the regulator. This effects the same results as shifting the brushes on the commutator, but it also reduces the voltage on the armature, which is necessary for the reason which we now develop.

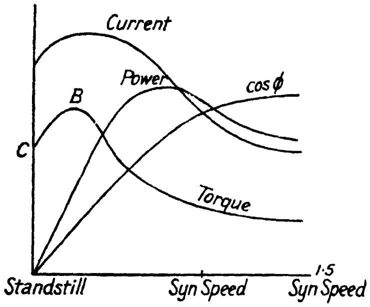


FIG. 8.5

The great drawback with these motors is the difficulty involved in commutation. In addition to the reactance voltage in the coil undergoing commutation, which is present in the d.c. machine, and which is due to the change of leakage flux as the current in the coil changes, when it passes the brush, i.e. when the coil passes the brush, the current in the coil changes from the value in one phase to that of the consecutive phase, there is an additional voltage, generated in the coil, due to the rotating field. This voltage, usually known as the transformer voltage, has its maximum value at standstill, viz.

$$e_t = 4{\cdot}44 \times \hat{\phi} \times T_c \times sf \times 10^{-8} \quad . \quad . \quad (8.18)$$

where T_c = turns per coil
s = slip
f = supply frequency

Its value is given in equation (8.18).

Now it is clear that unless special precautions are taken, such as the use of the Schwarz patent winding, or the use of another scheme by the author (now being patented), this transformer voltage must be kept low enough to allow good commutation to be obtained. In this case high contact-resistance brushes must be used and high-resistance connections between the coils and the commutator. This transformer voltage in the coil must, therefore, not exceed 3 V, approximately. This can only be obtained by using the following—

(*a*) a small flux per pole;
(*b*) single-turn coils in the armature;
(*c*) a low supply frequency;
(*d*) a large number of poles.

It will be seen that the voltage which can be supplied to the armature is limited, for on a given diameter of commutator, which is limited by the peripheral speed permissible, there are three sets of brushes per pole pair. On an eight-pole machine there will be twelve brush sets, and if we allow, say, 6 in. for the circumferential distance between brush sets, this will allow us about thirty segments between brushes, or about 100 V supply voltage. The diameter of the commutator, in this case would be 23 in. and the peripheral speed of the commutator about 4500 ft/min at 750 r.p.m., assuming the machine is running at synchronous speed on 50 c/s.

It is a characteristic of these machines to have a large diameter of commutator, with a large number of segments. A limit is set to the diameter by the peripheral speed of both armature and commutator, and the higher this is, the more difficult do the commutating conditions become. It must be remembered, too, that speeds of two or three times synchronous speed, or more, may be required. Thus, not only is the voltage supply to the commutator limited in magnitude, but the electric loading of the armature is also limited by the diameter limitation. It follows, therefore, that there is a definite limit to the output *per pole pair*, of the order of about 20 kW. This does not apply with the auxiliary winding used by Dr. Schwarz. With the Schwarz winding the limit to the voltage per segment is removed, and there is not the need to use a small flux per pole. Indeed, Dr. Schwarz has removed the one great drawback to these machines and has placed them on a more or less equal footing with the d.c. machines, as far as commutation is concerned. With the usual design, these machines are characterized by several features, common to all a.c. commutator motors, viz. large diameter, short axial length, large commutator diameter, large number of brush sets, and small radial depth of iron above the stator teeth and below the rotor teeth.

We have already mentioned the various voltages which cause commutation trouble. Consider again the reactance voltage, due to current change in the coil as it passes under the brush.

This is the voltage which is present in d.c. machines, but in the a.c. commutator motor, the current change on passing under the brush is different from that in the d.c. machine. Before the coil passes under the brush, it carries current of one phase of the armature, say, that of phase 3 for example; when it passes the brush it passes into the consecutive section of the armature, which is carrying current of phase 1, for example. Thus, the currents in the sections of the armature between brushes are assumed to be sinusoidal and differ by 120° in time phase.

The current will follow the values of the sine curve of one phase till it reaches the brush. After leaving the brush, the current, in the coil considered, must have the value appropriate to the values carried by the branch of the armature, into which it has entered.

Thus, the amount by which the current changes, during commutation, is equal to the difference of the instantaneous values of the current in two adjacent sections of the armature and is, therefore, equal to the instantaneous value of the brush current. The maximum change of current in the coil is equal to the maximum brush current. The e.m.f. of self-inductance in the short-circuited coil, due to the current change may be regarded as in phase with the brush current. The time of commutation, which is the time the coil is short-circuited by the brush, depends on the width of brush and the peripheral speed of the commutator and is very short, probably between $\frac{1}{1000}$ and $\frac{2}{1000}$ sec.

This e.m.f., due to current change in the coil undergoing commutation, is similar to that in the d.c. machine. It retards commutation. In the d.c. machine it is neutralized by a rotational e.m.f., generated by rotation in an interpole flux, and the same is also true in the a.c. single-phase commutator motor. This e.m.f. can be kept down to a low enough value by (*a*) using single-turn coils in the armature; (*b*) keeping the electric loading to a sufficiently low value; (*c*) using narrow brushes, so as not to short-circuit more than one coil at any time; and (*d*) keeping down the peripheral speed of the commutator.

The voltage, however, which is the most serious in this machine and which has been responsible for its limitations, both in design and application, is the voltage which is generated in the short-circuited coil by the *main* flux. At standstill this voltage is a maximum and becomes zero at synchronous speed.

Its value is given by equation (8.18).

The Schwarz Winding

To Dr. Schwarz we are indebted for one solution of the problem of the voltage generated in the short-circuited coil by the main flux. He uses another winding, called the "auxiliary" winding, in the same slots as the main winding. The main winding is usually an ordinary d.c. lap winding connected to commutator segments. The armature slots are deep, and between the main winding and the auxiliary winding, in the same slots, are placed sheet-iron interleaves, which form an additional magnetic path for the auxiliary field of the auxiliary winding. The auxiliary winding is a lap winding, connected to the same commutator segments as the main winding, but the auxiliary winding coils have twice as many turns as the coils of the main winding, but the coil span of the auxiliary coils is but one-third of the pole-pitch. The coil span factor for the auxiliary winding is $\cos 60 = \frac{1}{2}$, and so, for the same voltage per segment, the number of turns per coil must be twice that of the main coils. The two windings—main and auxiliary—are connected in parallel to the same commutator segments. The arrangement is shown in

Fig. 8.6. In Fig. 8.6 the auxiliary coils x_1 and x_2 are in the same slots, and form a transformer link with almost perfect magnetic coupling; with common fluxes ϕ_h' and ϕ_h''. x_2 is connected with segments 2 and 3, and is, therefore, in parallel with the main winding b_1, which is on the left of slots *Nb* and *Nb'*; x_1 is connected in parallel with a_2 and lies in slots *Nx* and *Nx'*. The energy of the leakage

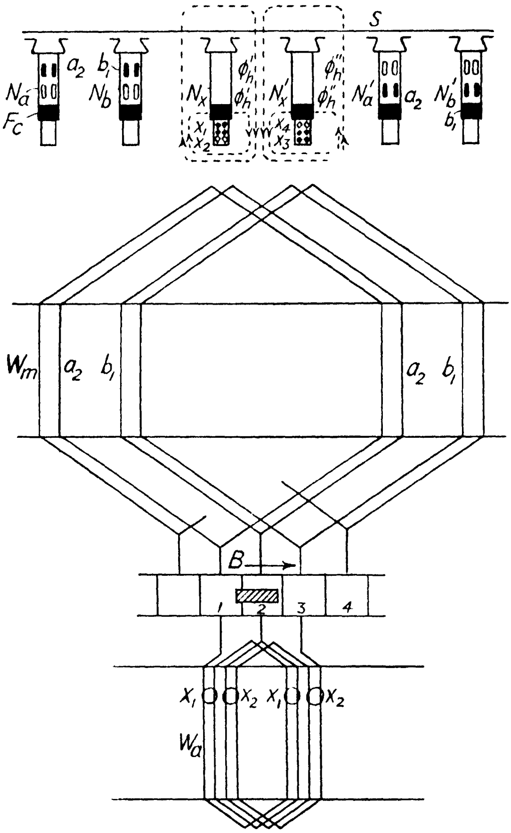


FIG. 8.6

field, which produces the sparking, is transferred from the coil a_2 to x_1 and x_2 and to b_1. The reactance of the winding a_2 and the reactance of b_1 are almost halved by the assistance of the transformer x_1 and x_2. Since b_1 lies in the same slot as b_2, there exists a transformer linkage, which continues through the next auxiliary groups.

In the words of the inventor: "In this way a parallel connection of windings is obtained in both directions round the armature, so that the operative reactance of the winding, while commutating is reduced to a fraction of its normal value."

This arrangement operates satisfactorily at the interruption of the transformer currents, during the commutation period. The net result is that the voltage per segment can be increased until it is no longer the voltage, but the losses produced by the transformer currents that determine the maximum limit for the segment voltage. The inventor has found it possible to increase the current density in normal graphite brushes from 7 or 8 A/cm^2 to 12 or 14 A/cm^2, without harm. Thus, he is able to use large flux per pole, reasonably large voltage per segment, and to reduce the size of the commutator. There is little doubt that this invention has largely removed the limits to design and application.

In some cases intermediate segments between the main segments are provided for the auxiliary winding. In Fig. 8.7 one turn of the main winding is shown, and the winding group of the auxiliary winding, which is connected in parallel with it. The main winding is connected to segments 1 and 2, and so are the beginning and end of the auxiliary winding, which consists of two windings in series. The connection between the first and second auxiliary windings is brought out and connected to segment 1′. The voltage between segments 1 and 2 is divided into two equal parts at segment 1′. The main current now flows through 1′, and the two auxiliary windings are connected in parallel, and have no reactance, since the magnetic effects of the component currents are cancelled.

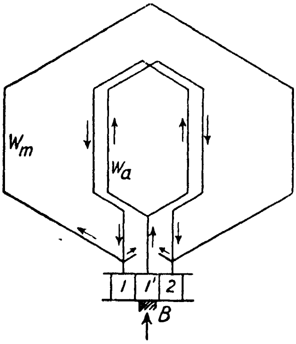


Fig. 8.7

By the use of these devices it is possible to build motors with a larger horse-power per pole.

The reader is referred to the paper by Dr. Ing. Benno Schwarz, Rheydt, in *Electrotechnik und Maschinenbau*, on "Recent Developments of the Stator-fed Polyphase Shunt-wound Commutator Motor," 24th February, 1935.

There is little doubt that this invention of Dr. Schwarz has profoundly affected these motors, removing limits which have, hitherto, prevented the more extensive use of these machines.

Another Method of Approach (due to Shuttleworth)

Shuttleworth has given another method of approach* which will, perhaps, give a clearer picture of this motor and its analysis. In this method one single-phase is used, which is permissible, if the other phases are not neglected in their effect. Briefly, he adopts the device, commonly used in the theory of the single-phase repulsion motor, of resolving the ampere-turns of the stator into two components—

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* Shuttleworth, N.: "Polyphase Commutator Machines and their Applications," *Journal I.E.E.*, March (1915).

(*a*) along the brush axis;

(*b*) at right angles to the brush axis (*see* Fig. 8.8).

The stator phase is displaced by angle α from the rotor phase and S = number of stator turns *per pole per phase* (effective)

R = number of rotor turns per pole per phase (effective)

p = number of poles

$\hat{I}$ = maximum current per phase

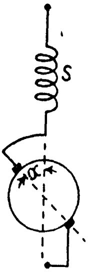

FIG. 8.8

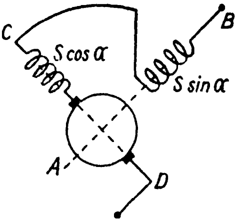

FIG. 8.9

Exciting ampere-turns along the brush axis $= K(S\cos\alpha - R)\hat{I}$, the factor K takes into account the effect of the other phases. $K = \frac{3}{2}$ for a three-phase machine.

It will be remembered that the amplitude of the resultant ampere-turns for m phases $= \frac{m}{2} \times$ amplitude for one phase. Now the turns $S\sin\alpha$ are called the exciting turns and the axis AB the "excitation" axis, and the corresponding flux along AB the "excitation" flux.

The total exciting ampere-turns $= KS\sin\alpha\hat{I}$.

The resultant of the two components of exciting ampere-turns, which are at right angles

$$= K\hat{I}\sqrt{S^2 + R^2 - 2SR\cos\alpha} \quad . \quad . \quad (8.19)$$

The flux per pole, due to this excitation, may be obtained from the open-circuit curve. If there is no saturation, then

$$\phi = K_0 K\hat{I}\sqrt{S^2 + R^2 - 2SR\cos\alpha} \quad . \quad . \quad (8.20)$$

where K_0 = constant

If the currents in the exciting winding alone produced the field, the e.m.f. between brushes, generated by rotation, would be exactly opposite to the current, i.e. 180° out of phase. There is another excitation at right angle, viz. $S\cos\alpha - R$ per phase.

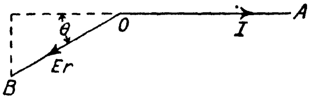

FIG. 8.10

This causes the rotational voltage to differ from the phase of the current by 180 — θ, as shown in Fig. 8.10, here $\tan\theta = \dfrac{S\cos\alpha - R}{S\sin\alpha}$

Fig. 8.10 shows the relation of voltage and current.

$E_r = OB$ = rotational voltage.

The voltage induced in the exciting winding of the stator per phase

$$= \sqrt{2}\pi \times f \times \hat{\phi} \times pS\sin\alpha \times 10^{-8} \quad . \quad . \quad (8.21)$$

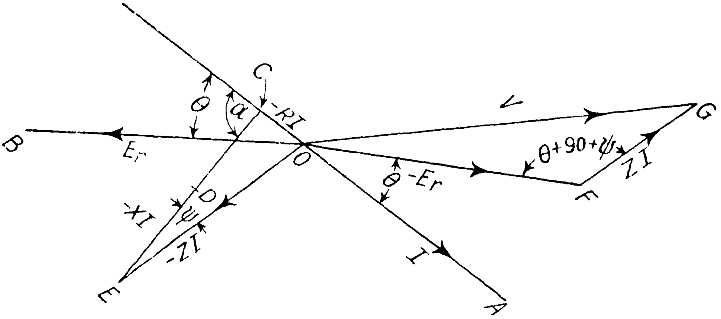

Fig. 8.11

The voltage induced in the rotor and the transformer axis of the stator winding

$$= \sqrt{2}\pi \times f \times \hat{\phi} \times p(S\cos\alpha - R) \times 10^{-8} \quad . \quad (8.22)$$

The resultant of these two voltage (which are at right angles)

$$= \sqrt{2}\pi \times \hat{\phi} \times f \times p\sqrt{S^2 + R^2 - 2SR\cos\alpha} \times 10^{-8} \quad (8.23)$$

where $\hat{\phi}$ is given by equation (8.20)

These voltages lag I by 90°.

To this resultant voltage we must add the leakage reactance voltages in stator and rotor.

In Fig. 8.11, OB = counter e.m.f. in armature

OA = current vector

OC = resistance volts (total for stator and rotor)

CD = sum of leakage reactance voltages in stator and rotor

DE = vector sum of voltages, equations (8.21) and (8.22), and given by equation (8.23)

OE = impedance voltage, due to self and mutual impedances

$OF = -E_r$ = component of supply volts to overcome E_r

$ZI = -OE = FG$

OG = supply volts per phase = constant

Now $OF = -E_r \propto$ speed of rotor $\times \phi \propto$ speed $\times$ current (if no saturation).

Now θ is constant for α constant and ψ is also constant.

Therefore, angle OFG is constant.

Therefore, the locus of F is a circle with $OG = V$ as chord.

The angle subtended by OG, at the centre $= 2[90 - (\theta + \psi)]$, and the radius of the circle R, for (θ positive),

$$= \frac{V}{2 \cos (\theta + \psi)} \qquad . \quad . \quad . \quad (8.24)$$

Fig. 8.12 shows the circle surrounding the points O, G, and F, but θ is made negative to improve the power factor, but

$$\tan \theta = \frac{S \cos \alpha - R}{\sin \alpha} \qquad . \quad . \quad . \quad (8.25)$$

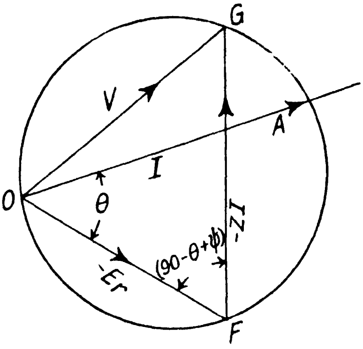

Fig. 8.12

Therefore, $R > S \cos \alpha$ when θ is negative.

Looking at Fig. 8.12, this will mean bringing OF below OA.

The radius of the circle (for θ negative)

$$= \frac{V}{2 \cos (\theta - \psi)} . \qquad . \quad . \quad . \quad (8.26)$$

At standstill OF is zero, and the current, which is proportional to FG, approaches a maximum value. As the speed increases, OF increases and GF decreases; I, therefore, decreases. At very high speed OF approaches OG and the current tends to zero. FG, taken to the proper scale, may represent the magnitude of the current, and the phase of the current is always θ with respect to $-E_r$ or OF. The value of the power factor is always known for all values of the current, for the current determines FG in Fig. 8.12, and once the point F is determined, $-E_r$ is known and I makes an angle of θ with $-E_r$ always.

It is clear that unity power factor, at normal speed, is obtainable by suitable design.

It is necessary to draw a separate diagram for each value of α, for θ depends on α, and each value of α gives a definite series of characteristics.

Now FG may be large, even with moderate value of the current. Since the reactance depends on $R^2 + S^2 - 2SR\cos\alpha$, when α is near 180°, this is large, when $\cos\alpha$ is negative. The current then is small, even at standstill. This is the usual starting position.

If we neglect the resistance drop ψ becomes 0 in Fig. 8.11 and the circle diameter

$$= \frac{V}{\cos\theta} \quad . \quad . \quad . \quad . \quad (8.27)$$

Let $\dfrac{R}{S} = m$

then
$$\tan\theta = \frac{S\cos\alpha - R}{S\sin\alpha} = \frac{\cos\alpha - m}{\sin\alpha} \quad . \quad . \quad (8.28)$$

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1 + \tan^2\theta}}$$

$$= \frac{1}{\sqrt{1 + \dfrac{\cos^2\alpha - 2m\cos\alpha + m^2}{\sin^2\alpha}}} \quad . \quad . \quad (8.29)$$

$$= \frac{\sin\alpha}{\sqrt{1 + m^2 - 2m\cos\alpha}} \quad . \quad . \quad . \quad (8.30)$$

The torque is proportional to $\phi \times I \times \cos\theta$ and

$$\phi = KK_0I\sqrt{R^2 + S^2 - 2RS\cos\alpha} \quad . \quad . \quad (8.31)$$

$$= K_2IS\sqrt{m^2 + 1 - 2m\cos\alpha}$$

$$\therefore \quad \underset{\text{torque}}{T} = \text{constant} \times I^2 \sin\alpha \quad . \quad . \quad (8.32)$$

This is a maximum when I is a maximum, i.e. when FG is a maximum in Fig. 8.12. FG is obviously a maximum when it becomes a diameter of the circle, i.e.

$$= \frac{V}{\cos\theta} \quad . \quad . \quad . \quad . \quad (8.33)$$

but FG (neglecting resistance drop) $= X_TI$

where $X_T = X_L + X_M$

$$= X_L + K_3 \times [R^2 + S^2 - 2RS\cos\alpha] \quad . \quad . \quad (8.34)$$

where X_L = sum of leakage reactances of stator and rotor

X_M = mutual reactance of both

$$X_M I = \sqrt{2}\pi \times \hat{\phi} \times f \times p\sqrt{S^2 + R^2 - 2SR\cos\alpha} \times 10^{-8} \quad . \quad (8.35)$$

$$= \sqrt{2} \times \pi \times K_0 K I\sqrt{S^2 + R^2 - 2SR\cos\alpha}$$

$$\times f \times p \times \sqrt{S^2 + R^2 - 2SR\cos\alpha} \times 10^{-8} \quad . \quad (8.36)$$

$$X_M = K_3 \times (R^2 + S^2 - 2RS\cos\alpha) \quad . \quad . \quad (8.37)$$

I_{max} = maximum r.m.s. current

$$= \frac{V}{\cos\theta[X_L + K_4(m^2 + 1 - 2m\cos\alpha)]} \quad . \quad . \quad (8.38)$$

where $K_4 = K_3 S^2$ . . . . . (8.39)

maximum torque

$$= \text{constant} \times I_{\max}^2 \sin\alpha$$

$$= \text{constant} \times \frac{V^2}{\cos^2\theta[X_L + K_4(m^2 + 1 - 2m\cos\alpha)]^2} \quad (8.40)$$

THE THREE-PHASE SHUNT COMMUTATOR MOTOR

We have already seen that the three-phase series motor varies its speed with the load, in a similar manner to the d.c. series motor. It has long been recognized, however, that a variable speed three-phase motor which has a shunt characteristic is desirable, i.e. when a given speed is obtained, it is desired that the speed shall not vary much with the load. In this respect, the three-phase shunt motor resembles the d.c. shunt motor. This motor has an outward appearance, similar to the three-phase series motor, i.e. a stator with a three-phase winding, like an induction motor, and a rotor with an ordinary d.c. winding connected to a commutator. The armature winding is usually of the lap type. The commutator has three brush sets per pole pair; similar brush sets are joined together to form a single terminal. There are three such terminals.

Both stator and rotor are supplied in parallel, with voltages of the same frequency, but of different magnitude. To vary the speed, we have seen that a voltage must be impressed on the rotor winding, having the same direction and frequency as the e.m.f. induced in the winding by the rotating field. If the impressed secondary voltage is in phase with the induced e.m.f., the speed will rise, for a reduced slip is required to produce the current and torque. If the impressed e.m.f. is 180° out of phase with the induced e.m.f., the slip will increase and the speed fall. By adjusting the magnitude and phase of the impressed voltage, one has control of speed and power factor. The impressed voltage must have the same frequency as the e.m.f. induced in the rotor.

But, it may be asked, how can one supply the rotor with a voltage having the same frequency as the stator supply voltage, and at the same time have slip-frequency voltage impressed on the rotor conductors?

The speed of the revolving field, in revolutions per minute

$$= \frac{120f}{p}$$

where f = frequency
and p = number of poles.

The frequency of the voltage generated at the brushes—*which are fixed in space—is the line frequency*, while the frequency of the e.m.f.s in the conductors of the rotor is of slip frequency, depending on the *relative motion* of field and rotor.

Thus, the brushes of a three-phase commutator motor may be connected to the same source of supply as the stator. A variable secondary voltage has generally been supplied by brush shifting, as in the Schräge motor. The constant secondary voltage of a single-stage rotary controller is not suitable for the whole range of speed, so it is usual with single-stage rotary controllers to provide brush shift in conjunction.

Brush shifting can be avoided by the methods described below.

The Single-stage Rotary Controller

In this single-stage rotary controller, the stator windings are A, C, E; B, D, and F are auxiliary windings, usually situated on the stator (*see* Fig. 8.13).

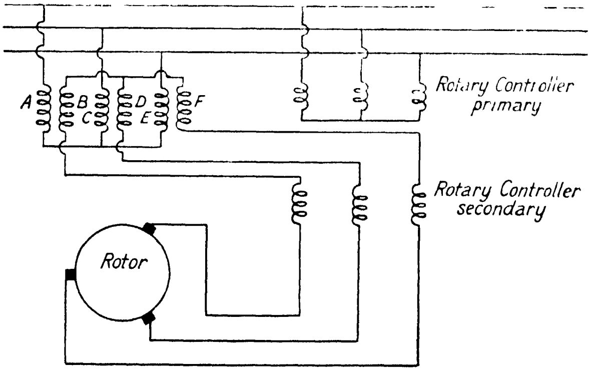

FIG. 8.13

The primaries of the stator, which are connected in star, and the primary of the rotary controller are connected to the supply lines. The secondaries of the rotary controller and the secondaries of the current transformer are connected in series to the rotor. The magnitude of the secondary voltage remains constant and only the

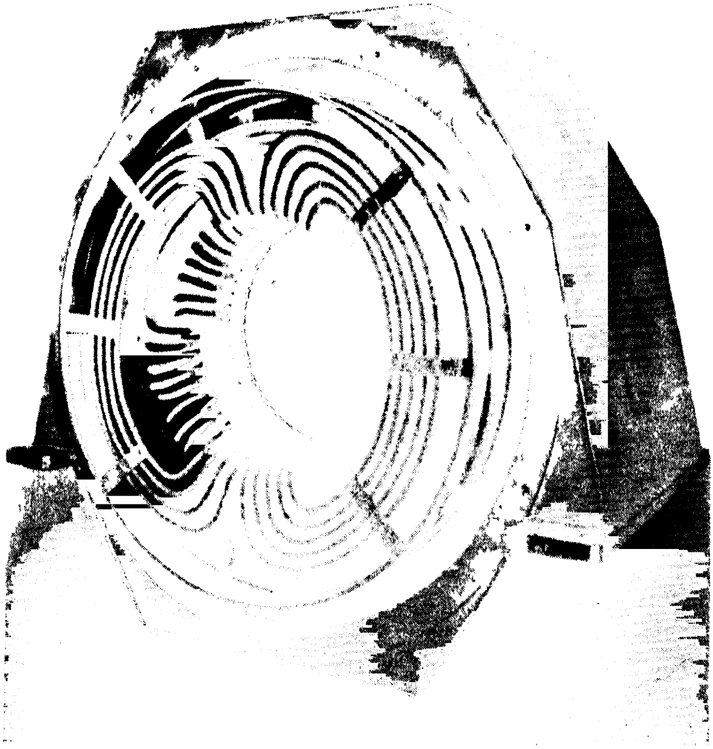

Plate XI. Three-phase, Three-plane, Split-phase, Hair-pin Winding

(Courtesy English Electric Co., Ltd.)

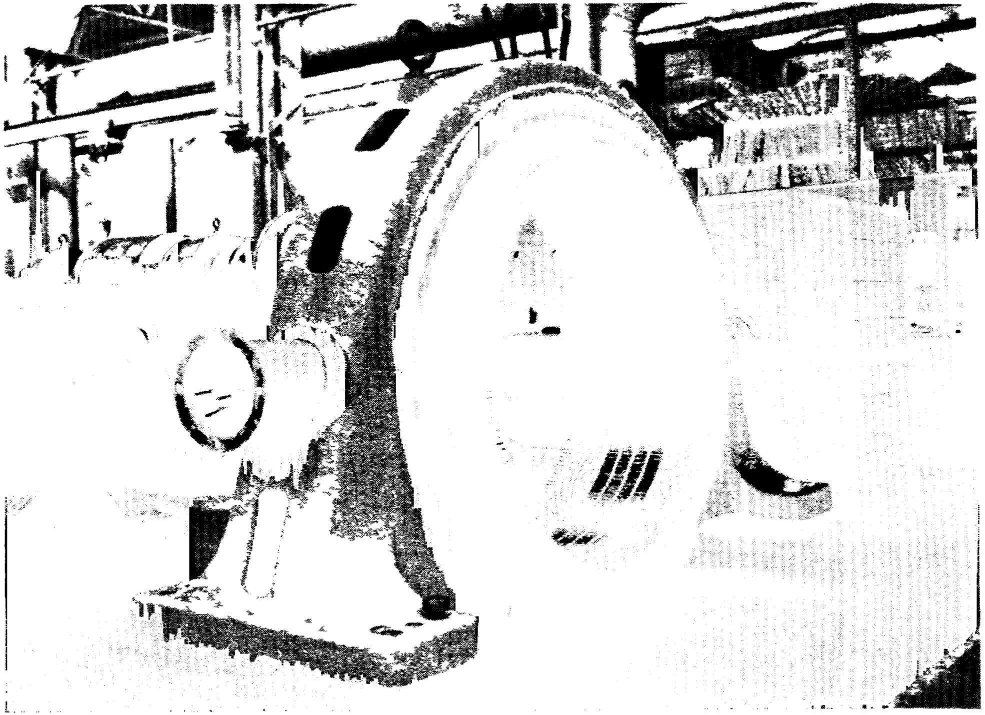

PLATE XII. A Two-bar-per-slot, Two-layer, Hair-coil, Wave-wound Stator Winding

[illegible]

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phase shifts, according as the point P moves over the circle (*see* Fig. 8.14).

Let $OA = E_{2s}$ $(s = 1)$, i.e. the voltage per rotor phase induced at standstill

AP = voltage in secondary of rotary transformer

AB = component of impressed voltage in phase opposition to E_2 and so speed reducing

PQ = voltage from secondary of current transformer B.

For the controlled range of speeds the wattless component of AP is much greater than is desirable, causing large magnetizing currents in the rotor. The demagnetizing component, introduced by the current transformer, viz. PQ, gives, with AP, the resultant voltage AQ, and the locus of Q is a circle, whose centre is displaced by the amount PQ from the circle followed by P.

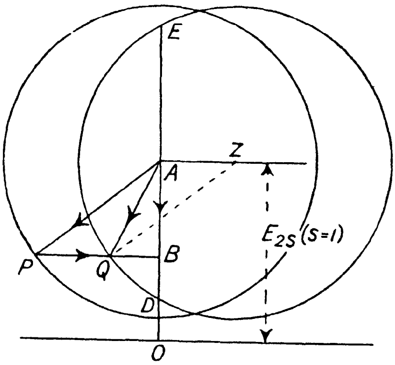

FIG. 8.14

The maximum speed corresponds to the point E, when Q falls on E, for then we have added to E_{2s} $(s = 1)$ the voltage AQ. The minimum speed is obviously at D, when Q falls on D. AB corresponds to the induced armature voltage.

If ω_s = synchronous speed

s = slip

and ω = actual speed

$$\omega = \omega_s(1 - s) \qquad (8.41)$$

Also

$$E_2 = sE_{2s} \qquad (8.42)$$

$$\therefore \quad E_{2s} - E_2 = E_{2s}(1 - s) \qquad (8.43)$$

$$\text{and } \therefore \quad \frac{E_{2s} - E_2}{E_{2s}} = 1 - s = \frac{\omega}{\omega_s} \qquad (8.44)$$

$$\therefore \quad \omega = \omega_s \times \frac{E_{2s} - E_2}{E_{2s}} \qquad (8.45)$$

and $E_{2s} - E_2 = OB$ (in Fig. 8.14).

The arc of the circle, between D and E in Fig. 8.14, represents the range of $\omega_{\min}$ to $\omega_{\max}$.

$$\omega_{\min} = \omega_s \frac{(E_{2s} - AD)}{E_{2s}} \qquad (8.46)$$

and

$$\omega_{\max} = \omega_s \frac{(E_{2s} + AE)}{E_{2s}} \qquad (8.47)$$

and $E_{2s} = OA$ (in Fig. 8.14).

Without brush shifting, Dr. Schwarz, states that a speed range of 1 : 2 to 1 : 25 is obtainable in an economical manner.

The Two-stage Rotary Controller

We have two rotary potential transformers, primaries *B* and *C* and the stator windings of the motor connected to the supply lines. The secondaries of the two rotary potential transformers *D* and *E*, and the secondaries of the current transformers *F*, *G*, and *H* are connected in series with the armature of the motor. (*See* Fig. 8.15.)

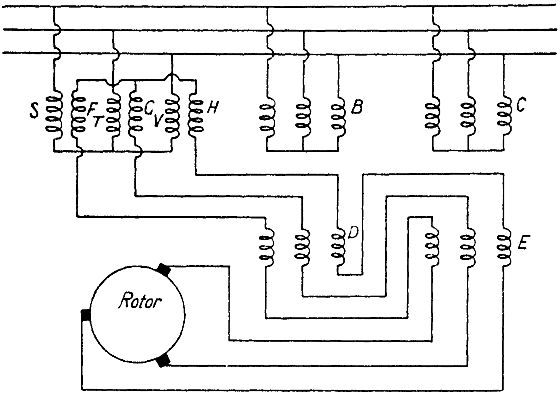

Fig. 8.15

In this case, the voltage AQ, which is the demagnetizing voltage, is constant over the whole range of speed. The idea, then, is to produce voltages in the rotary transformers, which have a speed controlling component, in phase, or in phase opposition with the induced rotor voltage, and also a demagnetizing component in quadrature with E_2.

Vector Diagram of Shunt Motor

Fig. 8.16, due to Blondel, shows the circle locus of the primary current, viz. the circle *HBP*, and the circle *HKd*, with centre Q the locus with brushes short-circuited.

OA = magnetizing current per phase and phase of the flux

OC = e.m.f. induced in the rotor per phase $= e_2$

$= 1{\cdot}84 Z_2 \times \hat{\phi} \times s \times f \times 10^{-8}$

where Z_2 = conductors in series per phase between two neighbouring brushes

$\hat{\phi}$ = maximum flux per pole, produced by the resultant of stator and rotor ampere-turns

$OE = V_2 =$ applied voltage per phase to the rotor

$OD = E_{2R} =$ resultant voltage of OE and OC

$I_2 =$ rotor current per phase lagging by an angle ϕ_2

$AB =$ rotor current, referred to the stator

$OB =$ stator current per phase

$OR = -e_1 =$ component of applied volts to the stator to overcome the induced voltage, due to flux ϕ

$= \sqrt{2}\pi \times \hat{\phi} \times f \times T_s \times K_1 \times K_3 \times 10^{-8}$ volts

where $T_s =$ turns in series per phase

$K_1 =$ breadth factor

$K_3 =$ coil span factor

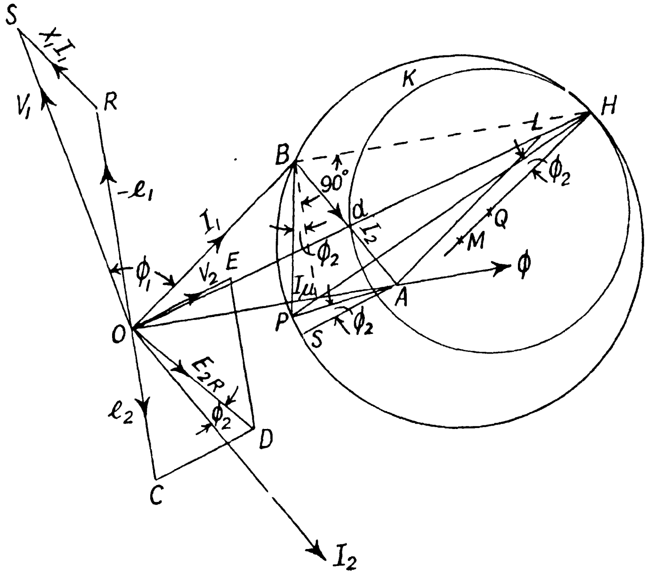

FIG. 8.16

The secondary current I_2 consists of two components, viz. BP and AP; BP is produced by e_2 and equals $\dfrac{e_2}{R_2 + jX_2}$

The r.m.s. value of this component

$$= \frac{1{\cdot}84 \times Z_2 \times \hat{\phi} \times s \times f \times 10^{-8}}{\sqrt{R_2^2 + X_2^2}}$$

The PA component is produced by V_2, the applied volts per phase, and $= \dfrac{V_2}{\sqrt{R_2^2 + X_2^2}}$

Assuming a constant secondary reactance, which is not strictly true, the point P is fixed for constant value of V_2 and brush-shift angle α. Draw BH and draw, also, OH making an angle of 90° with V_1 or OS. Then OH represents the ideal short-circuit current, and is proportional to the total leakage flux which balances the voltage V_1, assuming no resistance present. H is a fixed point since OH is fixed, and the angle $HBP = 90 + \phi_2$. Since P is a fixed point and H also fixed, PH is fixed, and it subtends an angle $90 + \phi_2$ at B; it follows the locus of the point B is the circle PBH, of which PH is a chord.

Let M be the centre of this circle. Then the centre is found by bisecting PB and BH and drawing perpendiculars to intersect at M. The centre clearly lies on the line MH, which makes an angle ϕ_2 with PH. AL, parallel to I_1, represents the primary leakage flux and is proportional to I; OL is proportional to the total flux linking the stator. In our diagram there appears the angle SOE between V_1 and V_2. Actually the same source supplies both stator and rotor, the rotor being supplied from a transformer. To obtain the line current, we must add vectorially the current in the primary of the transformer feeding the rotor. If m is the ratio of the transformer, we must take a current $= \frac{I_2}{m}$ and add it vectorially, in such a way that it makes the same angle with both V_1 and V_2.

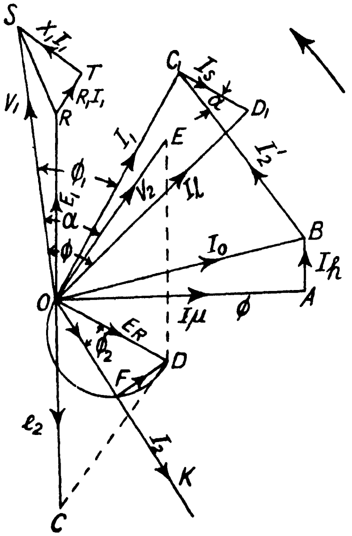

FIG. 8.17. CLEARER VECTOR DIAGRAM OF SHUNT MOTOR

So all we need to do is to add $\frac{I_2}{m}$ at the angle α of brush shift, to I_1. The total line current, obtained by this vector addition of I_1 and $\frac{I_2}{m}$ at angle α with I_2', gives another circle, and over the motor range, the total current is smaller than the stator current, since the rotor returns energy to the supply circuit.

Fig. 8.17 gives a clearer vector diagram for the shunt motor. In this diagram—

OA = vector of magnetizing current and direction of flux

$OC = e_2$ = e.m.f. induced in rotor per phase at slip s by ϕ, lagging 90° behind ϕ

$OE = V_2$ = applied voltage per phase to the rotor

OD = resultant voltage per phase of the rotor = vector sum of OE and $OC = E_R$

OF = secondary current in rotor, viz. I_2, lagging by angle ϕ_2 behind OD, and

$$= \frac{E_R}{Z_2}$$

where Z_2 = impedance of the rotor per phase

$$\tan \phi_2 = \frac{X_2}{R_2}$$

where X_2 = leakage reactance of the rotor per phase

R_2 = resistance per phase of rotor

$OF = R_2 I_2$

$FD = X_2 I_2$

$$BC_1 = I_2' = -\frac{T_2 f_2}{T_1 f_1} \times I_2$$

where T_2 = turns in series per phase in the rotor

T_1 = turns in series per phase in stator

f_1 = winding factor of stator = $K_1 \times K_3$

f_2 = winding factor of rotor = $K_2 \times K_4$

AB = watt component of no-load current to overcome hysteresis loss

I_0 = no-load current per phase = OB

OC_1 = primary current of motor

$C_1D_1 = I_s = I_2 \times \frac{V_2}{V_1}$ = current in primary of transformer supplying the rotor

C_1D_1 is added at an angle of α with I_2', therefore—

OD_1 = vector of *line* current, including the current taken by the stator and that taken by the primary of the transformer, supplying the rotor = I_l

OR = component of impressed volts to overcome the back e.m.f. generated in the stator per phase, due to ϕ

$R_1 I_1$ = resistance drop in the stator per phase

$X_1 I_1$ = leakage reactance drop in the stator per phase

$OS = V_1$ = applied volts per phase to the stator

$\cos \angle C_1OS$ = power factor of the stator

$\cos \angle SOD_1 = \cos \phi$ = power factor of the motor

Input $= \sqrt{3}\bar{V}_1\bar{I}_l \cos\phi$ (watts).

Electrical power delivered by the rotor to the line

$$= \sqrt{3}\bar{V}_2\bar{I}_2 \cos \angle EOK$$

Gross output of motor = input to stator — output of rotor (electrical) — I^2R loss in stator and rotor — iron loss in stator and rotor.

THE SCHRÄGE MOTOR

In this motor, e.m.f.s are injected into the secondary circuit of the motor, in accordance with the principle that speed control is effected when those e.m.f.s are either in phase, or in phase opposition, with the e.m.f.s in the rotor, induced by the revolving field. The primary of this machine is the rotor and the secondary the stator. The motor resembles the induction motor, but on the *rotor* are two sets of windings; one set, shown as a star-connected winding P in Fig. 8.18, is connected to three slip-rings, to which the a.c. voltage is brought. In the same rotor slots, is another winding; an ordinary d.c. lap winding, connected to a commutator and shown as winding C in our diagram. On the commutator there are three complete sets of brushes per pole pair, but each set is divided in two half-sets, and these half-sets are arranged to move in opposite directions on the commutator.

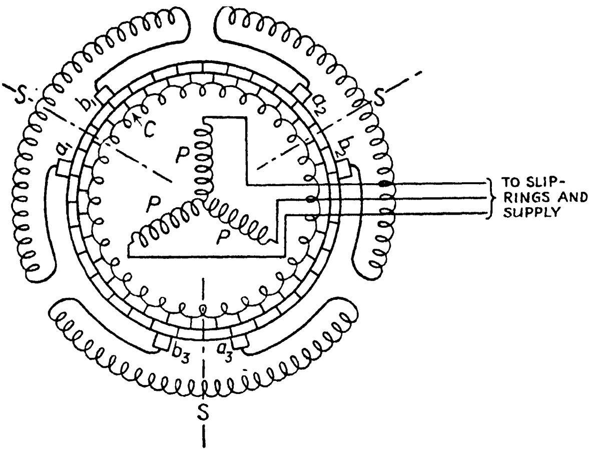

FIG. 8.18

Thus, on the two-pole machine, there are six half-sets of brushes. Imagine the two half-sets to be in line on the commutator and connected to separate brush spindles, which are moved by a lever mechanism in opposite directions on the commutator. The stator has a three-phase winding and each phase is connected to the two halves of a brush set. Thus, one phase on the stator is connected to half-brush sets a_1 and b_1, and similarly for the other phases. As before mentioned, these windings on the stator form the secondaries of the motor. Virtually it is an ordinary induction motor with an auxiliary winding on the rotor (i.e. the winding connected to the commutator).

A rotating field is produced by the three-phase currents, which rotate in the opposite direction to that of the rotor. E.m.f.s of slip frequency are generated in the secondary on the stator, as in any induction motor. Since the speed of the rotating field, relative to the rotor, is always the same, namely n_s, the frequency of the e.m.f.s induced in the auxiliary winding (i.e. the commutator winding) is independent of the speed of the rotor and always equal to line frequency. Therefore, the frequency of the e.m.f.s at the commutator brushes is always of slip frequency, and the frequency of the injected e.m.f. into the secondary is of slip frequency. This will be obvious if one remembers that for one definite speed, the distance of the brushes apart is fixed, and the brush position is fixed in space. Relatively to the brush half-sets, the relative speed of the field is the speed of the field—speed of rotor. Now the magnitude of the e.m.f. injected into the rotor depends on the distance apart of the half-brush sets. The greatest voltage is given when the brush sets a_1 and b_1, or a_2 and b_2, and a_3 and b_3 cover 180 electrical degrees on the commutator. When a_1 and b_1, and a_2 and b_2, and a_3 and b_3 are in line, i.e. a_1 in line with b_1, and a_2 in line with b_2, etc., the impressed voltage is zero, and the three stator windings are shorted. The machine in this case is an ordinary induction motor.

The speed control is continuous. The maximum speed range depends on the maximum voltage injected into the secondary winding from the auxiliary commutator winding, and this maximum voltage occurs when the half-brush sets (a_1 and b_1), (a_2 and b_2), (a_3 and b_3) embrace 180 electrical degrees on the commutator. For speed control only, the half-brush sets, when in line, must be at the centre of the phase of the secondary to which they are connected. In Fig. 8.18 the line bisecting (a_1 and b_1), (a_2 and b_2), and (a_3 and b_3) in each case lies at the centre of the stator phase considered. Thus, now we see why the brushes half-sets are moved in opposite directions simultaneously. It is to obtain voltages to inject into the secondary, and the magnitude of the voltages, thus injected, depends on the distances between a_1 and b_1, a_2 and b_2, a_3 and b_3. By insisting also on the condition that, when a_1 and b_1 are in line, a_2 and b_2 are also in line, and a_3 and b_3 in line, then the brush axis in each case must lie at the centre of the stator phase concerned, we ensure that the injected e.m.f. shall be in phase or in phase opposition with the induced e.m.f., due to the revolving field. Speed ranges of 3 to 1 and more may be obtained, but the output of this machine is limited by the voltage generated in the short-circuited coil by the revolving field. The e.m.f. of self-induction in the short-circuited coil is proportional to the specific electric loading and depends on the range of speed required. The output per pole is, therefore, limited and, indeed, this motor is at a disadvantage in respect to the ordinary shunt motor, fed through the commutator, for, in this Schräge motor, the e.m.f. generated in the short-circuited coil is of full supply frequency, and is constant and independent of the speed. I refer, of course, to the voltage generated due to the revolving field.

Power-factor Correction

This necessitates that the injected e.m.f.s must lead the voltages generated in the secondary by the revolving field. For this purpose, the axis of a brush pair must be displaced from the axis of a stator phase by a given angle. If unity power factor is to be obtained, at all speeds, both brush supports must be moved by different amounts. About 10 per cent difference in distance between the brush supports is necessary. In the neighbourhood of synchronous speed, where the commutator voltage is small, the power factor is almost the same as the induction motor, but above synchronism, the power factor is improved by the negative leakage reactance of the secondary winding.

SPECIMEN CHARACTERISTICS: SCHRÄGE TYPE THREE-PHASE COMMUTATOR MOTOR

The characteristics of this motor, which is of the shunt type are given below in a test carried out on a small motor in the laboratory of Electrical Engineering of the University of British Columbia, Vancouver, Canada.

Test Machine. ASEA, Sweden. Type FS-6. No. 27598.

Output, kW: 2·2–6·6
Output, h.p.: 3–9
R.p.m.: 600–1800
Primary volts: 220
Primary amperes: 17–24
Rating: continuous
Primary connection: Y
Cycles: 60 c/s
Secondary volts: 33·5
Secondary amperes: 57

Observations—

(1*b*) No-load Test

Brush setting	600	Start	1000	1200	1450	at 60 c/s
R.p.m.	650	359	1036	1175	1480	

(1*c*)

Load Test

	R.p.m.	Volts	Amperes	W_1	W_2	W_r	$\frac{W_2}{W_1}$	P.F.	Brake Load			B.h.p.	Efficiency (%)	f (c/s)
									W lb	S lb	Net lb			
Setting "600"	470	228	24·8	5770	450	6220	0·08	0·56	18	3·25	18·8	4·20	50·0	59
	500	229	22·0	4570	220	4790	0·05	0·54	14	2·75	15·3	3·60	56·0	59
	530	229	19·2	3830	0	3830	—	0·50	10	2·60	11·4	2·90	56·5	60
	550	230	16·8	3230	− 260	2970	− 0·08	0·44	6	2·20	7·9	2·10	52·7	60
	590	231	15·1	2630	− 600	2030	− 0·23	0·34	2	2·00	4·0	1·10	40·5	60
	610	231	14·2	2400	− 750	1650	− 0·31	0·29	0	1·80	2·2	0·64	29·0	60
Setting "1000"	940	231	25·5	5810	2160	8070	0·37	0·78	16	3·20	16·8	7·42	69·0	60
	930	231	22·0	5000	1520	6520	0·30	0·73	12	2·80	13·2	5·77	66·0	60
	960	232	19·0	4170	870	5040	0·21	0·66	8	2·60	9·5	4·26	63·0	60
	960	233	16·4	3430	170	3600	0·05	0·54	4	2·10	5·9	2·66	55·0	60
	1000	233	14·6	2680	− 450	2230	− 0·17	0·38	0	1·80	2·2	1·07	36·0	60
Setting "1450"	1400	232	24·8	5720	3160	8880	0·55	0·89	12	2·90	13·1	8·60	72·5	60
	1410	232	20·6	4820	2050	6870	0·42	0·82	8	2·80	9·2	6·10	66·0	60
	1415	233	17·5	3910	1020	4930	0·26	0·70	4	2·30	5·7	3·80	57·5	60
	1425	233	15·2	3100	− 40	3060	− 0·13	0·41	0	1·90	2·1	1·40	34·5	60

Effective weight of brake arm = 4·0 lb.
Length of prony brake arm = 30 in.

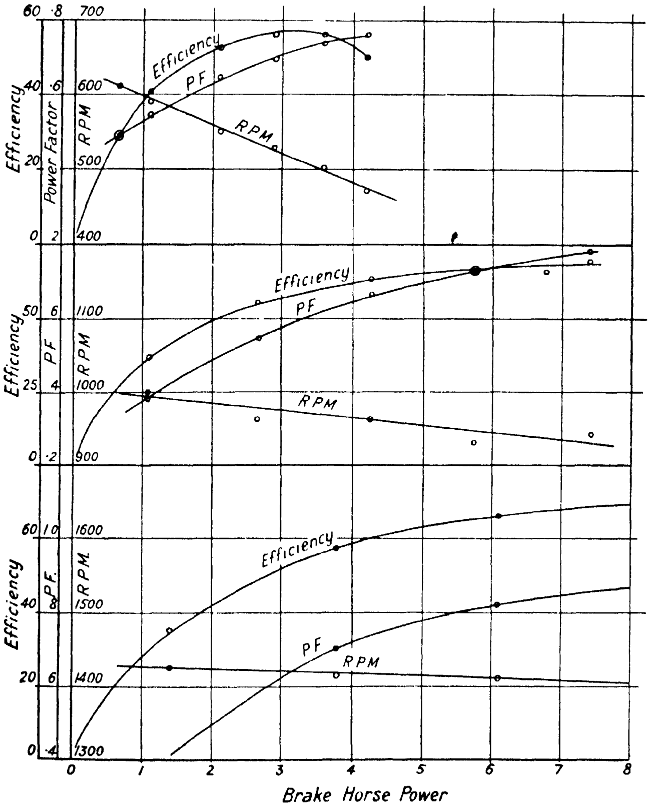

FIG. 8.19

(2*d*)

Rotor at Standstill

Brush Setting	600			1000			1200			1450		
Primary volts .	233	233	233	233	—	—	233	—	—	233	—	—
Secondary volts	36	35·6	36	36	—	—	36	—	—	36	—	—
Injected volts .	15·4	15·0	15·8	3·0	4·75	2·7	1·0	0·5	0·5	10·8	9·75	11·0

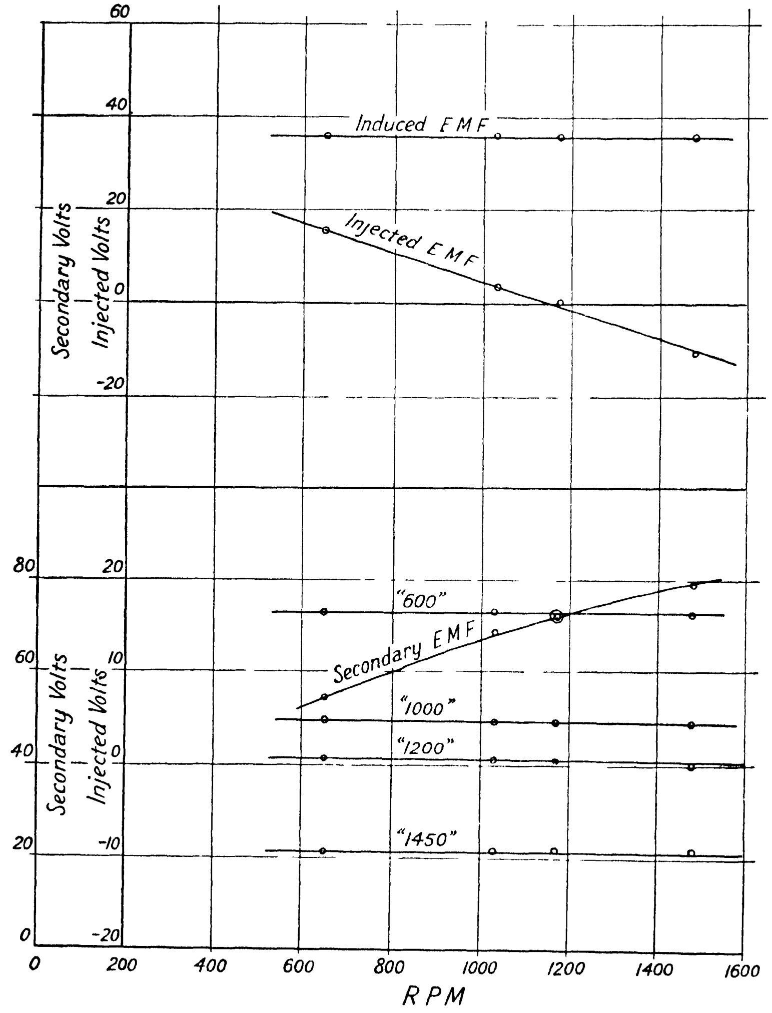

Fig. 8.20

(2*e*)

	R.p.m.	Brush Setting			
		600	1000	1200	1450
Primary volts .	650	232	233	232	232
Injected volts .		16·3	4·7	0·5	9·3
Secondary volts .		54	54	54	54
Primary volts .	1036	233	233	232	232
Injected volts .		16·3	4·6	0·5	9·2
Secondary volts .		68	69	68	68
Primary volts .	1175	233	232	233	233
Injected volts .		16	4·6	0·5	9·2
Secondary volts .		71	72	72	72
Primary volts .	1480	233	233	233	232
Injected volts .		16·4	4·3	0	9·1
Secondary volts .		79	79	79	79

Calculations

Power factors were determined from the ratio W_1/W_2.

$$\text{Net load on brake} = W - S + 4{\cdot}0$$

$$\text{Case (1) net load on brake} = 18 - 3{\cdot}25 + 4 \qquad = 18{\cdot}8 \text{ lb}$$

$$\text{Case (1) h.p.} = \frac{(2\pi)(470)(18{\cdot}8)}{33\,000} \qquad = 4{\cdot}2 \text{ h.p.}$$

$$\text{Case (1) efficiency} = \frac{\text{output}}{\text{input}} = \frac{(4{\cdot}2)(746)}{6220} \qquad = 50\%$$

CHAPTER IX

Phase Advancing

THE power factor of lightly-loaded induction motors is very low, and likewise that of fully-loaded slow-speed motors. The deleterious effects of low power factor on the voltage regulation of the system are well known; and, further, the losses in generators and motors and transmission lines are increased due to low power factor. Further, in the design of induction motors, especially if of low speed, the output for a given frame is frequently reduced by the necessity of using shallow slots, for it is very important to reduce the leakage reactance. Several different types of machine have been devised for injecting a leading e.m.f. into the rotor circuits and thus to improve the power factor.

Leblanc's System

Leblanc proposed an arrangement of exciters for this purpose. Considering a two-phase induction motor, two exciters, one for each phase, are connected in series with the rotor winding, and the field of one exciter is excited by the rotor current of the second. These machines were ordinary commutating a.c. generators. With this arrangement an e.m.f. is injected into one phase, which is 90° out of phase with the current carried by that phase; and if the polarity of the poles is properly arranged, this is a leading e.m.f. The objection to this scheme was the excessive cost. Leblanc has devised an exciter which embodies, in one machine, all the phases, and this is of a very simple nature.

The armature is made like an ordinary drum-wound continuous-current armature, and is surrounded by a simple ring of laminations having inwardly projecting poles, but without field windings. If such an armature is provided with four brushes, placed 90° apart on the commutator, and connected to the four slip-rings of a two-phase rotor of an induction motor, and is run at a speed which is high compared with the frequency of the rotor circuits, it will have the effect of producing leading currents in the rotor circuit. The beauty of the exciter is that the armature currents excite the field and produce a flux in the armature which is in such a phase as to generate an e.m.f. in each circuit which is in quadrature with the current carried by the circuit. By proper design and the use of carbon brushes, the commutation can be made sufficiently good. The rotors of induction motors of large power carry heavy currents, and this necessitates a commutator of considerable dimensions.

The Scherbius System

In the Scherbius phase advancer, the external ring of laminations is omitted, and the armature winding is embedded in slots, which are some distance from the periphery. The path for the magnetic field is, therefore, through the iron above the slots. When used in conjunction with a three-phase rotor, three sets of brushes are provided on the commutator at 120 electrical degrees apart. These brushes receive the rotor currents of slip frequency. The currents of slip frequency will produce a revolving field whose speed is

$$f/p \text{ revolutions per second}$$

where f = slip frequency
p = pairs of poles

Now suppose the rotor to be driven in the same direction as the field revolves. The speed of the field is independent of the speed of rotation of the armature. When the speed of the armature is equal to the speed of the rotating flux wave, no relative motion or cutting of the lines takes place. The self-induction effect disappears therefore. Above this speed, i.e. above the synchronous speed of the field, the reactance e.m.f. becomes negative; in other words, the current will lead.

Let ϕ = flux per pole in megalines
$2p$ = number of poles for which the armature is wound
f = slip frequency
n = number of revolutions per second at which the armature is driven

The star value of the e.m.f. injected for a wave-wound armature

$$e = \frac{p}{2} \times 0{\cdot}7 \times \phi \times \frac{Z}{100}(n - f/p)$$

where Z = the total number of active wires

The rating of the phase advancer is proportional to its volt-ampere capacity. The current is determined by the load, but the e.m.f. injected by the advancer is capable of adjustment. It is clear that if the motor is designed for a small natural slip, and the phase advancer has small ohmic and inductive losses, then the injected e.m.f. required becomes smaller, and the cost of the phase advancer will be less.

Fig. 9.1 shows the circle diagram for the motor with and without the phase advancer. It is taken from the article by the late Dr. Kapp.

In this figure, it is assumed that the ratio of turns in stator and rotor is 1 : 1. *ED* is proportional to the secondary or rotor current, and *EM* is proportional to the slip-ring voltage with the secondary open. The voltage drop in the rotor is $ER_0 = I_2R_2$, where $I_2 = ED$ and R_2 is the resistance of one phase of the rotor. Without the phase advancer the locus of *D* is the semicircle shown. With the phase advancer and at unity power factor, the point *D* moves to *A*, thereby reducing the stator current, but increasing the rotor current. The increased drop due to this increase in current is shown as R_0R. The resistance drop in the advancer is shown as R_0S. The slip voltage must have a power component *ES*. It must also have a wattless component equal to the e.m.f. introduced by the advancer minus the e.m.f. of self-induction lost in the advancer itself.

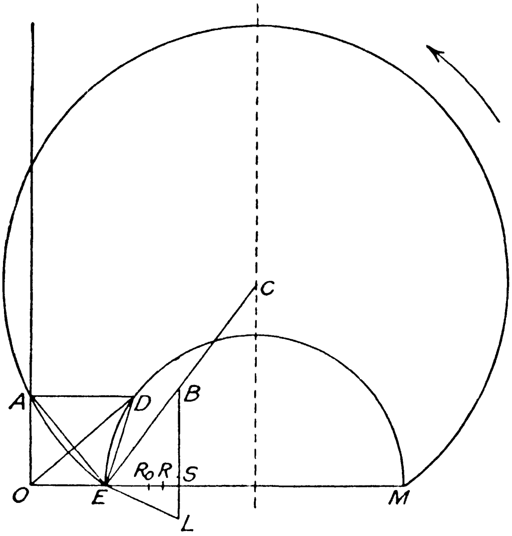

FIG. 9.1

In the diagram, *SL* = e.m.f. of self-induction of advancer

and *LB* = e.m.f. of advancer

The wattless component of the slip voltage is *SB*.

The total slip voltage is *EB*, which is greater than ER_0 and, therefore, the slip is increased by using a phase advancer.

Dr. Kapp lays especial emphasis on the importance of a small natural slip. He also states that the cost of the set, capable of giving unity power factor under such conditions, need not be greater than the cost of a motor designed to work alone. This may be true, but it is open to question. There is no doubt whatever that the use of the advancer will enable one to get a greater output from the motor frame, and in any case the extra cost of the advancer is probably justified. The locus of the primary current vector with phase advancer is the larger circle shown. In this it is assumed that the flux in the phase advancer is proportional to the rotor current, i.e. there is no saturation. If saturation sets in at the higher loads, injected e.m.f. and rotor current are no longer proportional, so that *EB* becomes less inclined and the centre *C* is lower. This enables compensation at, say, two-thirds of full load without over-compensating at overload.

Compensation is chiefly required at the lower values of the load, and so it may be advantageous to introduce a moderate degree of saturation in the advancer.

For small motors, the phase advancer may be mounted on the rotor shaft. It is then wound for a smaller number of poles than the main motor. In this case the speed n is fixed by the slip.

If n_s = synchronous speed of the set in revolutions per second

$$= \frac{f_1}{p_1}$$

where f_1 = supply frequency

p_1 = pairs of poles of induction motor

the speed of the induction motor

$$= (1 - s)\frac{f_1}{p_1} \qquad (9.1)$$

$$= (1 - s)n_s \qquad (9.2)$$

the speed of the field of the advancer

$$= \frac{sf_1}{p_2} \qquad (9.3)$$

$$= s\frac{f_1}{p_1} \times \frac{p_1}{p_2} = sn_s \times \frac{p_1}{p_2} \qquad (9.4)$$

therefore, the relative speed in this case

$$= (1 - s)n_s - sn_s\frac{p_1}{p_2} \qquad (9.5)$$

$$= n_s\left(1 - s - s\frac{p_1}{p_2}\right) \qquad (9.6)$$

and the star value of the injected e.m.f. for a wave-wound two-circuit advancer

$$= 0{\cdot}7\phi \times \frac{Z}{100} \times n_s\left(1 - s - s\frac{p_1}{p_2}\right) \times \frac{p_2}{2} \qquad (9.7)$$

where ϕ = flux in megalines
Z = total number of conductors on advancer
s = slip of main motor
p_1 = pairs of poles of main motor
p_2 = pairs of poles of the advancer

It will be seen that as the load decreases the slip decreases, and the term in brackets increases slightly with reduction of load. The injected e.m.f. increases slightly, therefore, at light load, in addition to the increase at light load compared with full load due to saturation effects. For large motors the phase advancer is driven from the main motor either by belt or by other means.

The Miles-Walker Advancer

Prof. Miles-Walker has designed a very ingenious type of advancer. In this machine two types of armature winding are used: in one, an open circuit star-winding with very wide brushes, suitable for low voltages and large currents; in the other, which is suitable for fairly high voltages and low current values, a closed-circuit winding with three sets of brushes, spaced 120 electrical degrees apart, is used. A diagrammatic scheme of connections is shown in Fig. 9.2. The inner circle represents the closed-circuit winding of the armature of the advancer, and the small letters *a*, *b*, and *c* show the three phases mesh-connected. Three brushes *P*, *Q*, and *R* bear on the commutator, and convey currents to the outer circle *A*, *B*, *C*, which represents the rotor windings of the induction motor, shown mesh-connected. The arrowheads show the direction along each conductor, which is taken as positive for the purpose of the clock diagram.

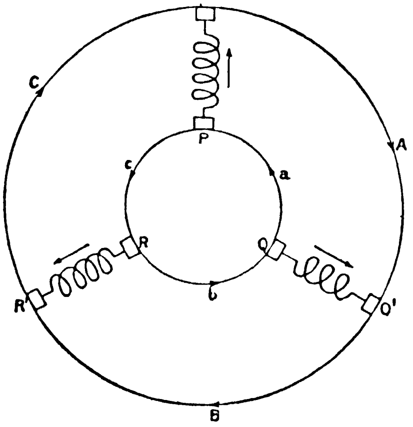

FIG. 9.2

In series with P, Q, and R, the series exciting coils are connected. A suitable e.m.f. to inject into phase A is an e.m.f. in phase with $(a - b)$. The current in brush Q is $(b - a)$ and, therefore, if the poles under which the coils in phase A are passing are excited with $- Q$, the required condition is satisfied. The span of the armature coils is almost a pole-pitch, so that the coils in phase A will be passing under adjacent poles.

Since return paths have to be arranged and also compensating windings, it is necessary in order to make a simple mechanical arrangement of the coils to excite these poles with exciting conductors carrying currents $+ Q + Q - P_1 - R$.

Since $P + Q + R = o$, it follows that the above arrangement

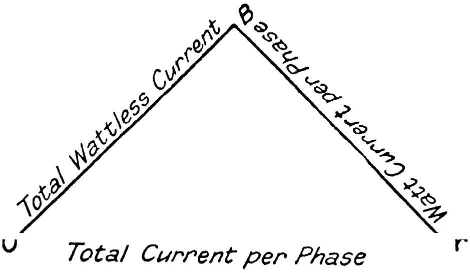

FIG. 9.3

will give $3Q$. The question whether $+ Q$ gives a forward or backward e.m.f. will depend on the direction of rotation and the hand of the winding.

To find the rating of the advancer, we must find the rotor standstill voltage per phase and the watt component of current in the rotor per phase. Knowing the power factor at full load, from the circle diagram we can determine the wattless current. If now the power factor is to be a leading one, we can find the amount of wattless current for this.

AB = watt current per phase
BC = total wattless current per phase
AC = total current of the phase advancer

The next thing to do is to determine the voltage of the advancer. From the value of the slip at full load we can determine the slip voltage at full load.

In Fig. 9.4, OE_a represents the slip voltage at full load, and Oa is set off at an angle from OE_a = angle CAB in Fig. 9.3. E_aR represents the resistance drop in the advancer and brushes, and RX the reactance drop in the advancer field coils. If, then, a voltage XV parallel to ba be added, a resultant voltage in phase with Oa is obtained.

This is the reason why the fields are excited with a current in phase with the sum of O_a and $- O_b$. The voltage to be generated is scaled off from the diagram. As shown, the projection of OV on the vertical line gives OV_x, which is greater than the slip voltage OE_a. If this voltage is greater than is required to drive the current through the rotor circuit, the slip will be reduced until the correct

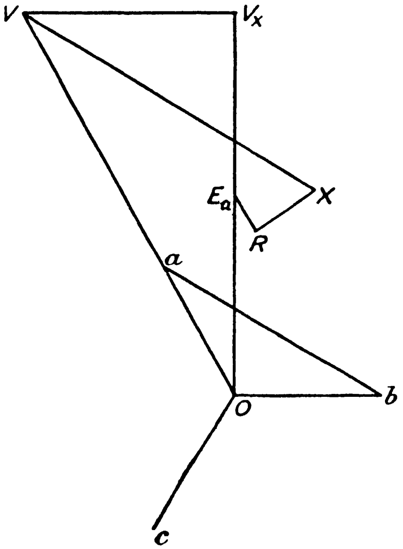

FIG. 9.4

working current for the load is reached. If OV_x is not sufficient, the slip is increased.

The Kapp Vibrator

The late Dr. Gisbert Kapp devised a type of phase advancer known as a vibrator. In this machine the principle is used that a conductor conveying an alternating current of low frequency, when placed in a magnetic field of constant strength, will vibrate and generate an e.m.f. leading the current by 90°.

The Kapp vibrator consists of a number of continuous-current armatures of the bi-polar type placed in bi-polar fields excited by direct current. The number of armatures required is equal to the number of phases in the rotor of the induction motor. Each armature is provided with a bi-polar closed-circuit winding and commutator, and each commutator has a pair of brushes. One brush of each commutator is connected to a slip ring of the induction motor, and the other brush is connected to a common neutral point when star connection is required.*

It is clear that the torque, acting on each armature, is proportional to the current, since the field is constant. The induced e.m.f.

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* The armatures may be connected in delta for large rotor currents

in the conductor is proportional to the velocity of the conductor in the field.

We have torque ∝ current

acceleration ∝ torque ∝ current

e.m.f. ∝ velocity

If the current follows a sine law with respect to time, the acceleration will follow a sine law and the velocity will follow a cosine law, and, therefore, the e.m.f. will follow a cosine law.

The acceleration is zero when the velocity is a maximum, and, therefore, the e.m.f. generated is in quadrature with the current. We have yet to show that it is *leading* the current.

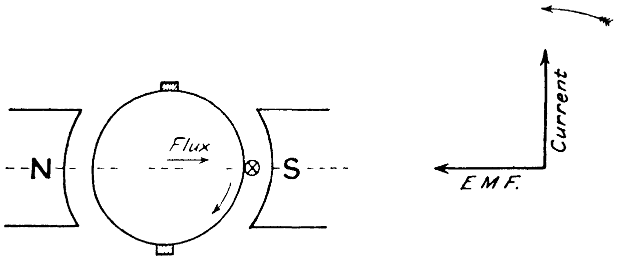

FIG. 9.5

In Fig. 9.5 let the field in which the armature vibrates have the direction from left to right, and let the direction of the current be considered positive when it flows away from the observer in the right-hand side of the armature. When the current is at its positive maximum, the torque is a maximum and the velocity zero, and the armature is momentarily at rest, but is on the point of moving in a clockwise direction. Now a clockwise movement will generate an e.m.f. in the right-hand half of the winding, which is towards the observer, i.e. which is negative. Thus, at this instant the e.m.f., which is zero, is about to assume a negative value. Hence, the vector diagram is as shown on the right, i.e. the e.m.f. leads the current by 90°. If Z = total number of conductors on one armature and ϕ = flux per pole in c.g.s. units, then the e.m.f. generated at any instant = e,

and
$$e = \phi Z \frac{n}{60} p/a \times \frac{1}{10^8} \quad . \quad . \quad . \quad . \quad (9.8)$$

$$= \phi Z \frac{\omega}{2\pi} \times \frac{1}{10^8} \text{ for two-pole armature} \quad . \quad . \quad (9.9)$$

where n = revolutions per minute

p = poles

a = circuits in parallel

$p = a$ for two poles

$\omega = \frac{2\pi n}{60}$ = angular velocity in radians per second

If i = instantaneous value of the current, the instantaneous power

$$= ei = i\frac{\omega}{2\pi}Z\frac{\phi}{10^8} \tag{9.10}$$

If T = torque in metre-kilograms,

$$ei = 9{\cdot}81\,T\omega = i\frac{\omega}{2\pi}Z\frac{\phi}{10^8} \tag{9.11}$$

$$\therefore \qquad i = T \times \frac{9{\cdot}81 \times 2\pi \times 10^8}{Z \times \phi} \tag{9.12}$$

Also

$$\omega = \frac{2\pi e \times 10^8}{Z\phi}$$

$$\int i\,dt = 9{\cdot}81 \times \frac{2\pi \times 10^8}{Z\phi} \times \int T\,dt \tag{9.13}$$

Let M = moment of inertia in kilogram-metres^2,

then

$$\int i\,dt = \frac{9{\cdot}81 \times 2\pi \times 10^8}{Z\phi} \times \frac{M\omega}{9{\cdot}81} \tag{9.14}$$

$$= \frac{9{\cdot}81 \times 2\pi \times 10^8}{Z\phi} \times \frac{M}{9{\cdot}81} \times \frac{e \times 2\pi \times 10^8}{\phi Z} \tag{9.15}$$

$$= M \times \frac{4\pi^2 \times 10^{16}}{Z^2\phi^2} \times e \tag{9.16}$$

It is clear that the vibrator behaves like a capacitor whose capacitance

$$= \frac{M \times 4\pi^2 \times 10^{16}}{Z^2\phi^2} \text{ farads} \tag{9.17}$$

Also since

$$e = \phi Z\frac{\omega}{2\pi} \times \frac{1}{10^8} \tag{9.18}$$

and

$$9{\cdot}81\,M\frac{d\omega}{dt}\,\omega = ei \tag{9.19}$$

$$= \frac{i\omega}{2\pi}Z\frac{\phi}{10^8} \tag{9.20}$$

$$\therefore \qquad \frac{d\omega}{dt} = \frac{iZ\phi}{2\pi \times 10^8 \times 9{\cdot}81 \times M} \tag{9.21}$$

but

$$i = \hat{I}\cos\omega_1 t \tag{9.22}$$

$$\int_0^{\pi/2} \frac{d\omega}{dt}\,dt = \frac{Z\phi}{2\pi \times 10^8 \times 9{\cdot}81 M}\hat{I}\int_0^{\pi/2} \cos \omega_1 t\,dt \qquad . \quad (9.23)$$

$$\omega_{\max} = \frac{Z\phi}{2\pi \times 10^8 \times 9{\cdot}81 M\omega_1}\hat{I} \qquad . \quad . \quad (9.24)$$

therefore, maximum value of e.m.f. injected

$$= \frac{Z^2\phi^2}{4\pi^2 \times 10^8 \times 9{\cdot}81 M\omega_1}\hat{I}\,. \qquad . \quad . \quad (9.25)$$

$$\omega_1 = 2\pi \times \text{slip frequency} \,. \qquad . \quad . \quad (9.26)$$

$$\therefore \qquad \text{e.m.f. injected} \propto \frac{I}{\omega_1} \propto \frac{\text{current}}{\text{slip}} \qquad . \quad . \quad (9.27)$$

Since this ratio decreases only slightly as the load increases, the e.m.f. injected does not fall off proportionately with the load, but at a much lower rate, with the result that the effect of the advancer is relatively greater at low values of the load, and this is what is required.

It will be seen from the expression above for the injected e.m.f. how important it is to keep the moment of inertia of the armatures low. The armature must, therefore, be small in diameter and great in length; the air-gap must be small and the flux high, so that considerable saturation is required in teeth and core. Such a vibrator would be difficult to design for very large motors of slow speeds where the rotor currents are of necessity exceedingly high, but for machines of moderate output it has achieved very remarkable results.

CHAPTER X

Induction Motor with D.C. Secondary Excitation

FREQUENTLY, where constant speed is required and where it is desirable to have a large starting torque and high power factor under running conditions, the induction motor with d.c. excitation is used. The normal synchronous motor, with salient poles and usual damping winding, has a very low starting torque. To reduce the large, wattless, starting current, it is necessary to supply the motor with reduced voltage, and since the damper must have fairly low resistance, to be effective, it follows that very low starting torque is obtainable, since the torque varies as the square of the applied volts to the machine. To meet this condition, an induction motor is used, which is started up in the usual way, by means of resistance in the rotor circuit. When the motor has reached full speed, the rotor is excited by a d.c. exciter, and, provided the slip is not too great, the machine drops into step and runs as a synchronous motor. The steps in the starting-up period are shown in Fig. 10.1.

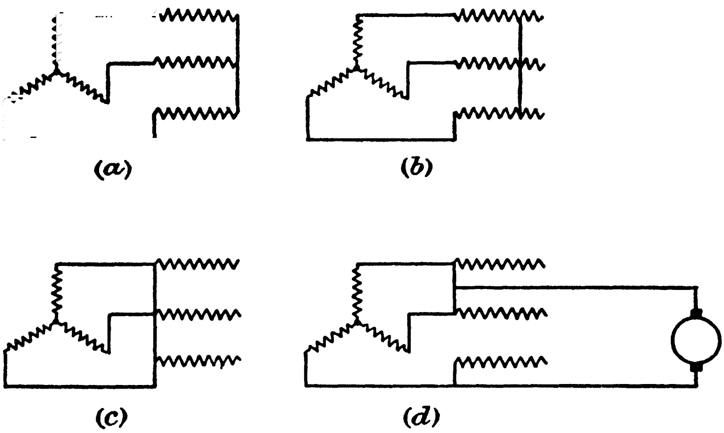

FIG. 10.1

With the connections shown in Fig. 10.1, one phase carries twice as much current as the other phases, and, therefore, the sectional area of the conductors in that phase must be increased. It is possible, however, to use a rotor winding in which all the copper is evenly loaded. If one were to take an induction motor of reasonably good characteristics, and convert it into a synchronous motor, with constant field excitation, then it would be found that its characteristics, as a synchronous motor, are very poor, for the no-load current will be nearly equal to the full-load current, and the power factor and apparent efficiency are very low, except in a very narrow range just below the maximum output point or pull-out point.

On the other hand, by converting a slow-speed induction motor of large magnetizing current into a synchronous motor, then the characteristics as synchronous motor, with constant excitation, are very much better than those of the motor as induction motor. The reason for the unsatisfactory behaviour of a good induction motor, when converted to the synchronous type, is due to the large value of its synchronous impedance. In the induction motor, large magnetizing action of the stator current is desirable to produce the field, with as small a value of the current as possible. In the synchronous motor, large magnetizing action, on the part of the stator currents, calls for corresponding changes in d.c. excitation, and the higher the armature reaction, the more must the field excitation be changed with load, to maintain unity power factor. For good synchronous motor operation, low armature reaction is necessary. For good induction motor operation, the armature reaction must be high. For stable operation of the synchronous motor, a high synchronous reactance is necessary; but this high value is far lower than that in a good induction motor. A synchronous reactance of 50 to 100 per cent is high for a synchronous motor, but the synchronous reactance of a good induction motor may exceed 300 per cent.

Performance of Synchronous Induction Motor

Let $\mathbf{V}$ = impressed volts per phase, chosen as datum vector

$\mathbf{Z} = R + jX$ = synchronous impedance

X_1 = leakage reactance per phase

b_1 = exciting susceptance per phase

$= \dfrac{\text{magnetizing current}}{\text{voltage induced by it}}$

The effective reactance of armature reaction $= \dfrac{1}{b_1}$

Then $$\mathbf{V} = \mathbf{E} + \mathbf{Z}\mathbf{I} \qquad (10.1)$$

$\mathbf{E}$ = back e.m.f. or nominal induced voltage

$$\mathbf{I} = i_1 - ji_2 \qquad (10.2)$$

$$\therefore \quad \mathbf{V} = \mathbf{E} + (R + jX)(i_1 - ji_2) \qquad (10.3)$$

$$X = X_1 + \frac{1}{b_1}$$

$$\therefore \quad \mathbf{E} = (V - Ri_1 - Xi_2) - j(Xi_1 - Ri_2) \qquad (10.4)$$

$$\therefore \quad |E|^2 = (V - Ri_1 - Xi_2)^2 + (Xi_1 - Ri_2)^2 \qquad (10.5)$$

$$\therefore \quad \mathbf{E} = e_1 - je_2 \qquad (10.6)$$

where

$$e_1 = V - Ri_1 - Xi_2 \qquad (10.7)$$

and

$$e_2 = Xi_1 - Ri_2 \qquad (10.8)$$

At unity power factor $i_2 = 0$,

and

$$E^2 = (V - Ri_1)^2 + X^2 i_1^2 \qquad (10.9)$$

At no-load and unity power factor $i_1 = 0$, $i_2 = 0$, and $E = V$.

The field excitation, required to keep $\cos \phi = 1$, at all loads, is given by equation (10.9).

From equation (10.9), we have

$$E^2 = V^2 - 2VRi_1 + R^2 i_1^2 + X^2 i_1^2 \qquad (10.10)$$

$$= V^2 - 2VRi_1 + Z^2 i_1^2$$

or

$$Z^2 i_1^2 - 2VRi_1 + V^2 - E^2 = 0 \qquad (10.11)$$

$$\therefore \quad i_1 = \frac{2VR \pm \sqrt{4V^2R^2 - 4Z^2(V^2 - E^2)}}{2Z^2}$$

$$= \frac{VR \pm \sqrt{Z^2E^2 - X^2V^2}}{Z^2} \qquad (10.12)$$

The minimum value of

$$E = \frac{XV}{Z} \qquad (10.13)$$

For a given value of E, i.e. for constant excitation

$$i_2 = \frac{XV}{Z^2} \pm \sqrt{\frac{E^2}{Z^2} - \left(i_1 - \frac{RV}{Z^2}\right)^2} \qquad (10.14)$$

If Z is approximately equal to X, R negligible,

$$i_2 \simeq \frac{V}{X} \pm \sqrt{\frac{E^2}{X^2} - i_1^2} \qquad (10.15)$$

Under the same assumption, the maximum value of $i_1 = \frac{E}{X}$ for the radical becomes unreal for greater values of i_1.

The power input $= Vi_1 \qquad (10.16)$

The gross power output

$$= e_1 i_1 + e_2 i_2 \qquad (10.17)$$

$$= i_1(V - Ri_1 - Xi_2) + i_2(Xi_1 - Ri_2) \quad . \quad (10.18)$$

$$= Vi_1 - Ri_1^2 - Xi_1i_2 + Xi_1i_2 - Ri_2^2 \quad . \quad . \quad (10.19)$$

$$= Vi_1 - Ri^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.20)$$

where $i^2 = i_1^2 + i_2^2$

Therefore, equation (10.20) = input — copper losses.

The current in the field

$$= E \times b \quad . \quad . \quad . \quad . \quad (10.21)$$

The copper loss in the field

$$= E^2 \times b^2 \times r_f \quad . \quad . \quad . \quad (10.22)$$

where r_f = field resistance

The iron loss $= E^2g \simeq V^2g \quad . \quad . \quad . \quad (10.23)$

where g = conductance

Therefore, the net mechanical output

$$= Vi_1 - i^2R - E^2b^2r_f - E^2g - (\text{friction} + \text{windage}) \quad . \quad (10.24)$$

From these equations, it is simple to calculate the performance as a synchronous motor.

These machines are, usually, designed to give an overload capacity of 70 per cent for $\cos \phi = 1$, or 90 per cent for a leading power factor of 0·9. Should the machine fall out of step as a synchronous motor, it will continue to run as an induction motor, since the maximum overload capacity as induction motor is greater than that as synchronous motor, and it will automatically re-synchronize when the peak load is passed.

Further Analysis

We will now enquire into the phenomena which arise during the starting period, and how the design of the motor affects such phenomena. Consider first the expressions for the torque in the machine as induction motor and also as synchronous motor. We have seen that, in the induction motor, the torque in synchronous watts

$$= \frac{m_2 s \bar{E}_2^2 R_2}{R_2^2 + s^2 L_2^2 \omega^2} \quad . \quad . \quad . \quad (10.25)$$

where m_2 = number of phases in the rotor

R_2 = rotor resistance per phase

$\bar{E}_2$ = volts per rotor phase at standstill

L_2 = coefficient of self-inductance of each rotor phase in henrys due to leakage flux

s = slip

$\omega = 2\pi \times f$

where f = frequency

Maximum torque occurs at a slip

$$s_m = \frac{R_2}{L_2\omega} \quad . \quad . \quad . \quad . \quad . \tag{10.26}$$

$$s_m = \frac{\text{rotor resistance per phase}}{\text{rotor reactance per phase at standstill}}$$

Therefore, maximum torque per phase

$$= \frac{\bar{E}_2{}^2}{2L_2\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{10.27}$$

$$= \frac{\text{square of rotor volts per phase at standstill}}{2 \times \text{rotor reactance per phase at standstill}}$$

This is obtained, of course, by substituting for s the value $\frac{R_2}{L_2\omega}$ in equation (10.25).

The maximum torque per phase, viz. $\frac{\bar{E}_2{}^2}{2L_2\omega}$, can also be written as

$$\frac{\bar{E}_2}{2} \times \frac{\bar{E}_2}{L_2\omega} = \frac{\bar{E}_2}{2} \times \bar{I}_{scr} \quad . \quad . \quad . \quad . \tag{10.28}$$

where $\bar{I}_{scr}$ = ideal rotor short-circuit amperes per phase.

Now assume this induction motor, with wound rotor, is operated as an auto-synchronous motor. As synchronous motor, the gross mechanical power developed per phase

$$= \frac{\bar{E}_2[\bar{E}_1 \cos(\gamma - \theta) - \bar{E}_2 \cos \gamma]}{\sqrt{R_1{}^2 + L_1{}^2\omega^2}} \quad . \quad . \quad . \tag{10.29}$$

where $\bar{E}_2$ = generated e.m.f. per phase in stator (r.m.s. value)
$\bar{E}_1$ = applied volts per phase in stator (r.m.s. value)
R_1 = resistance, in ohms, per phase of the stator
$L_1\omega$ = *synchronous* reactance per phase, in ohms, of the stator

$$\tan \gamma = \frac{L_1\omega}{R_1} \quad . \quad . \quad . \quad . \quad . \tag{10.30}$$

Now, generally, $\lambda \bumpeq 90°$, i.e. the synchronous reactance is very large compared with the resistance.

In this case, the gross mechanical power developed per phase

$$\bumpeq \frac{\bar{E}_2\bar{E}_1 \sin \theta}{\sqrt{R_1{}^2 + L_1{}^2\omega^2}} \quad . \quad . \quad . \quad . \quad . \tag{10.31}$$

$$\bumpeq \frac{\bar{E}_2\bar{E}_1 \sin \theta}{L_1\omega} \bumpeq \bar{E}_2\bar{I}_{sc} \sin \theta \quad . \quad . \quad . \tag{10.32}$$

The angle θ is, of course, the angle between the applied volts per phase and the component of applied volts, which is equal and opposite to the back e.m.f.; $180 - \theta$ is the angle between the applied volts and back e.m.f.

The maximum, gross, mechanical power per phase as synchronous motor (assuming $\lambda = 90°$)

$$= \bar{E}_2 I_{sc} \text{ watts}$$

The maximum torque, per phase, will also be proportional to this expression, and is equal to it when expressed in synchronous watts per phase. Now let us assume that this machine is operating as a synchronous motor, and let the direct current in one phase $= \sqrt{2}\bar{I}_r$, and $\frac{\bar{I}_r}{\sqrt{2}}$ in each of the other phases, where $\bar{I}_r =$ full-load rotor current as induction motor. The copper losses in the two cases are then equal, i.e. when the machine is working as induction motor, or as synchronous motor. The magnetizing current, as induction motor, to produce normal flux, as a fraction of full-load current, will depend on the number of poles for which the motor is wound and will vary from 25 per cent on high speed motors to 60 or 70 per cent or more on very slow speed machines. Let us assume that it is one-third of full-load current.

The corresponding d.c. excitation for normal flux will be $\sqrt{2} \times \frac{1}{3}\bar{I}_r = 0{\cdot}472\bar{I}_r$. This leaves $0{\cdot}942\bar{I}_r$ for producing leading current, and the r.m.s. value of such leading current will be $0{\cdot}666\bar{I}_r$. Assuming the watt component of the current remains unaltered, the power factor will be 0·830 leading. The total stator current will not be altered greatly, from the value it had as induction motor. Assuming a leakage reactance per stator phase of 10 per cent, the value of $\bar{E}_2$ will be roughly $1{\cdot}07\ \bar{E}_1$.

It is clear that such a machine, thus excited, could be operated as a synchronous motor and work in the neighbourhood of 0·8 leading power factor, with the same rotor heating in each case. Now what about the maximum mechanical power exerted in this case? The ideal short-circuit current in the stator will be but equal to the full-load current, and $\bar{E}_2$ (approximately) $= 1{\cdot}07\bar{E}_1$.

Therefore, maximum mechanical power as synchronous motor, or maximum torque in synchronous watts $= 1{\cdot}07\bar{E}_1 \times \bar{I}_1$ per phase. Now, as induction motor, this same machine will have a maximum torque, in synchronous watts per phase

$$= \frac{\bar{E}_2}{2} \times \bar{I}_{scr}$$

$$= \frac{\bar{E}_1}{2} \times \bar{I}_{scs} \quad . \quad . \quad . \quad . \quad (10.33)$$

where $\bar{I}_{scs} =$ ideal short-circuit current in the stator per phase.

Now $\bar{I}_{scs}$ may be roughly five times full-load current (this depends on the number of poles). Therefore, maximum torque in synchronous watts, *per phase*

$$= 2 \cdot 5 \bar{E}_1 \bar{I}_1$$

where $\bar{I}_1$ = full-load stator current

It is clear, therefore, that such an induction motor, when working as synchronous motor, has much too small a synchronizing power and also too little overload torque. In other words, it would pull out of step with small overload. Let us examine the conditions under which this machine will come into step, during the period of synchronizing. Let us assume that the machine is started up, under load, as an induction motor, and that it is running steadily, as such, at a slip s. The corresponding torque is, say, T_M, which is assumed constant. The relation between slip and torque is a linear one in the region from slip s to zero slip. The machine has to be accelerated from slip s to zero slip, during synchronizing. The synchronizing torque must have sufficient value to overcome the constant load torque, and also be sufficiently great to accelerate the rotor from slip s to zero slip.

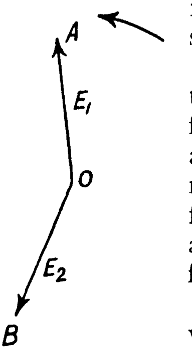

FIG. 10.2

Let OA (Fig. 10.2) represent the vector of applied volts per phase and OB represent the vector of generated e.m.f. per phase due to the field excitation. OA rotates at a speed corresponding to the supply frequency, whereas OB rotates at a speed corresponding to frequency of rotation, or the frequency of generated e.m.f.

The relative angular velocity is proportional to the slip, and the time of one revolution of relative velocity is the time of one slip period, i.e. E_1 is assumed stationary and $\bar{E}_2$ revolves at slip frequency. When the vector E_2 is in the first and fourth quadrants, we have generator action, and when E_2 is in the second and third quadrants, motor action. Whether the machine will pull into step, or not, will depend on whether, or not, the synchronizing torque action, during the half-cycle of motor action is sufficient, or not, to accelerate the machine to full speed. It will be noted that the average value of the synchronizing torque, at any speed, other than synchronous speed, over any number of slip periods, is zero.

Let s = slip

f = supply frequency

The slip frequency = sf and the time of one slip period = $\frac{1}{sf}$

The synchronizing torque $\propto \bar{E}_1\bar{E}_2 \sin \theta$, say, $\hat{T}_s \sin \theta$.

Let M = moment of inertia of the rotor

T_M = torque of the induction motor at slip s

T_l = torque of the load, assumed constant
s_l = slip corresponding to load torque

Then we have the equation of motion,

$$T_M + \hat{T}_s \sin\theta = T_l + M\frac{d\omega}{dt} \quad . \quad . \quad (10.34)$$

Now
$$\omega = \omega_s(1 - s)$$

$$\frac{d\omega}{dt} = -\omega_s\frac{ds}{dt} \quad . \quad . \quad . \quad (10.35)$$

Also
$$\frac{T_M}{T_l} = \frac{s}{s_l} \quad . \quad . \quad . \quad (10.36)$$

$\therefore$
$$T_M = T_l\frac{s}{s_l} \quad . \quad . \quad . \quad . \quad (10.37)$$

Now the relative angular velocity of the rotor, with respect to the revolving field of the stator

$$= s\omega_s = \frac{d\theta}{dt \times p/2}$$

here θ is an electrical angle, and we must divided θ by the number of pairs of poles to get the mechanical angle.

$\therefore$
$$M\frac{d\omega}{dt} = -M\omega_s\frac{ds}{dt} \quad . \quad . \quad . \quad (10.38)$$

but
$$s\omega_s = \frac{d\theta}{dt \times p/2} \quad . \quad . \quad . \quad (10.39)$$

$\therefore$
$$M\frac{d\omega}{dt} = -\frac{M}{p/2}\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad (10.40)$$

Also
$$T_M = \frac{T_l}{s_l \times \omega_s} \times \frac{d\theta}{dt} \times \frac{1}{p/2} \quad . \quad . \quad (10.41)$$

$\therefore$
$$\frac{T_l}{s_l \times \omega_s \times p/2}\frac{d\theta}{dt} + \hat{T}_s \sin\theta + \frac{M}{p/2}\frac{d^2\theta}{dt^2} = T_l \quad . \quad (10.42)$$

or
$$A\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + T_s \sin\theta - T_l = 0 \quad . \quad . \quad (10.43)$$

where A and B are the coefficients of $\frac{d^2\theta}{dt^2}$ and $\frac{d\theta}{dt}$ in equation (10.42).

The solution of equation (10.42) involves the use of elliptic functions. It is similar to the equation for the oscillation of a pendulum with large amplitude of swing.

An approximate solution can be obtained by a process of successive approximations, by using the expansion for sin θ, viz.

$$\sin\theta = \theta - \frac{\theta^3}{\angle 3} + \frac{\theta^5}{\angle 5}$$

The equation is first solved by using the first term θ for sin θ, and the value of θ, obtained from this solution, is substituted in the differential equation with the second term added, viz. $\sin\theta = \theta - \frac{\theta^3}{\angle 3}$, and a second solution of the differential equation is then obtained. This process is repeated. This step-by-step method is long and laborious.

We may get some idea as to whether the machine will come into step for any given value of the slip in the following manner. Assume the machine is started up as an induction motor under a constant load torque T_L. The slip will be known, for this torque, for any motor.

Let T_{S_a} = average value of the synchronizing torque during one-quarter of a slip period

T_{M_a} = induction motor torque, average value during the speed increase from slip s to slip 0

M = moment of inertia of revolving parts

Then, assuming average values of T_S and T_M during the accelerating period from speed $\omega_0(1 - s)$ to ω_0, we have

$$T_{S_a} + T_{M_a} - T_L = M\frac{d\omega}{dt} \quad . \quad . \quad (10.44)$$

$$\int_{\omega_0(1-s)}^{\omega_0} d\omega = \frac{T_{S_a} + T_{M_a} - T_L}{M}\int_0^{t/4} dt \quad . \quad . \quad (10.45)$$

$$t = \text{time of one slip period} = \frac{1}{sf} \quad . \quad . \quad (10.46)$$

$$\therefore \quad s\omega_0 = \frac{T_{S_a} + T_{M_a} - T_L}{M} \times \frac{t}{4} \quad . \quad . \quad (10.47)$$

$$= \frac{T_{S_a} + T_{M_a} - T_L}{M} \times \frac{1}{4sf} \quad . \quad . \quad (10.48)$$

from which

$$s = \sqrt{\frac{T_{S_a} + T_{M_a} - T_L}{4Mf\omega_0}} \quad . \quad . \quad . \quad (10.49)$$

This is the limiting value of the slip, corresponding to an average synchronizing torque, and an average induction motor torque, during the one-quarter of a slip period in which the speed rises from $\omega_0(1 - s)$ to ω_0. The machine is assumed to come into step without oscillation. The synchronizing power per phase = $E_2 I_{sc}$ sin θ and the average value of sin θ, over a quarter of a slip period (for constant E_2 and I_{sc})

$$= \frac{4}{\pi}\int_0^{\pi/4} \sin\theta d\theta = \frac{4}{\pi}\left[-\cos\theta\right]_0^{\pi/4}$$

$$= \frac{4}{\pi}\left[-\frac{1}{\sqrt{2}} + 1\right] = 0{\cdot}373 \quad . \quad . \quad . \ (10.50)$$

Therefore, the average value of the synchronizing power per phase, during one-quarter of a slip period $= 0{\cdot}373E_2I_{sc}$ and taking the case of $E_2 = 1{\cdot}07E_1$ and $I_{sc} = I_1$.

The average value of the above *per phase* $= 0{\cdot}4E_1I_1$.

$$\frac{T_{S_a}\omega_0}{550} \times 746 = 0{\cdot}4E_1I_1 \times m_1$$

ω_0 = angular velocity of rotor in radians per second

$$\therefore \qquad T_S = \frac{550 \times 0{\cdot}4 \times E_1I_1m_1}{\omega_0 \times 746}$$

$$= \frac{0{\cdot}295E_1I_1m_1}{\omega_0} \text{ lb-ft} \quad . \quad . \quad . \ (10.51)$$

The limiting value of the slip to come into step is given by equation (10.49), when this value of T_S is substituted and also the average induction motor torque at a slip $= \frac{s}{2}$

One fact emerges very clearly, and that is that the induction motor, used as auto-synchronous motor has very low synchronizing power, and, indeed, will not pull into step against any appreciable resisting torque.

The question arises then as to how the design of the motor should be modified to make it suitable for the purpose of running as an auto-synchronous machine. A glance at the equation for the gross mechanical power developed, and the corresponding torque will show that it is necessary to increase the value of I_{sc}. This can be done, of course, by increasing the ratio of field ampere-turns to armature ampere-turns per pole, and this involves an increase in the air-gap length.

It would be desirable to have a maximum torque, or pull-out torque, as a synchronous motor equal to twice the full-load torque. The maximum power, developed by the synchronous motor, per phase $= \frac{E_1^2}{4R}$, but this value is obtained only when $\theta = \gamma$, and $E_2 = \frac{E_1}{2\cos\gamma}$, i.e. for ranges of excitation outside practical limits.

The power developed *per phase* $= \bar{E}_2 I_{sc} \sin\theta$ and the maximum value of this, when $\lambda = 90°$

$$= \bar{E}_2 I_{sc} \qquad (10.52)$$

The average flux density, in the teeth of induction motors, is probably about 12 000 to 13 000 lines per cm², and so it will not be possible to increase this by more than 30 per cent without introducing excessive saturation. Assume $\bar{E}_2 = 1{\cdot}3\bar{E}_1$, then it will be necessary to make I_{sc} equal to $1{\cdot}535 I_1$ to obtain a maximum power per phase equal to twice the normal input per phase at unity power factor. This means an increase of 53·5 per cent in the value of the r.m.s. current in the rotor, or a corresponding d.c. current of 2·17 times the r.m.s. value of the normal induction motor current in one phase. In the other two phases, which will be connected in parallel, the d.c. current will be 1·085 times the normal r.m.s. rotor current. More room will be needed, in the rotor slots, i.e. deeper slots must be used, which will necessitate a larger rotor diameter, than would be required in an ordinary induction motor, for the same output. It might be thought desirable to increase the number of rotor turns per coil, and so keep down the current for exciting purposes, but the voltage generated between the slip-rings at the time of start, limits this increase. The torque in synchronous watts, as an induction motor

$$= m_2 \bar{E}_r I_r \cos\phi_r$$

where $\bar{E}_r$ = volts per rotor phase at standstill

$$\therefore \qquad m_2 \bar{E}_r I_r \cos\phi_r = \frac{\text{b.h.p.} \times 746}{(1 - s)} \qquad (10.53)$$

i.e.

$$\bar{E}_r I_r = \frac{\text{b.h.p.} \times 746}{m_2(1 - s)\cos\phi_r} \qquad (10.54)$$

Taking a value of $s = 0{\cdot}025$ at full load

and $\cos\phi_r = 0{\cdot}95$

and $m_2 = 3$

we have $\bar{E}_r I_r = 268$ b.h.p. $\qquad (10.55)$

The r.m.s. value of induction motor, rotor, current

$$= \frac{268 \times \text{b.h.p.}}{\bar{E}_r} \qquad (10.56)$$

The corresponding direct current, for excitation, at unity power factor for the overload capacity of twice

$$= \frac{2{\cdot}17 \times 268 \times \text{b.h.p.}}{\bar{E}_r}$$

$$= \frac{582 \times \text{b.h.p.}}{\bar{E}_r} \qquad (10.57)$$

and in the other two phases

$$= \frac{296 \text{ b.h.p.}}{\bar{E}_r} \quad . \quad . \quad . \quad . \quad (10.58)$$

The power required for excitation in watts (approximately)

$$= \frac{s\bar{E}_r \times 582 \times \text{b.h.p.}}{\bar{E}_r} \times 1\cdot5$$

$$= s \times 870 \times \text{b.h.p.}$$

$$= s \times 0\cdot87 \times \text{b.h.p. (in kW)} \quad . \quad . \quad (10.59)$$

where s = slip

The I^2R loss in the rotor, for the overload capacity stated, will be 2·36 times that in the rotor, when running under full-load current, as an induction motor. Clearly the rotor slots will need to be much deeper to accommodate the necessary copper. This will mean increasing the diameter of the rotor, except perhaps on large motors. At unity power factor in the stator, considerable reduction in the stator copper losses will result, over those in the induction motor.

Pulling into Step

The most favourable position of the rotor, with respect to the application of the d.c. excitation is that in which the synchronous torque is positive, i.e. when the synchronous torque reduces the slip. Should the d.c. excitation be applied at an instant of time when the synchronous torque is negative, the slip is increased, and if the synchronous torque is not sufficient, during the positive half-wave to reduce the slip from its increased value to zero, the machine will not synchronize.

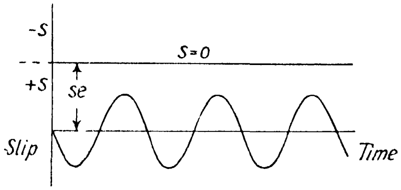

FIG. 10.3

Fig. 10.3 shows variation of slip, when the d.c. excitation is applied, when $\theta = 180$. Fig. 10.4 shows good position for rotor, when d.c. excitation is supplied.

We may plot the various torques graphically as ordinates and θ as abscissae.

The induction motor torque

$$T_M = T_e \times \frac{s}{s_l}$$

where T_l = load torque, assumed constant

s = slip

T_M = induction motor torque at slip s

s_l = load slip

$T_s = T_s \sin \theta$ = synchronous torque

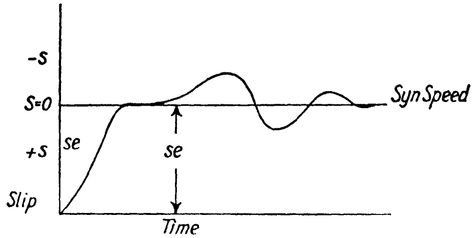

FIG. 10.4

The curve *OPFE* (Fig. 10.5) represents the synchronizing torque as a function of θ,

$$T_M = \text{induction motor torque} = cb \text{ for } \theta_1 = OA$$
$$T_s = Ad = \text{synchronizing torque}$$
$$Ab = Ac - cb = T_e - T_M$$

At angle $\theta = \theta_1$, Ab is the portion of the synchronizing torque delivered to the load.

The synchronous torque bd accelerates the rotor. For any value of θ the vertical intercept between T_s and T_M gives the accelerating or retarding torques.

When $\theta = 0$ the d.c. field is applied and the slip is s_e.

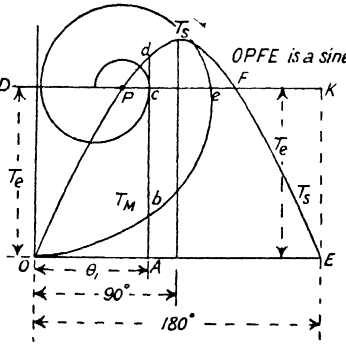

FIG. 10.5*

When synchronous speed is reached we reach point e, where $T_M = 0$.

When the induction motor torque curve intersects the load line

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* See *Electrical Machinery*, vol. 2, by Dr. Liwschitz Garrik. Van Nostrand (New York: 1946)

DF at a point *e*, which is to the left of *F*, i.e. when $De < DF$, the synchronous torque is > the load torque and the slip becomes negative. If T_M intersects T_s below the point *F*, the machine will not come into step. If β is the angle *DP*, then *De* must be not larger than $\pi - \beta$, or else there will be slipping of a pole.

When $De = \pi - DP$ the point *e* falls on *F*, and the machine will pull into step, but will be unstable. By equating the amount of work needed to accelerate the rotor from load slip to synchronous speed, and equating the area between the curves T_s and T_M for $\theta = 0$ and $\theta = \pi - DP = \pi - \beta$, it will be possible to determine the field amperes necessary to obtain synchronism in the best position of the rotor for the application of d.c. excitation

$$- M \frac{\omega_s}{p/2} \frac{ds}{dt} s\omega_s dt = - M \frac{\omega_s^2}{p/2} sds$$

and work to accelerate the moving mass from s_e to 0

$$= \frac{1}{2} M \frac{\omega_s^2 s_l^2}{p/2}$$

The area enclosed between the curves T_s and T_M (between $\theta = 0$ and $\theta = \pi - \beta$)

$$= \hat{T}_s (1 + \cos \beta) - \frac{T_l}{3} (\pi - \beta) \quad . \quad . \quad (10.61)$$

where the curve of T_M is taken as parabolic

but $$T_l = \hat{T}_s \sin \beta$$

where $\beta = DP$

$$\therefore \qquad \sin \beta = \frac{T_l}{\hat{T}_s}$$

where $\hat{T}_s$ = maximum value of synchronous torque

let $$\frac{\hat{T}_s}{T_l} = \alpha \quad . \quad . \quad . \quad . \quad (10.62)$$

$$\sin \beta = \frac{1}{\alpha} \text{ and } \cos \beta = \frac{\sqrt{\alpha^2 - 1}}{\alpha} \quad . \quad . \quad (10.63)$$

$$\therefore \quad \frac{1}{2} M \frac{\omega_s^2 s_l^2}{p/2} = T_l \left\{ \alpha + \sqrt{\alpha^2 - 1} - \frac{1}{3} \left(\pi - \sin^{-1} \frac{1}{\alpha} \right) \right\} \quad (10.64)$$

for given values of s_l, T_l, and *M* it is possible, from this equation to determine α, and thus $\hat{T}_s$ and the field current to pull the machine into step, under the most favourable conditions. Also the maximum value of the slip, corresponding to load torque T_l in the induction motor is obtainable from this equation.

Solution of the Equation of Motion for θ not Equal to 0

H. E. Edgerton and F. J. Zak have investigated the pulling into step when θ is not equal to zero. They show that

$$\left(\frac{T_l}{T_s}\right) < 1{\cdot}057Kd = \frac{1{\cdot}057 \times T_l}{s_l\omega_s\sqrt{\frac{M}{p/2}}\,\hat{T}_s} \quad . \quad . \quad (10.65)$$

The above is given as the condition which must be satisfied, if the machine is to pull into step when the excitation is applied at the most unfavourable rotor angle.

If the power is in kilowatts and M is expressed as WR^2, the slip

$$s_l = \frac{620}{n_s}\sqrt{\frac{CP_n}{WR^2 \times f_1}} \quad . \quad . \quad . \quad (10.66)$$

where $C = \frac{\hat{T}_s}{T_n}$

P_n = power in kilowatts

n_s = synchronous revolutions per minute

WR^2 = weight × (radius of gyration)^2 in lb-ft^2

f_1 = supply frequency

T_s = maximum value of synchronous torque

Power Factor Correction

The auto-synchronous motor is useful for improving the power factor of the line to which it is connected. Let us consider a three-phase machine, which is required to carry full load, with a leading power factor of 0·9.

Let $\bar{I}$ = full-load current in the stator.

The wattless current

$$= I \sin \phi = 0{\cdot}435\bar{I}$$

If q_1 = number of stator slots per pole per phase

z_1 = number of conductors per stator slot, in series

The amplitude of the *ampere-turn* wave of the stator

$$= 1{\cdot}28q_1z_1\bar{I}$$

Since the wattless current $= 0{\cdot}435\bar{I}$, the amplitude of the ampere-turn (fundamental) wave, due to the wattless current

$$= 0{\cdot}557q_1z_1\bar{I}$$

If q_2 and z_2 = number of rotor slots per pole per phase and the number of conductors per slot in the rotor, then

$$0{\cdot}91 q_2 z_2 I_2 = 0{\cdot}557 q_1 z_1 I \quad . \quad . \quad . \quad (10.60)$$

where I_2 = d.c. current to neutralize the wattless current.

Assume the machine to be designed with a synchronous reactance of 50 per cent, i.e. the short-circuit current is twice the normal at full supply voltage and $E_2 = 1{\cdot}3E_1$ for this condition. The machine will need to be excited so that the generated e.m.f. per phase is 30 per cent greater than the supply volts per phase. The magnetizing current in the induction motor, as before stated, may vary from 25 per cent to 60 per cent of full-load current and even more, depending on the number of poles. Taking an average value of 35 per cent of full-load current with normal gap, we shall need to double the air-gap length and we shall need practically full-load current of the induction motor to excite the synchronous motor. The demagnetizing ampere-turns have been shown to be $0{\cdot}557 q_1 z_1 I_1$. The rotor will then need a d.c. exciting current approximately equal to 2·2 times the r.m.s. value of the full-load rotor current as an induction motor, since the d.c. current must be equal to the maximum value of the a.c. current. It has been stated that the first cost of the machine is increased 33 per cent, or roughly the capital cost per leading kilovolt-ampere is one-third of the capital cost per kilowatt, and the cost of the auto-synchronous motor is from 30 to 40 per cent greater than a plain induction motor for a leading power factor of 0·9. This increase includes switch-gear, exciter, etc. The natural field of these motors is from 150 h.p. upwards; below this output the cost of power-factor correction is too great.

The stator windings have semi-formed coils in semi-closed slots, and large spans are braced. For low-tension circuits, the winding usually consists of the double-layer bar type, consisting of half-coils joined by copper clips at each end. For voltages of 2000 to 11 000, the winding is of the hair-pin type, one end being completely formed and insulated before insertion. The open end of the coil is formed to shape, electrically butt-welded and insulated after insertion. Rotors are usually slotted for a three-phase winding, but only two phases are wound, and remainder of the slots are left empty. Uniform current loading for the rotor is ensured by series connection. Shunt-wound exciters are provided.

CHAPTER XI

The Revolving Field

Consider a coil on the machine, of full pitch and having z_1 turns, and let it carry an alternating current $i = \hat{I}\sin\omega t$. The m.m.f. of such a coil is rectangular in shape, and the height of the rectangle is proportional to the current at the instant considered. If there were but one slot per pole per phase, with one coil side per slot, the winding thus formed by connecting consecutive coils of one phase in series, the end of one coil being joined to the beginning of the next, we should have a single-phase winding with as many coils as there are pole pairs in series. The m.m.f. diagram, for the whole phase, would then consist of a number of rectangles, positive and negative m.m.f.s following each other in succession. These rectangles would rise and fall and reverse with the current values, but still remain rectangular in shape. The m.m.f. *per pole*

$$= 0{\cdot}4\pi z_1 \hat{I}\sin\omega t$$

Now this is a periodic function and single valued, and we can resolve this rectangular m.m.f. shape into a fundamental wave and higher harmonics. The amplitude of the fundamental sine wave of m.m.f.

$$= \frac{4}{\pi} \times 0{\cdot}4\pi \frac{z_1}{2} \hat{I}\sin\omega t \quad . \quad . \quad . \quad (11.1)$$

$$= 0{\cdot}8 z_1 \hat{I}\sin\omega t \quad . \quad . \quad . \quad . \quad (11.2)$$

$$= 1{\cdot}13 z_1 I \sin\omega t \quad . \quad . \quad . \quad . \quad (11.3)$$

where $I = \dfrac{\hat{I}}{\sqrt{2}}$ = r.m.s. value

$\hat{I}$ = maximum value of the current

Now, if there are q_1 slots per pole per phase, we shall need to combine vectorially q_1 such vectors as given by equation (11.3), and the angle between the various vectors will be

$$\frac{180}{\text{number of slots per pole}} \text{ electrical degrees}$$

The resultant fundamental sine wave for one phase

$$= 1{\cdot}13 z_1 I q_1 \sin \omega t \times K_1 \quad . \quad . \quad . \quad (11.4)$$

where $K_1 = \dfrac{\sin q_1 \lambda/2}{q_1 \sin \lambda/2}$

where $\lambda = \dfrac{180}{\text{slots per pole}}$

If the co-ordinate axis is taken at the pole centre and the distance measured along the circumference to the right of the origin is called x, then if we call the m.m.f. at distance x from the origin ϕ_x, we have

$$\phi(x) = \phi(-x) \text{ and } \phi(x + \pi) = -\phi(x)$$

then only odd harmonics are present.

$$\therefore \qquad \phi_r = \Sigma a_n \cos \frac{n\pi x}{\tau} \quad . \quad . \quad . \quad (11.5)$$

n is the order of the harmonic

$$\phi_x = a_1 \cos \frac{x\pi}{\tau} + a_3 \cos \frac{3x\pi}{\tau} + a_5 \cos \frac{5x\pi}{\tau} + \dots \quad (11.6)$$

From $x = 0$ to $x = \frac{\tau}{2}$, where τ = pole pitch,

$$\phi_x = 0{\cdot}2\pi z_1 \sqrt{2} I \sin \omega t \quad . \quad . \quad . \quad (11.7)$$

for $q_1 = 1$.

From $x = \frac{\tau}{2}$ to $x = \frac{3\tau}{2}$

$$\phi_x = -\,0{\cdot}2\pi z_1 \sqrt{2} I \sin \omega t \quad . \quad . \quad . \quad (11.8)$$

From $x = \frac{3\tau}{2}$ to $x = 2\tau$,

$$\phi_x = +\,0{\cdot}2\pi z_1 \sqrt{2} I \sin \omega t \quad . \quad . \quad . \quad (11.9)$$

$$\int_0^{2\tau} \phi_x \cos \frac{x\pi}{\tau}\, dx$$

$$= \left[\int_0^{\tau/2} \phi_x \cos \frac{x\pi}{\tau}\, dx - \int_{\tau/2}^{3\tau/2} \phi_x \cos \frac{x\pi}{\tau}\, dx + \int_{3\tau/2}^{2\tau} \phi_x \cos \frac{x\pi}{\tau}\, dx\right]$$

$$= a_1 \int_0^{2\tau} \cos^2 \frac{x\pi}{\tau}\, dx + a_3 \int_0^{2\tau} \cos \frac{3x\pi}{\tau} \cos \frac{x\pi}{\tau}\, dx + \dots \quad . \quad (11.10)$$

$$= a_1 \tau$$

since
$$\int_0^{2\tau} \cos \frac{ax\pi}{\tau} \cos \frac{bx\pi}{\tau}\, dx = 0 \quad . \quad . \quad (11.11)$$

for a and b unequal integers.

$$\therefore\ a_1\tau = \phi_x\left[\int_0^{\tau/2}\cos\frac{x\pi}{\tau}\,dx - \int_{\tau/2}^{3\tau/2}\cos\frac{x\pi}{\tau}\,dx + \int_{3\tau/2}^{2\tau}\cos\frac{x\pi}{\tau}\,dx\right] \tag{11.12}$$

$$= \phi_x\frac{\tau}{\pi} + \phi_x\frac{2\tau}{\pi} + \phi_x\frac{\tau}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \tag{11.13}$$

$$= 0{\cdot}2\pi z_1\sqrt{2}I\sin\omega t\left[\frac{\tau}{\pi} + \frac{2\tau}{\pi} + \frac{\tau}{\pi}\right] \quad . \quad . \quad . \tag{11.14}$$

$$= \frac{4\tau}{\pi} \times 0{\cdot}2\pi z_1\sqrt{2}I\sin\omega t \quad . \quad . \quad . \quad . \quad . \quad . \tag{11.15}$$

$$\therefore \qquad a_1 = 1{\cdot}13z_1I\sin\omega t \quad . \quad . \quad . \tag{11.16}$$

In the same way, we have

$$a_n\tau = \phi_x\left[\int_0^{\tau/2}\cos\frac{nx\pi}{\tau}\,dx - \int_{\tau/2}^{3\tau/2}\cos\frac{nx\pi}{\tau}\,dx + \int_{3\tau/2}^{2\tau}\cos\frac{nx\pi}{\tau}\,dx\right] \tag{11.17}$$

It will be noted that in equation (11.17)

$$\phi_x = +\,0{\cdot}2\pi z_1\sqrt{2}I\sin\omega t \quad . \quad . \tag{11.18}$$

but between $\frac{2}{\tau}$ and $\frac{3\tau}{2}$, ϕ_x is negative, and hence the negative sign before the second term.

$$\text{Now the } \int_0^{\tau/2}\cos\frac{nx\pi}{\tau}\,dx = \frac{\tau}{n\pi}\left[\sin\frac{nx\pi}{\tau}\right]_0^{\tau/2} = \frac{\tau}{n\pi}\sin\frac{n\pi}{2} \quad . \tag{11.19}$$

$$-\int_{\tau/2}^{3\tau/2}\cos\frac{nx\pi}{\tau}\,dx = 2\sin\frac{n\pi}{2}\times\frac{\tau}{n\pi} \quad . \quad . \quad . \tag{11.20}$$

$$\int_{3\tau/2}^{2/\tau}\cos\frac{nx\pi}{\tau}\,dx = \sin\frac{n\pi}{2}\times\frac{\tau}{n\pi} \quad . \quad . \quad . \tag{11.21}$$

$$\therefore \qquad a_n = \frac{1}{n\pi}\left[4\times 0{\cdot}2\pi z_1\sqrt{2}I\sin\omega t\right]\sin\frac{n\pi}{2} \quad . \tag{11.22}$$

$$= \frac{1}{n}\left[1{\cdot}13z_1I\sin\omega t\right]\sin\frac{n\pi}{2} \quad . \quad . \quad . \tag{11.23}$$

$$\therefore\ \underset{(q_1=1)}{\phi_x} = 1{\cdot}13z_1I_1\sin\omega t\left[\cos\frac{\pi x}{\tau} - \frac{1}{3}\cos\frac{3\pi x}{\tau} + \frac{1}{5}\cos\frac{5\pi x}{\tau} - \frac{1}{7}\cos\frac{7\pi x}{\tau} + \ldots\right] \tag{11.24}$$

Note that the value of the m.m.f. at a point, distant x from the origin, consists of a series of pulsating m.m.f.s of different frequencies, which are odd multiples of the fundamental frequency.

The fundamental of the single-phase m.m.f., for $q_1 = 1$,

$$= 1{\cdot}13 z_1 I_1 \sin \omega t \cos \frac{\pi x}{\tau} \quad . \quad . \quad . \quad . \quad . \quad (11.25)$$

$$= 1{\cdot}13 z_1 I_1 \frac{\left[\sin\left(\omega t + \frac{\pi x}{\tau}\right) + \sin\left(\omega t - \frac{\pi x}{\tau}\right)\right]}{2} \quad . \quad (11.26)$$

The latter equation shows that the pulsating wave can be replaced by two rotating waves, of equal amplitude, rotating in opposite directions at synchronous speed. So we have the equivalence of the pulsating wave, viz.

$$1{\cdot}13 z_1 I_1 \sin \omega t \cos \frac{\pi x}{\tau}$$

and the two rotating waves, viz.

$$\frac{1{\cdot}13 z_1 I_1 \sin\left(\omega t + \frac{\pi x}{\tau}\right)}{2}$$

and

$$\frac{1{\cdot}13 z_1 I_1 \sin\left(\omega t - \frac{\pi x}{\tau}\right)}{2}$$

The pulsating wave of m.m.f. is distributed in space sinusoidally, and its ordinates vary sinusoidally with time. The axes of the poles, in the pulsating wave, are fixed in space. This equivalence has been long known and used by Ferraris and others in connection with the theory of single-phase motors, and is an application of Fresnel's theorem.

Note. The amplitude of the fundamental of the m.m.f. wave is $1{\cdot}13 z_1 I_1$. The amplitude of the *ampere-turn wave* $= 0.92 z_1 I_1$ for one slot per pole per phase in all the above cases.

Now if there are q_1 slots per pole per phase, we shall have to compound q_1 sinusoidal vectors at the correct angles with each other.

The resultant of q_1 such vectors

$$= q_1 \text{ times one vector} \times \text{breadth factor}$$

$$= \frac{q_1 \sin q_1 \lambda/2}{q_1 \sin \lambda/2} \text{ for the fundamental} \quad . \quad . \quad (11.27)$$

and

$$= \frac{q_1 \sin n q_1 \lambda/2}{q_1 \sin n \lambda/2} \text{ for the } n\text{th harmonic} \quad . \quad . \quad (11.28)$$

When there are q_1 slots per pole per phase in the single-phase machine, the resulting m.m.f. per pole—

$$\phi(x) = 1{\cdot}13 z_1 q_1 I_1 \sin \omega t \left[\cos \frac{\pi x}{\tau} K_1 - \frac{1}{3} \cos \frac{3\pi x}{\tau} K_3 + \frac{1}{5} \cos \frac{5\pi x}{\tau} K_5 - \frac{1}{7} \cos \frac{7\pi x}{\tau} K_7 + \ldots\right] \quad (11.29)$$

where $K_1 = \dfrac{\sin q_1\,\lambda/2}{q_1 \sin \lambda/2}$

$$K_3 = \frac{\sin 3q_1\,\lambda/2}{q_1 \sin 3\,\lambda/2}$$

$$K_n = \frac{\sin nq_1\,\lambda/2}{q_1 \sin n\,\lambda/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.30)$$

$$\text{where } \lambda = \frac{180}{\text{slots per pole}} \quad . \quad . \quad . \quad . \quad . \quad (11.31)$$

K_1, K_3, K_5, K_n are the breadth factors for the fundamental, third, fifth, and nth harmonics. Note that the amplitude of the nth harmonic is $\frac{1}{n}$ of the amplitude of the fundamental, and the pole pitch of the nth harmonic is $\frac{1}{n}$ of that of the fundamental.

If the winding is chorded, or of fractional pitch, the expression for a_n, viz.

$$\frac{1}{n}\,(1{\cdot}13 z_1 I_1 \sin \omega t)\,\sin n\,\frac{\pi}{2} \quad . \quad . \quad . \quad (11.32)$$

becomes
$$\frac{1}{n}\,(1{\cdot}13 z_1 I_1 \sin \omega t)\,\sin\left(n\,\frac{\beta}{\tau}\,\frac{\pi}{2}\right) \quad . \quad . \quad (11.33)$$

where β = span of coil

and τ = pole pitch

so that, for the chorded winding, with q_1 slots per pole per phase, the m.m.f.

$$\phi(x) = a_1 \cos\frac{\pi x}{\tau} + a_3 \cos\frac{3x\pi}{\tau} + a_5 \cos\frac{5x\pi}{\tau} + \ldots \quad (11.34)$$

where
$$a_n = \frac{1}{n}\left[(1{\cdot}13 z_1 I_1 \sin \omega t)\,\sin\left(n\,\frac{\beta}{\tau}\,\frac{\pi}{2}\right) q_1 K_n\right] \quad . \quad (11.35)$$

by putting $n = 1, 3, 5, 7$, etc., in a_n, we have $\phi(x)$.

Now we need to take into account the effect of the various phases.

For the three-phase machine the m.m.f. at the point distant x from the origin, is determined by the combined action of the three phases. Phase 2 is displaced from phase 1 by 120 electrical degrees on the stator, and the current it carries differs in phase by 120 electrical degrees in time. Phase 3 is displaced from phase 1 by 240°, and its current differs in time phase by 240°.

It is clear that each phase will be represented by a series of harmonics of similar type.

The m.m.f. at point x, due to phase 2, with full pitch coils

$$= 1\cdot13 z_1 q_1 I_1 \sin(\omega t - 120)\Big[\cos\left(\frac{\pi x}{\tau} - 120\right)K_1 - \frac{1}{3}\cos 3\left(\frac{\pi x}{\tau} - 120\right)K_3 + \frac{1}{5}\cos 5\left(\frac{\pi x}{\tau} - 120\right)K_5 \ldots\Big] \quad (11.36)$$

The m.m.f. at point x, due to phase 3,

$$= 1\cdot13 z_1 q_1 I_1 \sin(\omega t - 240)\Big[\cos\left(\frac{\pi x}{\tau} - 240\right)K_1 - \frac{1}{3}\cos 3\left(\frac{\pi x}{\tau} - 240\right)K_3 + \frac{1}{5}\cos 5\left(\frac{\pi x}{\tau} - 240\right)K_3 \ldots\Big] \quad (11.37)$$

Now we may combine the fundamental m.m.f.s of the three phases, viz.

$$= 1\cdot13 z_1 q_1 I_1 K_1\left[\sin\omega t\cos\frac{\pi x}{\tau} + \sin(\omega t - 120)\cos\left(\frac{\pi x}{\tau} - 120\right) + \sin(\omega t - 240)\cos\left(\frac{\pi x}{\tau} - 240\right)\right] \quad (11.38)$$

$$= 1\cdot13 z_1 q_1 I_1 K_1\left[\frac{1}{2}\sin\left(\omega t + \frac{\pi x}{\tau}\right) + \frac{1}{2}\sin\left(\omega t - \frac{\pi x}{\tau}\right) + \frac{1}{2}\sin\left(\omega t + \frac{\pi x}{\tau} - 240\right) + \frac{1}{2}\sin\left(\omega t - \frac{\pi x}{\tau}\right) + \frac{1}{2}\sin\left(\omega t + \frac{\pi x}{\tau} - 480\right) + \frac{1}{2}\sin\left(\omega t - \frac{\pi x}{\tau}\right)\right] \quad . \quad . \quad (11.39)$$

$$= 1\tfrac{1}{2} \times 1\cdot13 z_1 q_1 K_1 \sin\left(\omega t - \frac{\pi x}{\tau}\right) \times I_1 \quad . \quad . \quad . \quad . \quad . \quad (11.40)$$

Therefore, the fundamental m.m.f. of the three-phase winding, at the point x,

$$= \phi(x) = 1\cdot695 z_1 q_1 K_1 I_1 \sin\left(\omega t - \frac{\pi x}{\tau}\right) \quad . \quad (11.41)$$

The amplitude of the fundamental m.m.f. wave of the three-phase machine

$$= 1\cdot695 z_1 q_1 K_1 I_1 \quad . \quad . \quad . \quad (11.42)$$

and, taking the value for $K_1 = 0\cdot96$, the amplitude of the fundamental of the m.m.f. wave

$$= 1\cdot625 z_1 q_1 I_1 \quad . \quad . \quad . \quad (11.43)$$

Frequently we require the amplitude of the *ampere-turn* wave of the fundamental, it is

$$1\cdot29 z_1 q_1 I_1 \quad . \quad . \quad . \quad . \quad (11.44)$$

Thus, the fundamental *ampere-turn* amplitude per pole

$$
\begin{aligned}
&= 1{\cdot}29 z_1 q_1 I_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.45)\\
&= 1{\cdot}29 \times \text{conductors in series per slot}\\
&\qquad\quad \times \text{slots per pole per phase}\\
&\qquad\quad \times \text{r.m.s. current per phase}
\end{aligned}
$$

In this same way the resultant third harmonic is zero for the three-phase machine, and all multiples of three are also zero. For any given harmonic, the amplitude of the resultant $= \frac{m}{2}$ times the amplitude for one phase, where m = number of phases, and the same applies to the fundamental.

It will be noted that in the *polyphase* machine, the fundamental m.m.f. wave is constant in amplitude, and rotates in one direction only.

The *m.m.f.* wave for the three-phase winding

$$
\begin{aligned}
\phi_x = 1{\cdot}695 z_1 q_1 I_1 \Bigg[&\sin\left(\omega t - \frac{\pi x}{\tau}\right) K_1 + \frac{\sin\frac{5\pi}{2}}{5} \sin\left(\omega t + \frac{5\pi x}{\tau}\right) K_5\\
&+ \frac{1}{7} \sin\frac{7\pi}{2} \sin\left(\omega t - \frac{7\pi x}{\tau}\right) K_7\\
&+ \frac{\sin\frac{\pi \times 11}{2}}{11} \sin\left(\omega t + \frac{11\pi x}{\tau}\right) K_{11} + \ldots \Bigg] \quad . \quad (11.46)
\end{aligned}
$$

$$
\begin{aligned}
= 1{\cdot}695 z_1 q_1 I_1 \Bigg[&K_1 \sin\left(\omega t - \frac{\pi x}{\tau}\right) + \frac{K_5}{5} \sin\left(\omega t + \frac{5\pi x}{\tau}\right)\\
&- \frac{K_7}{7} \sin\left(\omega t - \frac{7\pi x}{\tau}\right) - \frac{K_{11}}{11} \sin\left(\omega t + \frac{\pi x 11}{\tau}\right) + \ldots \Bigg] \quad (11.47)
\end{aligned}
$$

If the winding is of fractional pitch, instead of the breadth factor K_n, we must use the winding factor, which is the product of the breadth factor and the coil span factor.

For example, for K_n we must use, with the chorded winding,

$$
\frac{\sin nq\,\lambda/2}{q \sin n\,\lambda/2} \times \sin\left(n \frac{\beta}{\tau} \frac{\pi}{2}\right)
$$

The terms involving

$$
\sin\left(\omega t - \frac{\pi x}{\tau}\right),\ \sin\left(\omega t - \frac{7\pi x}{\tau}\right),\ \sin\left(\omega t - \frac{13\pi x}{\tau}\right),\ \sin\left(\omega t - \frac{19\pi x}{\tau}\right)
$$

represent waves which travel in the same direction as the fundamental, i.e. first, seventh, thirteenth, nineteenth harmonics. The terms

$$
\sin\left(\omega t + \frac{5\pi x}{\tau}\right),\ \sin\left(\omega t + \frac{11\pi x}{\tau}\right),\ \sin\left(\omega t + \frac{17\pi x}{\tau}\right)
$$

i.e. the fifth, eleventh, seventeenth, etc., travel in the opposite direction to the fundamental.

In general, with a whole number of slots per pole per phase, the following harmonics are possible

$$n = l'm \pm 1 \qquad . \quad . \quad . \quad (11.48)$$

where l' is any positive or negative integer and includes zero; m is the number of phases in the winding.

Thus, with $m = 3$, we have—

n	l'	n	l'
1	0	-11	-4
$+4$	$+1$	$+16$	$+5$
-2	-1	-14	-5
$+7$	$+2$	$+19$	$+6$
5	-2	-17	-6
$+10$	$+3$	$+22$	$+7$
-8	-3	-20	-7
$+13$	$+4$		

The table shows the harmonics that an integral slot winding may produce. In the three-phase winding, with 60° phase belt, l' is confined to positive and negative *even* integers. The positive values of n indicate harmonics which travel in the same direction as the fundamental, the negative values of n indicate waves travelling in the opposite direction to the fundamental.

Therefore, for the three-phase machine, with 60° phase belt, we have the following values for n: $1, -5, +7, -11, +13, -17, +19$, etc.

For the two-phase machine, the values of n are: $1, -3, +5, -7, +9, -11, +13, -15, +17$, and the amplitude of the resultant m.m.f. for the two phases is the same as the amplitude due to one phase.

We have seen that a single-phase winding, with a whole number of slots per pole, has an m.m.f. distribution given by equation (11.5), viz.

$$\phi_x = \Sigma a_n \cos \frac{n\pi x}{\tau}$$

The nth harmonic $(q_1 = 1)$

$$= \frac{1{\cdot}13 z_1 I_1}{n} \sin \omega t \cos \frac{n\pi x}{\tau} \qquad . \quad . \quad (11.49)$$

which we may write, for short,

$$= A_{1n} \sin \omega t \cos \frac{n\pi x}{\tau} \qquad . \quad . \quad . \quad (11.50)$$

where $$A_{1n} = \frac{1{\cdot}13 z_1 I}{n} \text{ [here } q_1 = 1] \qquad . \quad . \quad (11.51)$$

The fundamental wave length $= 2\tau$ and that of the nth harmonic $= \dfrac{2\tau}{n}$

Let there be m phases and let us find the resultant m.m.f. for the nth harmonics of the m phases.

For phase 1, we have the nth harmonic m.m.f. at point x

$$= A_{1n} \sin \omega t \cos \frac{n\pi x}{\tau} \quad . \quad . \quad . \quad (11.52)$$

For phase 2

$$= A_{2n} \sin \left(\omega t - \frac{2\pi}{m}\right) \cos \left(\frac{n\pi x}{\tau} - \frac{n2\pi}{m}\right) \quad . \quad (11.53)$$

For phase 3

$$= A_{3n} \sin \left(\omega t - \frac{4\pi}{m}\right) \cos \left(\frac{n\pi x}{\tau} - \frac{n4\pi}{m}\right) \quad . \quad (11.54)$$

For phase m

$$= A_{mn} \sin \left(\omega t - \overline{m-1}\,\frac{2\pi}{m}\right) \cos \left(\frac{n\pi x}{\tau} - \overline{m-1}\,\frac{2\pi}{m}\right) \quad (11.55)$$

Replacing $\sin \omega t \cos \dfrac{n\pi x}{\tau}$ by $\dfrac{1}{2}\left[\sin\left(\omega t - \dfrac{n\pi x}{\tau}\right) + \sin\left(\omega t + \dfrac{n\pi x}{\tau}\right)\right]$ and so for all the products of sine and cosine in equations (11.53), (11.54), and (11.55) we have, for the Kth phase

$$A_{Kn} \sin \left(\omega t - \overline{K-1}\,\frac{2\pi}{m}\right) \cos \left(\frac{n\pi x}{\tau} - \overline{K-1}\,\frac{n2\pi}{m}\right) \quad (11.56)$$

$$= \frac{A_{Kn}}{2}\left[\sin\left\{\left(\omega t - \frac{n\pi x}{\tau}\right) + (n-1)\,\overline{K-1}\,\frac{2\pi}{m}\right\}\right.$$

$$\left. + \sin\left\{\omega t + \frac{n\pi x}{\tau} - (n+1)\,\overline{K-1}\,\frac{2\pi}{m}\right\}\right] \quad (11.57)$$

$$= \frac{A_{Kn}}{2}\left[\sin\left(\omega t - \frac{n\pi x}{\tau}\right) \cos (n-1)\,\overline{K-1}\,\frac{2\pi}{m}\right.$$

$$+ \cos\left(\omega t - \frac{n\pi x}{\tau}\right) \sin (n-1)\,\overline{K-1}\,\frac{2\pi}{m}$$

$$+ \sin\left(\omega t + \frac{n\pi x}{\tau}\right) \cos (n+1)\,\overline{K-1}\,\frac{2\pi}{m}$$

$$\left. - \cos\left(\omega t + \frac{n\pi x}{\tau}\right) \sin (n+1)\,\overline{K-1}\,\frac{2\pi}{m}\right] \quad . \quad . \quad (11.58)$$

Summing up for the m phases, i.e. for $K = 1$ to $K = m$, the *resultant m.m.f. nth harmonic* for the m phases at a point distant x from the origin at the centre of phase 1,

$$\phi_R x = \tfrac{1}{2}A_{1n}\left[\sin\left(\omega t - \frac{n\pi x}{\tau}\right)\sum_{K=1}^{K=m}\cos(n-1)\overline{K-1}\,\frac{2\pi}{m}\right.$$

$$+\cos\left(\omega t - \frac{n\pi x}{\tau}\right)\sum_{K=1}^{K=m}\sin(n-1)\overline{K-1}\,\frac{2\pi}{m}$$

$$+\sin\left(\omega t + \frac{n\pi x}{\tau}\right)\sum_{K=1}^{K=m}\cos\overline{n+1}\,\overline{K-1}\,\frac{2\pi}{m}$$

$$\left.-\cos\left(\omega t + \frac{n\pi x}{\tau}\right)\sum_{K=1}^{K=m}\sin\overline{n+1}\,\overline{K-1}\,\frac{2\pi}{m}\right] \quad . \quad (11.59)$$

where, of course, $A_{1n} = A_{2n} \ldots = A_{Kn} \ldots = A_{mn}$.

We must sum the cosine series and sine series.

Now $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + \overline{m-1}\,\beta)$

$$= \cos\left(\alpha + \frac{(m-1)\beta)}{2}\right)\sin\frac{m\beta}{2}\operatorname{cosec}\frac{\beta}{2} \quad . \quad (11.60)$$

and $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots + \sin(\alpha + \overline{m-1}\,\beta)$

$$= \sin\left\{\alpha + \frac{(m-1)\beta}{2}\right\}\sin\frac{m\beta}{2}\operatorname{cosec}\frac{\beta}{2}. \quad . \quad (11.61)$$

Now our first series

$$= \cos 0 + \cos\overline{n-1}\,\frac{2\pi}{m} + \cos(n-1)2\,\frac{2\pi}{m} + \ldots \cos\overline{n-1}\,\overline{m-1}\,\frac{2\pi}{m} \quad . \quad . \quad (11.62)$$

here $\alpha = 0$, $\beta = \overline{n-1}\,\frac{2\pi}{m}$

So our first summation

$$= \cos\left\{\overline{(m-1)}\,\overline{\frac{n-1}{2}\,\frac{2\pi}{m}}\right\}\sin m\left.\overline{\frac{n-1}{2}\,\frac{2\pi}{m}}\right\}\operatorname{cosec}\left(\overline{\frac{n-1}{2}\,\frac{2\pi}{n}}\right) \quad (11.63)$$

Our second summation $\Sigma \sin\overline{n-1}\,\overline{K-1}\,\frac{2\pi}{n}$

$$= \sin\left\{(m-1)\,\overline{\frac{n-1}{2}\,\frac{2\pi}{m}}\right\}\sin\left(m\,\overline{\frac{n-1}{2}\,\frac{2\pi}{m}}\right)\operatorname{cosec}\left(\overline{\frac{n-1}{2}\,\frac{2\pi}{n}}\right) \quad (11.64)$$

Let us write these sums as d, e, f, g, i.e.

$$d = \sum_{K=1}^{K=m}\cos(n-1)(K-1)\,\frac{2\pi}{m} \quad . \quad . \quad (11.65)$$

$$e = \sum_{K=1}^{K=m} \sin (n-1)(K-1)\frac{2\pi}{m} \qquad . \qquad . \ (11.66)$$

$$f = \sum_{K=1}^{K=m} \cos\left(\overline{n+1}\times\overline{K-1}\,\frac{2\pi}{m}\right) \qquad . \qquad . \ (11.67)$$

$$g = \sum_{K=1}^{K=m} \sin \overline{n+1}\,(K-1)\frac{2\pi}{m}. \qquad . \qquad . \ (11.68)$$

Then the resultant nth harmonic, for the m phases, of the m.m.f. at x

$$= \phi_{R}x = \tfrac{1}{2}A_{in}\sqrt{d^2+e^2}\sin\left(\omega t - \frac{n\pi x}{\tau} + \delta_1\right)$$

$$+ \tfrac{1}{2}A_{in}\sqrt{f^2+g^2}\sin\left(\omega t + \frac{n\pi x}{\tau} + \delta_2\right) \ (11.69)$$

where $$\tan\delta_1 = \frac{e}{d} \text{ and } \tan\delta_2 = \frac{g}{f} \qquad . \qquad . \ (11.70)$$

Equation (11.69) represents two rotating waves of equal amplitude, rotating in opposite directions at a speed $\frac{1}{n}$ of that of the fundamental.

The polyphase winding produces only one travelling wave for each harmonic, therefore, $\sqrt{f^2+g^2}$ must vanish, i.e. f and g must be zero for one direction of rotation, and d and e must be zero for the other direction.

For f and g to vanish and d and e not to vanish

$$\sum_{K=1}^{K=m} \cos \overline{n+1}\,\overline{K-1}\,\frac{2\pi}{m} = 0 \qquad . \qquad . \ (11.71)$$

and $$\Sigma \sin \overline{n+1}\,\overline{K-1}\,\frac{2\pi}{m} = 0 \qquad . \qquad . \ (11.72)$$

The condition is when

$$n+1 = l$$

where l is a positive integer

$$f = \Sigma\cos\overline{n+1}\,(\overline{K-1})\frac{2\pi}{m}$$

$$= \sum_{K=1}^{K=m} \cos l\,\overline{K-1}\,\frac{2\pi}{m} = 0 \qquad . \qquad . \ (11.73)$$

and $$(n-1)\frac{2\pi}{m} = l'2\pi$$

where l' is a positive integer

then $$d = \sum_{K=1}^{K=m} \cos (n-1)(K-1)\frac{2\pi}{m}$$

$$= \sum_{K=1}^{K=m} \cos (K-1)l'2\pi = m \qquad . \qquad . \ (11.74)$$

and

$$e = \sum_{K=1}^{K=m} \sin (n - 1)(K - 1)\frac{2\pi}{m}$$

$$= \sum_{K=1}^{K=m} \sin (K - 1)l'2\pi = 0 \quad . \quad . \quad (11.75)$$

and

$$\sqrt{d^2 + e^2} = m$$

We have two conditions for d and e to be zero and $\sqrt{f^2 + g^2}$ to be other than zero, viz. when

$$n + 1 = l \quad . \quad . \quad . \quad . \quad (11.76)$$

and

$$\frac{(n \pm 1)}{m} = l' \quad . \quad . \quad . \quad . \quad (11.77)$$

where l and l' are positive integer

These conditions determine the existence of the nth harmonic in the m.m.f. curve.

If the minus sign is used, the wave travels in the same direction as the fundamental. The positive sign in equation (11.77) indicates that the nth harmonic travels in the opposite direction to the fundamental. Equation (11.76), viz. $n = l - 1$, relates to a single-phase and gives an infinity of harmonics, while equation (11.77) gives the condition, for polyphase windings, for the existence of the nth harmonic, viz.

$$n = l'm \pm 1 \quad . \quad . \quad . \quad . \quad (11.78)$$

The speed of the nth harmonic m.m.f., relative to the stator, is obtained by finding the distance moved by the harmonic in unit time relative to some point on the stator. The m.m.f.

$$= \phi_{Rx} = M \sin \left(\omega t - \frac{n\pi x}{\tau} + \delta_1\right) \quad . \quad . \quad (11.79)$$

where $M = \frac{1}{2}A_{in}\sqrt{d^2 + e^2}$

$= \frac{1}{2}A_{in}m$

Now, taking some point on the stator distant x from the origin, the m.m.f. originally at x, will remain constant in magnitude, but will travel at the speed of the wave, i.e. $\sin \left(\omega t - \frac{n\pi x}{\tau} + \delta_1\right)$ = constant.

Now $\omega t - \frac{n\pi x_1}{\tau}$ is constant, therefore, differentiating with regard to t, we have

$$\omega - \frac{n\pi}{\tau}\frac{dx_1}{dt} = 0 \quad . \quad . \quad . \quad (11.80)$$

$$\therefore \quad \frac{dx_1}{dt} = \frac{\tau\omega}{n\pi} = \frac{\tau \times 2\pi f}{n\pi} = \frac{\tau \times 2f}{n} \quad . \quad . \quad (11.81)$$

Therefore, velocity of the nth harmonic $= \frac{1}{n}$ of the velocity of the fundamental.

Rotor Harmonic M.M.F.s

Each stator harmonic m.m.f. produces its harmonic flux, and each harmonic flux has relative motion with respect to the rotor. Corresponding e.m.f.s and currents will be produced in the rotor. These rotor currents will produce m.m.f., just like the stator currents, and hence there will be a whole series of harmonics produced by the rotor currents, which, in turn, are due to the stator harmonics. However, with m_2 phases in the rotor the order of the harmonics which can exist, due to rotor currents is given by the following equation—

$$n_2 = K_2 m_2 \perp n_1 \quad . \quad . \quad . \quad (11.82)$$

where K_2 has all positive and negative integral values, including zero, and n_1 = order of the stator harmonic and n_2 = order of the rotor harmonic.

The procedure in proving this relation is simply to find, say, the vth harmonic due to the pth phase, and sum up over the m_2 phases.

The nth stator harmonic produces the rotor current

$$\hat{I}_2 \sin s_n \omega t \quad . \quad . \quad . \quad . \quad (11.83)$$

where s_n is the slip with respect to nth stator harmonic.

The vth harmonic m.m.f. in phase 1 of the rotor at point distant x_2 from the mid-point of phase 1,

$$\phi_v = A_v \sin (s_n \omega t) \cos \frac{v x_2 \pi}{\tau} \quad . \quad . \quad (11.84)$$

For phase 2 of the rotor, the space angle is $v \frac{2\pi}{m_2}$

The time angle is determined by the nth harmonic of the stator, so for phase 2 at x_2

$$\phi_{v2} = A_v \sin \left(s_n \omega t - n \frac{2\pi}{m_2}\right) \cos \left(\frac{v x_2 \pi}{\tau} - v \frac{2\pi}{m_2}\right) \quad . \quad (11.85)$$

for phase 3

$$\phi_{v3} = A_v \sin \left(s_n \omega t - n \, . \, 2 \frac{2\pi}{m_2}\right) \cos \left(\frac{v x_2 \pi}{\tau} - v \, . \, 2 \frac{2\pi}{m_2}\right) \quad (11.86)$$

for phase m

$$\phi_{vm} = A_v \sin \left((s_n \omega t - n \, \overline{m - 1} \, \frac{2\pi}{m_2}\right) \cos \left(\frac{v x_2 \pi}{\tau} - v \, \overline{m - 1} \, \frac{2\pi}{m_2}\right) \quad (11.87)$$

Now, summing up, we have $\sum_{m=1}^{m-m_2} \phi_{vm}$

$$\phi_{vm} = \tfrac{1}{2} A_v \left[\sin \left\{s_n \omega t - \frac{v x_2 \pi}{\tau} + (v - n) \, \overline{m - 1} \, \frac{2\pi}{m_2}\right\} \right.$$
$$\left. + \sin \left\{s_n \omega t + \frac{v x_2 \pi}{\tau} - (v + n) \, \overline{m - 1} \, \frac{2\pi}{m_2}\right\}\right] \quad (11.88)$$

$$
\begin{aligned}
&= \tfrac{1}{2}A_v \left[\sin\left(s_n\omega t - \frac{vx_2\pi}{\tau}\right)\cos\,(v-n)\,\overline{m-1}\,\frac{2\pi}{m_2}\right. \\
&\quad + \cos\left(s_n\omega t - \frac{vx_2\pi}{\tau}\right)\sin\,(v-n)\,\overline{m-1}\,\frac{2\pi}{m_2} \\
&\quad + \sin\left(s_n\omega t + \frac{vx_2\pi}{\tau}\right)\cos\,(v+n)\,\overline{m-1}\,\frac{2\pi}{m_2} \\
&\quad \left. - \cos\left(s_n\omega t + \frac{vx_2\pi}{\tau}\right)\sin\,(v+n)\,\overline{m-1}\,\frac{2\pi}{m_2}\right] \qquad . \quad (11.89)
\end{aligned}
$$

Equation (11.89) gives the vth harmonic m.m.f. in phase m, and to find the resultant vth harmonic m.m.f., due to m_2 phases, we must sum up over the m_2 phases.

The Resultant M.M.F. vth Harmonic

$$
\begin{aligned}
\phi_{Kv} &= \tfrac{1}{2}A_v \left[\sin\left(s_n\omega t - \frac{vx_2\pi}{\tau}\right)\sum_{m=1}^{m=m_2}\cos\,(v-n)\,\overline{m-1}\,\frac{2\pi}{m_2}\right. \\
&\quad + \cos\left(s_n\omega t - \frac{vx_2\pi}{\tau}\right)\sum_{m=1}^{m=m_2}\sin\,(v-n)\,\overline{m-1}\,\frac{2\pi}{m_2} \\
&\quad + \sin\left(s_n\omega t + \frac{vx_2\pi}{\tau}\right)\sum_{m=1}^{m=m_2}\cos\,(v+n)\,\overline{m-1}\,\frac{2\pi}{m_2} \\
&\quad \left. - \cos\left(s_n\omega t + \frac{vx_2\pi}{\tau}\right)\sum_{m=1}^{m=m_2}\sin\,(v+n)\,\overline{m-1}\,\frac{2\pi}{m_2}\right] \quad (11.90)
\end{aligned}
$$

Since v and n are integers, the sums are zero, except when

$$(v-n)\frac{2\pi}{m_2} = K_2 2\pi$$

where K_2 = integer including 0

or

$$(v+n)\frac{2\pi}{m_2} = K_2 2\pi$$

$$\therefore \qquad v = K_2 m_2 \pm n \qquad . \quad . \quad . \quad (11.91)$$

CHAPTER XII

Noise—Causes and Treatment[*]

THE causes of noise are mechanical and electrical. Chief mechanical causes of noise are dynamic unbalance, eccentricity of air-gap, loose cores, and noises due to windage. These can be eliminated fairly simply, but there are many electrical causes of noise, and the elimination of noise, due to electrical causes, demands from the designer a much greater insight into theory than is necessary for ordinary design work. That, perhaps, is not surprising, for the subject is by no means easy.

Magnetic Humming

First we have magnetic humming, which is due to vibration of the steel laminations, caused by the rapid pulsation of flux in the teeth of stator and rotor. When rotor teeth are opposite stator teeth, the magnetic reluctance is a minimum; when rotor slots are opposite stator teeth we have maximum reluctance. Thus, we have the flux varying rapidly in stator and rotor teeth as the rotor revolves. Since the pull on the teeth varies as the square of the flux density, it is clear they are subjected to a succession of rapidly varying mechanical impulses, which causes vibration of the laminations. The frequencies of these vibrations are in the region to which the ear is most sensitive.

The design of the teeth in both stator and rotor, and especially the design of the lip of the teeth, with semi-enclosed slots, is all important. They must be sufficiently rigid to withstand the forces to which they are subjected without vibration, and obviously the cores must be very firmly held in position. In this case, skewing of the slots is very helpful in such noise elimination.

If very quiet motors are required, one must not only use skewing of slots, but the normal flux density in the teeth and gap must be kept low, and the air-gap must be moderately large to cut down the influence of parasitic fluxes and torques. Extreme quietness means, in effect, a very liberal design of motor.

---

[*] The substance of this chapter is due to Mr. Riggenbach (see *Brown-Boveri Journal*, October, 1933)

Noise due to Asynchronous and Synchronous Torques

Since noise is caused by vibration, it will be present when vibratory forces are present, and noise caused in this way occurs during starting and running. The noise due to asynchronous and synchronous torques is most evident during starting. The vibratory forces are caused when the numbers of pairs of poles of the higher harmonic fields in stator and rotor, differ by 1, or 2, etc. We have already dealt with the expressions for the harmonic fields set up in both stator and rotor. When the pole pairs differ by 1, a magnetic pull is set up in a certain direction, and this travels around the machine, tending to make the stator and rotor oscillate. If the difference of the pole pairs is more than 1, there will be several unbalanced radial forces, which travel round the rotor and cause vibration and noise.

The polyphase winding, which is symmetrical, produces the following harmonics $n = \pm (2lm \pm 1)$, l is a positive integer.

With $m = 3$, we have

$$\begin{aligned} n_1 = {} & (-5),\ (+7),\ (-11),\ (13+),\ (-17), \\ & (+19), (-23), (+25), (-29), (+31), \\ & (-35), (+37), (-41), (+43) \qquad \cdot \quad \cdot \quad \cdot \ (12.1) \end{aligned}$$

The stator *slotting* gives rise to higher harmonic fields in the air-gap, the numerical order of which

$$n_1' = \pm \left\{ \frac{N_s l}{p'} \pm 1 \right\} \qquad \cdot \quad \cdot \quad \cdot \ (12.2)$$

where N_s = number of stator slots

l = integer

p' = *pairs* of poles

The rotor slotting gives rise to higher harmonic fields, which are produced by the fundamental stator field, as well as stator higher harmonic fields.

These harmonic fields of the rotor are given

$$n_2 = \pm \left(\frac{N_r l'}{p'} \pm n_1 \right) \qquad \cdot \quad \cdot \quad \cdot \ (12.3)$$

where N_r = number of rotor slots

l' = positive integer

The most important rotor higher harmonic fields are those which correspond to the fundamental (i.e. $n_1 = 1$) and for this case ($n_1 = 1$)

$$n_2 = \pm \left\{ \frac{N_r l'}{p'} \pm 1 \right\} \qquad \cdot \quad \cdot \quad \cdot \ (12.4)$$

We have

$$p'n = \pm p'(2lm \pm 1) \quad . \quad . \quad . \quad (12.5)$$

and

$$p'n_1' = \pm (N_s l \pm p') \quad . \quad . \quad . \quad (12.6)$$

and

$$p'n_2 = \pm (N_r l' \pm n_1 p') \quad . \quad . \quad . \quad (12.7)$$

for a pole-pair difference of 1,

$$p'n - p'n_2 = 1 \quad . \quad . \quad . \quad . \quad (12.8)$$

i.e.

$$\pm p'(2lm \pm 1) - \pm \{N_r l' \pm r_1 p'\} = 1 \quad . \quad . \quad (12.9)$$

$$\therefore \qquad N_r = \frac{p'\{\pm (1 + n_1) - 2lm\} - 1}{l'} \quad . \quad . \quad (12.10)$$

For three-phase motors $m = 3$, and n_1 is always odd, so that the numerator will be an odd number.

Since l' is an integer, the number of rotor slots will be odd. The smaller n, the greater will be the vibration forces. In a paper given by Riggenbach, in 1933, on noise, examples are given of this condition. He shows that in two- and four-pole motors having odd numbers of rotor slots, pole-pair differences of 1, referred to a stator higher harmonic, appear with the rotor teeth harmonics.

Thus, $n_1 = (-5)$, $(+7)$, (-11), $(+13)$, (-17), $(+19)$, (-23), $(+25)$, (-29), $+31$ and the pole pairs due to these harmonics, are $p'n_1 = 2n_1 = (-10)$, $(+14)$, (-22), $(+26)$, (-34), $(+38)$, (-46), $(+50)$, (-58), $+(62)$.

The fundamental stator field $n_1 = 1$, creates higher rotor harmonics, given by

$$n_2 = \pm \left\{\frac{N_r l'}{p'} \pm 1\right\}$$

with pole pairs

$$p'n_2 = \pm Nrl' \pm p'$$

for $N_r = 25$ and $p' = 2$ (i.e. four poles), we have

$$p'n_2 = (+27), (-23), (+52), (-48), (+77), -(73), \text{etc.}$$

We see there are several rotor harmonics having pole pairs differing by 1. Thus, we have

— 22, due to the eleventh stator harmonic
and — 23, the rotor harmonic
+ 26, due to the thirteenth stator harmonic
and + 27, the rotor harmonic

Now if we make $n_1 = 5$, we have

$$p'n_2 = \pm [Nrl' \pm n_1 p']$$

since $n_1 = 5$ and $p' = 2$, we have

$$p'n_2 = (-15), +(35), (-40), (+60)$$

Here we have, due to the fifth stator harmonic, two fields differing by one pole pair, viz. $(+14)$, (-15), and (-34), $+(35)$.

Again, when $n_1 = 7$, the stator seventh harmonic produces rotor harmonics with pole pairs of 11 and $+39$, and, again, there are two pole pairs, viz. -10, -11, and $+38$, $+39$, differing by one pole pair. Thus, a four-pole, three-phase motor with twenty-five rotor slots produces a series of higher harmonic fields with pole pairs differing by 1. This motor, as stated by Riggenbach, is very noisy, and he cites the following frequencies as detected.

$$f = 500,\ 525,\ 605,\ 685,\ 750,\ 860,\ 900,\ 965,\ 1165,\ 1280,\ 1315, \text{ and } 1355$$

The conclusion seems to be that odd numbers of rotor slots should be avoided; but it is easy in any case to tabulate, as done by Riggenbach, the harmonics which cause vibration and noise for any given slot numbers.

Synchronous Torques

With certain slot ratios in stator and rotor, higher harmonic fields of the same order, and having the same direction of rotation can be present in stator and rotor. We have shown that the speed of rotation is inversely as the order of the harmonic. On starting up such a motor, one passes through the synchronous speed of the harmonic considered. Below the synchronous speed of the harmonic, we have a motoring torque; above synchronous speed a retarding torque. This gives rise to the well-known saddle-backs in the starting torque curve. The number of rotor slots, which is responsible for this condition is given by equating $p'n$ to $p'n_2$,

i.e. $$\pm p'(2lm \pm 1) = \pm[N_r l' + n_1 p'] \qquad (12.11)$$

i.e. $$N_r = \frac{p'\{2lm \pm 1 \pm n_1\}}{l'} \qquad (12.12)$$

where, of course, l and l' are integers.

Clearly when the number of rotor slots N_r is a multiple of the number of pole pairs p', we get these effects. When $n_1 = 1$ and also $l' = 1$, we get the worst condition,

i.e. $$N_r = 2p'[lm \pm 1] \qquad (12.13)$$

This shows that in two-pole motors, every even number of slots gives rise to these synchronous torques. In these motors (two-poles), the number of rotor slots must be chosen so that synchronous effects occur only with harmonics of high order. These effects may also be due to harmonics of stator and rotor slottings.

This occurs when

$$\frac{N_s l}{p'} \pm 1 = \frac{N_r l'}{p'} \pm n_1 \quad . \quad . \quad . \quad (12.14)$$

from which

$$N_r = \frac{N_s l \pm p'(1 \pm n_1)}{l'} \quad . \quad . \quad . \quad (12.15)$$

and when $n_1 = 1$ and $l' = 1$, we again have the worst condition, viz.

$$N_r = N_s l \pm 2p' \quad . \quad . \quad . \quad (12.16)$$

We see, therefore, that the number of rotor slots must not be a multiple of the number of stator slots increased or decreased by $\pm\ 2p'$.

When $l = 1$, we get the greatest synchronous effect, i.e. when the number of rotor slots differs, by the number of poles, from the number of stator slots.

Asynchronous Torques due to Higher Harmonics

The higher harmonics of the stator teeth produce supplementary torques. These higher stator harmonics, due to stator teeth, are

$$n_1' = \pm \left\{ \frac{N_s l}{p'} \pm 1 \right\} \quad . \quad . \quad . \quad (12.17)$$

and the number of pole pairs in the fields

$$= p' n_1' = \pm\ (N_s l \pm p') \quad . \quad . \quad . \quad (12.18)$$

The most important is given by $l = 1$, i.e.

$$p' n_1' = + (N_s \pm p') \quad . \quad . \quad . \quad (12.19)$$

When the number of rotor bars is the same as the number of poles in the harmonic field, i.e. when $N_r = 2(N_s \pm p)$, the induction motor torque, or asynchronous torque, is greatest.

Here the beneficial effect of skewing can be clearly seen, for the effect of skewing is to weaken the effect of these harmonic fields.

These asynchronous torques cause marked indentations in the speed-torque curve, and in some cases may even reduce the starting torque very greatly. It is clear that the choice of the number of slots in stator and rotor is a very important one. In all cases where noise is to be eliminated one must have—

(*a*) Skewed slots in the rotor.
(*b*) Reasonably large air-gap length.
(*c*) Avoid odd numbers of slots in the rotor.
(*d*) Attend to the rules given above.

The reader is referred to the article, in the *Brown-Boveri Journal*, by Riggenbach, to whom the writer is indebted. It should be stated that with the squirrel-cage rotor, since the number of phases

$$= \frac{N_r}{p'}$$

the order of the harmonics for the rotor

$$n_2' = \frac{N_r l'}{p'} \pm n_1 \quad . \quad . \quad . \quad (12.20)$$

The slot harmonics of stator and rotor, viz.

$$\pm \frac{N_s}{p'} + 1 \text{ and } \pm \frac{N_r}{p'} + 1$$

are important, for they have the same distribution factor and same coil-span factor as the fundamental. They are particularly disturbing.

Let $\frac{N_s}{p'} = Q_1$ = slots per *pole pair* in stator

$\frac{N_r}{p'} = Q_2$ = slots per pole pair in rotor

Then the distribution factor for the stator harmonics

$$= \frac{\sin(\pm Q_1 + 1)q_1 \lambda/2}{q_1 \sin(\pm Q_1 + 1)\lambda/2} \quad . \quad . \quad . \quad (12.21)$$

Now $$\lambda/2 = \frac{2\pi}{2Q_1} = \frac{\pi}{Q_1}$$

therefore, breadth factor for the harmonic $\pm \frac{N_s}{p'} + 1$

$$= \frac{\sin(\pm Q_1 + 1)q_1 \frac{\pi}{Q_1}}{q_1 \sin(\pm Q_1 + 1)\frac{\pi}{Q_1}} = \frac{\sin q_1 \frac{\pi}{Q_1}}{q_1 \sin \frac{\pi}{Q_1}} = \frac{\sin q_1 \lambda/2}{q_1 \sin \lambda/2} \quad (12.22)$$

and $$\cos n\frac{\varepsilon}{2} - \cos \pm (Q_1 + 1)\frac{\varepsilon}{2} \quad . \quad . \quad (12.23)$$

Now ε = deficiency of pitch from full pitch with short-span coils $= x\lambda$, where x is the number of slots by which the coil span, in slots, differs from the number of slots per pole.

$$\therefore \quad \frac{\varepsilon}{2} = \frac{x\lambda}{2} = \frac{x\pi}{Q_1} \quad . \quad . \quad . \quad . \quad (12.24)$$

$$\therefore \quad \cos n\frac{\varepsilon}{2} = \cos \pm (Q_1 + 1)\frac{x\pi}{Q_1} = \cos\frac{x\pi}{Q_1} = \cos\frac{\varepsilon}{2} \quad (12.25)$$

Thus, the coil-span factor for the $\pm Q_1 + 1$ harmonics is the same as that of the fundamental.

Thus, slot distribution and span, i.e. the winding factor, does not reduce this harmonic.

CHAPTER XIII

Testing of Induction Motors

THE usual tests carried out in practice are—

(1) The no-load test
(2) The short-circuit test
(3) The ratio of transformation
(4) Load tests
(5) Resistance tests

In the no-load test, the motor is connected to the a.c. supply and started in the usual way. If of the three-phase type, the two-wattmeter method is used for measuring the power input to the motor. An ammeter and voltmeter are also required. The motor with its rotor short-circuited is run light at various voltages above and below the normal, the frequency being kept constant throughout the test and the input to the motor measured; this input represents the total power required to supply the iron losses, the friction and windage loss, and the ohmic losses in the stator and rotor. The latter are usually small at no load and may often be neglected.

Now plot the watts input as ordinates and the voltage as abscissae, and extend the curve so as to intersect the vertical axis through the origin. The power at this point represents the constant friction losses. Also the current taken at no load at normal voltage and frequency, and the power factor should be carefully noted.

Clamp the rotor and apply varying voltages to the stator. The maximum value of the applied voltage in the short-circuit test should not usually exceed 25 per cent of normal voltage, and take a series of readings of amperes and watts and voltage.

It is advisable in this test to turn the rotor round slowly to give a true average short-circuit test. An ammeter and wattmeter, whose current-carrying capacity is about 25 to 50 per cent greater than normal full-load current, is required, and a low-reading voltmeter.

Plot the mean short-circuit current as ordinates and applied voltage as abscissae. The curve so obtained will be a straight line, and probably bending upwards as normal voltage is reached. The value of the short-circuit current at normal voltage is usually obtained by assuming the line straight, i.e. by assuming the short-circuit current is strictly proportional to the applied voltage. The power factor at short circuit being known from the ratio of watts and volt-amperes, the circle diagram can be constructed.

Next measure the resistances of the rotor and stator per phase by means of direct current.

The mean transformation ratio is also required, and may be found with sufficient accuracy by measuring the voltages on the stator and rotor circuits when the latter is open circuited. The rotor should be moved into various positions relative to the stator windings; also the variation of the ratio with applied p.d. should be noted. The transformation ratio, as above determined, is inaccurate to the extent that a fraction of the magnetic flux, generated by the primary winding in this test, does not link the secondary, but instead leaks across the slots, the stator inner periphery, around the winding end connections, and also zigzags along the tops of the teeth, the total leakage amounts to 3 or 4 per cent, depending upon the design. A leakage factor by which the transformation ratio must be corrected may be found as follows—

(*a*) Apply a p.d. to the stator E_1 and observe the rotor voltage E_2 on open circuit.

(*b*) Apply a p.d. to the rotor E_2' and observe the stator voltage E_1' on open circuit; then the true transformation ratio

$$= \frac{E_1}{E_2} \times v \quad . \quad . \quad . \quad . \quad (13.1)$$

where

$$v = \sqrt{\frac{E_1 E_2'}{E_1' E_2}} \quad . \quad . \quad . \quad . \quad (13.2)$$

because in test (*a*) and (*b*)

$$\frac{E_1}{vE_2} = \frac{vE_1'}{E_2'} \quad . \quad . \quad . \quad . \quad (13.3)$$

Load Tests

Whenever possible, a motor should be tested on load and running under normal conditions. For the determination of its performance, namely, efficiency, power factor, slip, and torque, an accurate form of dynamometer or brake is necessary. After running the motor sufficiently long to heat up the windings and bearings, it may be tested for efficiency and performance. All instruments should be carefully calibrated.

With constant line p.d. and frequency, the tests should be carried out from no-load to the maximum torque the motor can develop. Observations should be made of—

(*a*) The torque. (*b*) The speed.

In practice, the best way of measuring the slip is to use the stroboscopic method. A cardboard disc divided into a number of black and white sectors alternately, the total number of sectors being equal to twice the number of motor poles, is attached to the motor shaft. The disc is illuminated by an arc lamp or incandescent lamp, which receives its current from the same source as the motor.

Now, an arc is extinguished twice during each complete wave of current, so that the light coming from it will consist of a succession of flashes.

If the motor were driven at synchronous speed, since the time taken by a white sector to move into the position occupied by the white sector ahead of it is equal to half a period, it follows that as each flash reaches its maximum brightness, the white and black sectors are in the same relative position, and the disc appears stationary. If the motor slips, the positions of the sectors will be retarded relatively to the flashes, and the disc will appear to rotate. By counting the number of apparent revolutions of the disc in any convenient time, we obtain the slip revolutions. If we determine the motor revolutions during the same time interval, then the slip

$$= \frac{\text{slip revolutions}}{\text{motor revolutions} + \text{slip revolutions}}$$

This method is all right for small values of the slip, but it becomes difficult to count the revolutions of the disc at high slips. The difficulty can be overcome by using a direct-reading slip indicator of the Drysdale type. The instrument consists of a boxwood cone mounted on a spindle whose end may be applied to the motor shaft, so that the cone is driven at the same speed as the motor. Resting on the cone is a pivoted disc mounted at the end of a light fork which is pivoted in a slider. The position of the slider may be varied by means of a screw, and may be read off on a scale which gives the slip directly. A disc of paper is attached to the pivoted disc, which has a number of alternate black and white sectors as before mentioned, and this is illuminated as before mentioned.

If the position of the disc is such that its diameter is equal to that of the cone, the disc and cone will revolve at the same speed; and if this speed were the synchronous speed, the disc would appear at rest. This gives the zero of the scale. Since the speed of the motor is less than synchronous speed, the disc will appear to rotate. The speed of the disc may be increased by moving it along to the thicker end of the cone until it appears stationary. The method is a zero method.

The value of the slip corresponding to a given displacement of the disc from the zero position can be calculated from the angle of the cone.

Intermittent Contact Method of Measuring Slip

In this method a disc of paper, with a suitable angle cut out of it, is pasted on the end of the motor shaft, and a contact brush is pressed against it. The brush forms one contact of the intermittent contact circuit, and the circuit is completed through a polarized ammeter. This circuit is supplied from a transformer connected to the source of supply. It is clear that contact will be made and broken at different points of the stator p.d. wave if the motor is rotating asynchronously, and consequently the frequency of oscillations of the ammeter equals the frequency of the rotor slip currents.

Numerous other methods are also used.

Efficiency Tests

Two similar induction motors, of about the same output, may be tested for efficiency by a modification of the Hopkinson test for d.c. machines. The two machines are belted together, but the sizes of the pulleys are chosen dissimilarly, so that the machine with the smaller pulley is driven above synchronous speed and hence acts as induction generator. The difference in the diameters of the pulleys must be such as to give slips corresponding to the maximum load when the rotors are short-circuited. As in the corresponding d.c. test, the mains supply only the losses in the two machines, and the power required for driving the motor is derived from the generator. The machines are connected up as shown in Fig. 13.1.

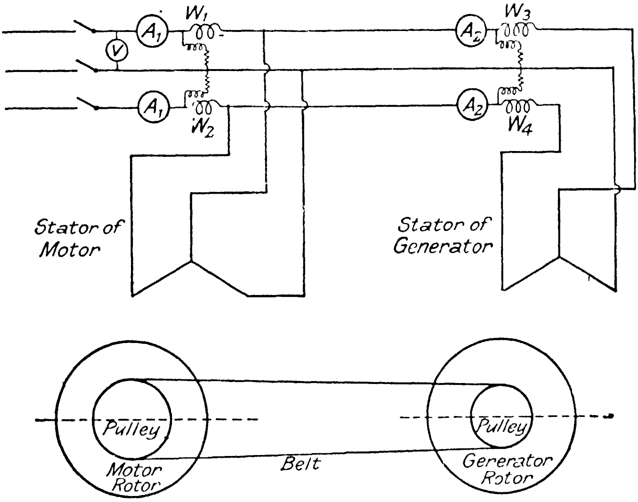


Fig. 13.1

Let w = power from mains = algebraic sum of the readings of wattmeters W_1 and W_2

= total losses in both machines

ω_b = power required to bend the belt and drive it against air friction

the actual power wasted *in* the machines = $w - \omega_b$

The power wasted in each machine is proportional to the slip of each.

Let s_g = slip of the generator

s_m = slip of the motor

then the power lost in the motor

$$= \frac{s_m}{s_m + s_g}(\omega - \omega_b) \quad . \quad . \quad . \quad (13.4)$$

and the power lost in the generator

$$= \frac{s_g}{s_m + s_g}(\omega - \omega_b) \quad . \quad . \quad . \quad (13.5)$$

Let W_g = power developed by the generator

= algebraic sum of readings of wattmeters W_3 and W_4.

The input to the motor

$$= W_g + (w - \omega_b) \quad . \quad . \quad . \quad (13.6)$$

the output of the motor

$$= W_g + \omega - \omega_b - \frac{s_m}{s_m + s_g}(\omega - \omega_b) \quad . \quad . \quad (13.7)$$

$$= W_g + (w - \omega_b)\frac{s_g}{s_m + s_g} \quad . \quad . \quad . \quad . \quad (13.8)$$

and the efficiency of the motor

$$\frac{W_g + (\omega - \omega_b)\dfrac{s_g}{s_m + s_g}}{W_g + \omega - \omega_b} \quad . \quad . \quad . \quad (13.9)$$

Similarly the efficiency of the generator is obtained by dividing its output W_g by the power transmitted to it from the motor. The power for bending the belt and driving it against air friction may be obtained as follows—

(1) Read the wattmeters W_1 and W_2 when both stators are connected to the mains and the rotor of the motor short-circuited, the motor driving the generator by belt, but the rotor of the generator is open-circuited.

(2) Read the wattmeters W_1 and W_2 when both machines are running light with the belt off.

The difference of the power taken in the two cases is ω_b. It may be necessary to apply a correction for belt slipping if such occurs. This may be applied as follows—

Let D_g = diameter of generator pulley in any suitable units

D_m = diameter of motor pulley in same units

t = belt thickness

$$\frac{\text{speed of generator}}{\text{speed of motor}} = \frac{D_m + t}{D_g + t} \quad . \quad . \quad (13.10)$$

By reason of belt slip, this ratio is reduced in ratio

$$\frac{1-\alpha}{1} \quad . \quad . \quad . \quad . \quad (13.11)$$

where α may be termed the belt slip

$$\therefore \quad \frac{\text{speed of generator}}{\text{speed of motor}} = \frac{D_m + t}{D_g + t}(1-\alpha) = \frac{1+s_g}{1-s_m} \quad . \quad (13.12)$$

where s_g and s_m = slips of generator and motor respectively.

Since the speed of the generator is reduced in the ratio $\frac{1-\alpha}{1}$ the power transmitted to it by the motor is reduced in the same ratio. A fraction α of the total power developed by the motor is lost in producing heat at the pulleys. If η = motor efficiency, calculated approximately as before, then in order to balance the loss due to belt slip, an extra amount of power must be drawn from the mains

$$= \alpha \frac{W_m}{\eta}$$

where W_m = power developed by the motor

$\therefore$ the power wasted in the machines

$$= w - \omega_b - \frac{\alpha W_m}{\eta}$$

Alexanderson has devised an approximate method of determining the efficiency of an induction motor.

The induction motor is belted to a separately-excited d.c. generator. In calculating the efficiency, the torque is assumed proportional to the slip, which is near enough to the truth for loads up to full load. The output of the generator is measured by voltmeter and ammeter, and the copper loss in the armature and brushes calculated.

Let W = total mechanical power of the motor

W_1 = power to balance the rotor-bearing friction and windage, belt losses, and friction and core losses of the generator

T = torque corresponding to W

T_1 = torque corresponding to W_1

then $$\frac{W}{W_1} = \frac{T}{T_1} = \frac{s}{s_1} = \frac{\text{slip at torque } T}{\text{slip at torque } T_1} \quad . \quad . \quad (13.13)$$

(the speed is assumed practically constant).

The slip is measured on load by the stroboscope or other way, and also with the load switched off the d.c. generator, the excitation being maintained constant.

$$\therefore \qquad \frac{W - W_1}{W} = \frac{s - s_1}{s} \qquad . \quad . \quad . \ (13.14)$$

$$\text{i.e.} \qquad W - W_1 = W \times \left(\frac{s - s_1}{s}\right) \qquad . \quad . \ (13.15)$$

$$W - W_1 = \text{output of generator} + \text{copper loss in the armature} \quad (13.16)$$

$$\therefore W = \frac{s}{s - s_1} (\text{output of generator} + \text{armature copper loss}) \quad (13.17)$$

The brake horse-power is obtained by subtracting the rotor friction loss at the same speed.

To determine the latter, the belt is slipped off, and the slip measured with the motor running light.

Let s_f = slip of the motor running light
W_f = loss due to rotor friction

$$\text{then} \qquad \frac{W}{W_f} = \frac{s}{s_f}$$

$$\therefore \qquad W - W_f = (s - s_f) \frac{W_f}{s_f} \qquad . \quad . \quad . \ (13.18)$$

and the net output of the motor

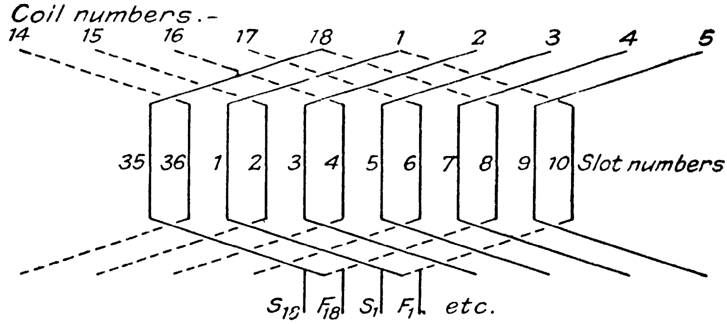
$$= \frac{s - s_f}{s - s_1} \begin{pmatrix} \text{out of generator} \\ + \text{ armature copper loss} \end{pmatrix}$$

The input being known, the efficiency is easily calculated. The method is only approximate.

CHAPTER XIV

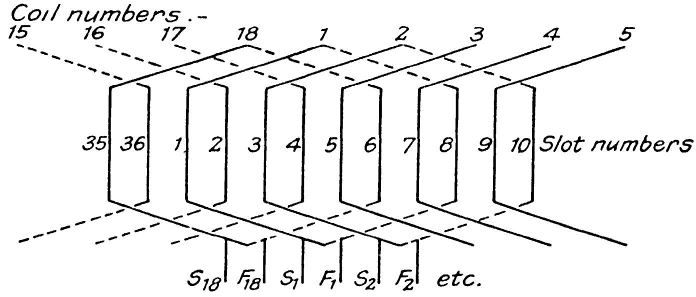
Induction Motor Windings

WINDINGS for a.c. motors are either of the single-layer type, i.e. with one coil side per slot, or double-layer windings. The single-layer windings for small and medium sizes are usually of the mush type.



Position of slot 1 in each case to be on the left hand side of stator and below the horizontal centre line by an amount equal to one *coil span*

6 pole. 36 slots. 7 teeth embraced.

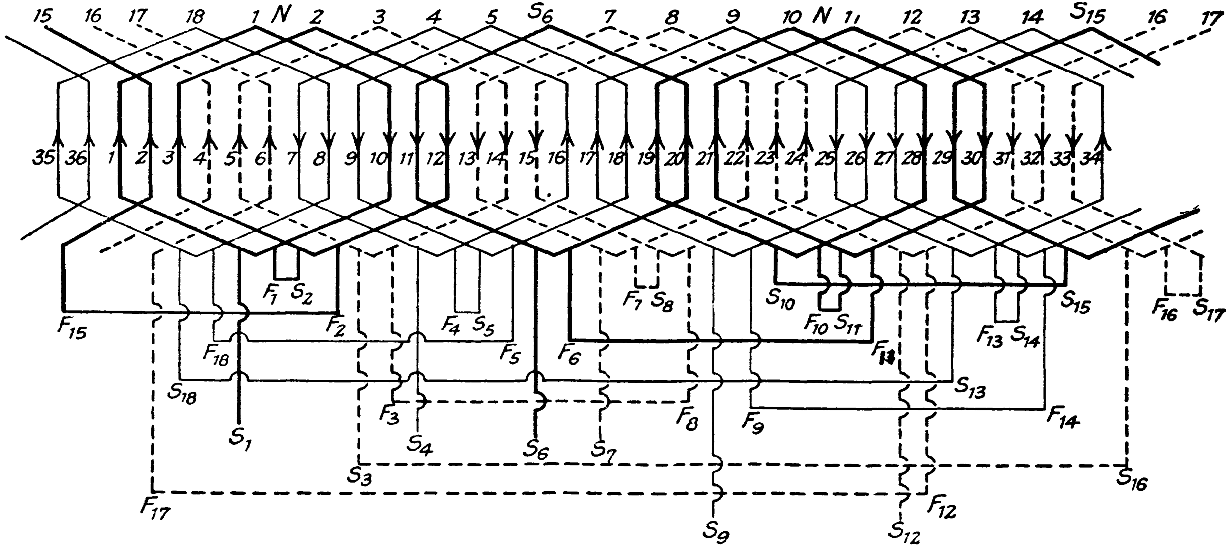


Coils numbered round stator in clockwise direction looking from connection side

FIG. 14.1

Plate X (facing page 213) shows a mush winding for the stator of a three-phase motor, in process of winding, and also gives a view of a mush-wound *rotor* in process of winding.

This mush winding, also called a basket-type winding, is a very common type. It has one coil side per slot, and it will be observed that coil ends, from coils situated in consecutive slots cross one another, i.e. proceed to left and right alternately.


15
16
17
18
1
N
2
3
4
5
S6
7
8
9
10
N
11
12
13
14
S15
16
17
35
36
1
2
3
4
5
6
7
8
9
10
11
12
13
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F1
S2
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S5
F7
S8
S10
F10
S11
S15
F16
S17
F15
F2
F18
F5
F6
F11
F13
S14
S18
S13
F3
F8
F9
F14
S1
S4
S6
S7
S3
S16
F17
F12
S9
S12

Fig. 14.2

In Fig. 14.1 is shown a basket-type winding for 4 poles, 36 slots, and also a layout of coils for 6 poles, 36 slots. The method of numbering the slots and coils is clearly indicated. The pitch of the coils, for the three-phase *mush* winding must always be odd; and for the two-phase type the pitch must be even. Thus, for the four-pole, three-phase, 36-slot mush winding, we have 18 coils, and the coil lies in slots 1 and 10, i.e. 9 teeth embraced. For 6 poles, 36 slots, three-phase, it will be noticed that the coil lies in slots 1 and 8, i.e. 7 teeth embraced.

It is *very important* to remember that the number of parallel circuits per phase, possible in the mush winding is equal to the number of pole *pairs* in the machine.

This is a trap for the unwary designer—especially the young designer. For example, with a four-pole machine, it is only possible, with the mush type, to get *two* parallel circuits per phase. Of course, the same thing is true also for the hemi-tropic, concentric winding. The designer should remember that there is *one coil group per pair of poles* in this mush type of winding. In the four-pole, 36-slots, three-phase mush winding, there are 18 coils, or 6 per phase, and it is not possible to connect it to have 4 parallel circuits per phase. It will be noticed also that coils, whose end connections leave the core in the *same* direction are numbered consecutively, thus coil in slot 1 is called 1, and the coil in slot 3 is called 2. The method of connecting such a winding for three phases is clearly shown in Fig. 14.2.

In setting out such a winding, it is advisable to set out, as in Fig. 14.2, the slot numbers. In Fig. 14.2, the slots are numbered 1 to 36. Now draw in the coils as shown, and number them as shown at the top of the end windings, viz. 1 to 18. Now separate, or mark out, the slots which belong to phase 1. In our diagram they are slots 1, 2, 3, 10, 11, 12, 19, 20, 21, 28, 29, 30. The coil numbers are 1, 15, 2, 10, 11, 6. For 4 poles, the currents must flow in the conductors in the slots of phase 1, thus—

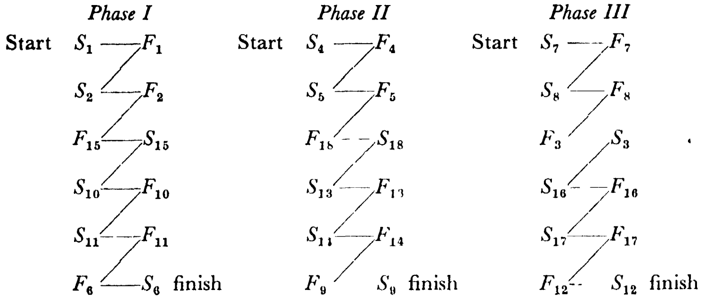
↑ ↓ ↑ ↓

Thus, in slots 1, 2, and 3 we mark the direction of the current ↑; in slots 10, 11, 12 ↓; in slots 19, 20, 21 ↑; and in slots 28, 29, 30 ↓. The winding may then be connected up quite readily. Thus, if we call the beginning of coil 1 S_1, and the end of coil 1 F_1, our connection proceeds as shown on page 296. The method of connecting such coils in parallel circuits per phase is obvious.

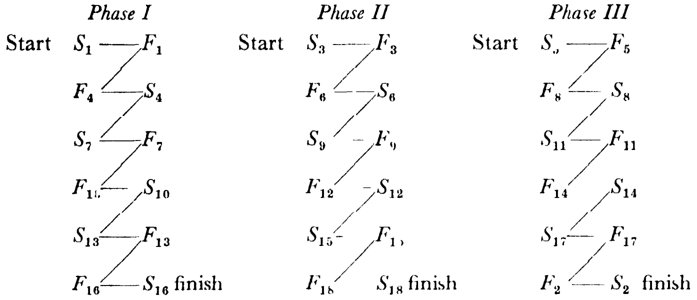
Hemi-tropic or Undivided Coil or Half-coil Windings

These windings are also used, but not so much for motors as in former days. They are known as concentric windings. These windings have one coil group per *pole pair* per phase. Such a winding

Mush Winding for 4-poles, 36-slots, 7 teeth Embraced



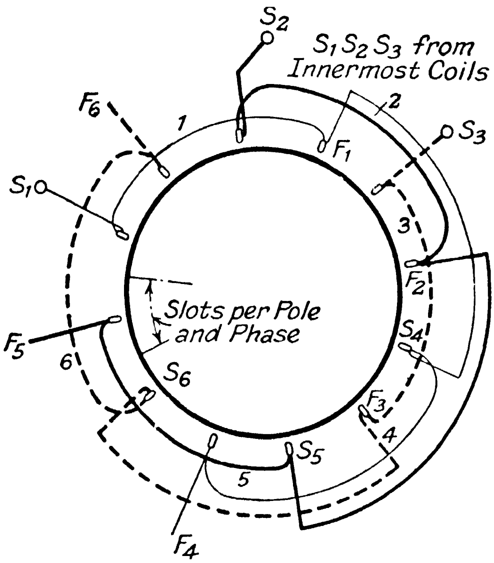
Mush Winding for 6-poles, 36-slots, 7 teeth Embraced



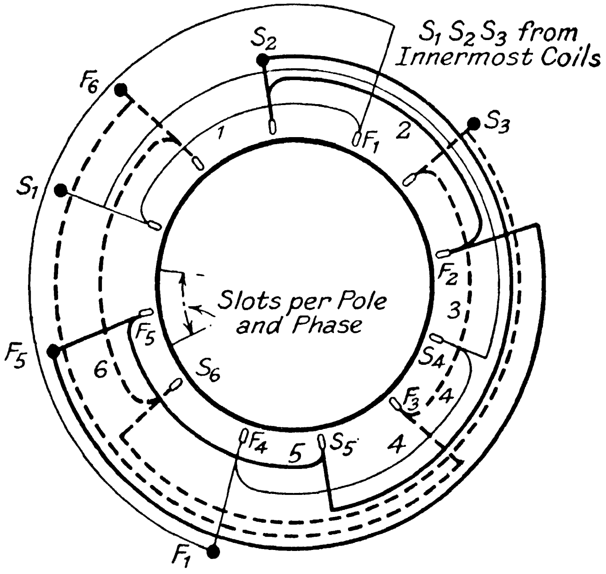
is shown in Fig. 14.3. It is a four-pole winding and the upper part of the diagram shows the two coil groups in series, i.e. one circuit per phase. Below are connections for two parallel circuits per phase.

If the coils are numbered in a clockwise direction, as shown in the top diagram of Fig. 14.3, then for 4 poles we will have 6 coil groups for the 3 phases. Three of these groups have bent coils and three groups have straight coils. In other words, in all cases, where the number of pole pairs is even, half the coils have end connections lying in one plane, and the other half have end connections lying in a plane at right angles to the first, or in a plane inclined to the first. In those cases in which the number of pole *pairs* is odd, we have one *cranked* coil, i.e. a coil group, whose ends lie partly in one plane and partly in the other plane. It will be seen that, with this winding, we always move through the winding in the same sense, i.e. always clockwise, or always anti-clockwise. The coils are formed on a former outside the machine, and can be thoroughly insulated and impregnated before winding. It is rather an expensive type of winding, and, perhaps, a bit inflexible, in the sense that, with a given number of poles only one-half the number of parallel circuits per phase can be used in comparison with the double-layer winding.

Furthermore it is not possible to obtain the advantages which short-pitch or chorded windings possess, for these windings have

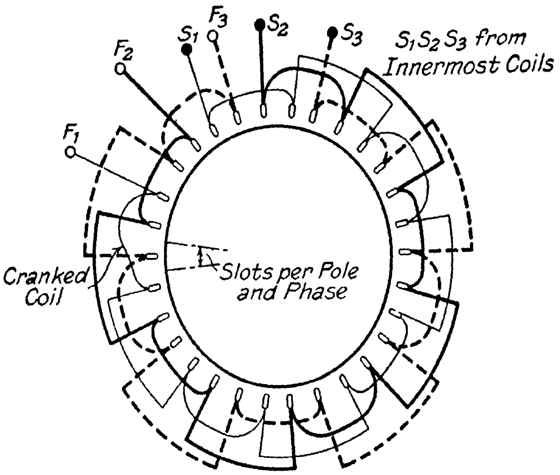


No. 1. Two-Pole Pair in Series

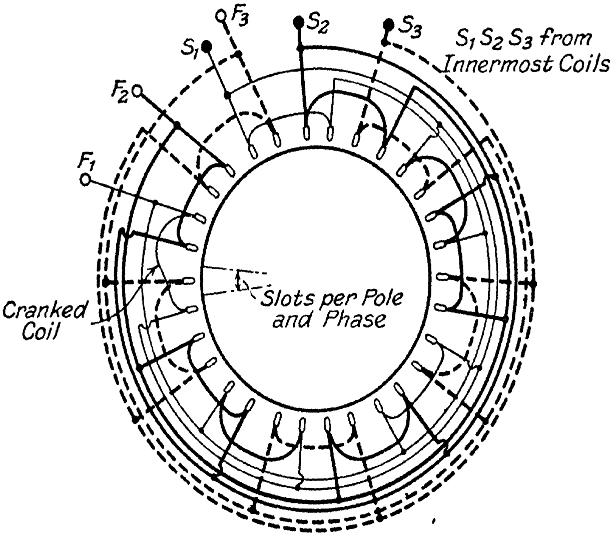


No. 2
1 Pole-Pair in Series
2 Circuits in Parallel

FIG. 14·3



No. 1. 5-Pole Pairs in Series

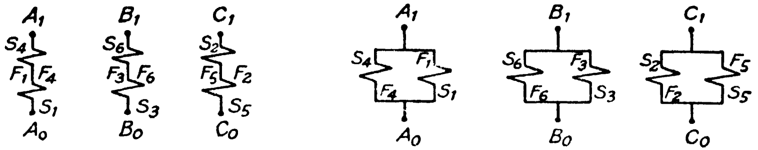


No. 2. 1-Pole Pair in Series.
5 Circuits in Parallel

Fig. 14.4

always an integral number of slots per pole and per phase, and the average span is full pitch.

Furthermore, unless drop-in windings are used, open slots are necessary, which decidedly increase the iron losses, lower the efficiency, and introduce higher harmonics in the flux and torque. Fig. 14.4 shows a hemi-tropic ten-pole, three-phase winding with (*a*) five pole pairs in series, i.e. one circuit per phase, and (*b*) five circuits in parallel.



(1) Coils in Series (2) 2 Circuits in Parallel

Coil Connections

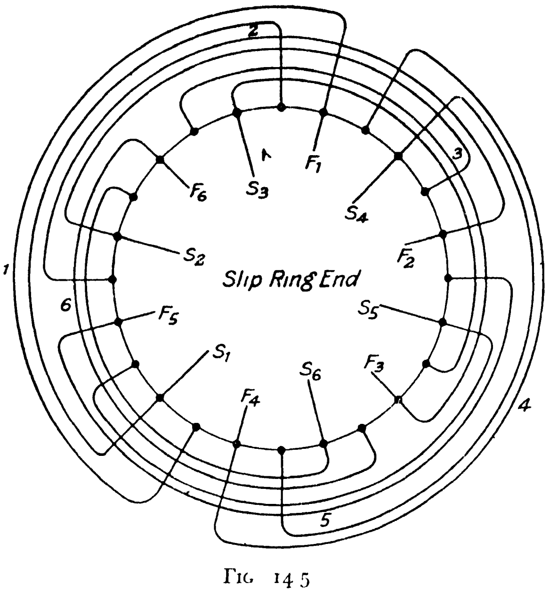


Fig. 14.5

The cranked coil is clearly shown. It is very easy to connect such a winding. The simplest way to proceed is to lay down a plan of the winding and mark on the conductors the current directions under each pole.

Whole-coil Windings

In the category of single-layer windings are the "whole-coil," or divided-coil, windings. In these windings, we have one coil group per pole per phase. They are used usually on two-pole machines. The ends of the coils are arranged to lie in three planes. The length of the mean turn is reduced, in comparison with the hemi-tropic winding, and the end connection inductance is also reduced.

Fig. 14.5 shows a three-phase stator winding for two poles, using divided coils, with the connections for one circuit per phase, and also two circuits in parallel per phase.

With the divided coil winding, in proceeding through the winding, one proceeds clockwise and anti-clockwise through consecutive sections of the winding. Thus, in Fig. 14.5, the beginning and end of the coil group are called S_1 and F_1 for coil group 1, and the connection proceeds as follows—

$$\begin{tikzcd} S_1 \arrow[r, dash] & F_1 \arrow[dl, dash] \\ F_4 \arrow[r, dash] & S_4 \end{tikzcd}$$

This is simply illustrated in Fig. 14.6. Plate XI (facing page 228) shows a three-phase, three-plain, split-phase, hair-pin winding.

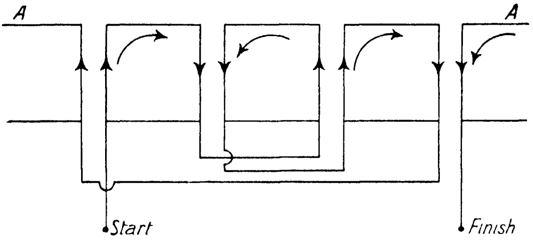


FIG. 14.6. FOUR-POLE DIVIDED COIL WINDING FOR ONE PHASE
There are two slots per pole per phase

It is possible to arrange the overhang in two planes, by the use of a cranked coil as shown in Fig. 14.7, which shows a two-pole, three-phase, stator winding with overhang in two planes.

In all cases, whether divided or undivided coils are used, for symmetry, the start of the phases and also the ends, must differ by $\frac{2\pi}{m}$, where m is the number of phases. Thus, in the three-phase machine, the start of phase 2 must differ by 120 electrical degrees from the start of phase 1, i.e. by two-thirds of the pole pitch, and the same is true of the difference between the starts and finishes of phases 2 and 3. In the two-phase machine, the starts of phases differ by 90 electrical degrees. In all these windings, each pole pitch is divided into a number of belts, equal to the number of the phases, and each phase occupies the section $\frac{\text{pole pitch}}{\text{number of phases}}$ of the pole pitch.

It is desirable to bring out the leads as closely as possible to the terminals; thus, the starts of the phases are taken out in one double pole pitch, near the terminals.

Double-layer Windings

The double-layer winding is very commonly used for induction motor stators and wound rotors. The coils may consist of a number of turns of round wire, and there may be three or four or even five coils per slot. On the larger stators, the conductor may consist of rectangular copper strap, usually not more than 12 to 14 mm deep, and the width of strip (if the section is large) will usually be split up into several thin strips each, say, 1 to 1·5 mm wide. The idea is to reduce the eddy-current losses in the strap. A reference to Field's curves for eddy-current losses in deep and wide conductors will show the importance of laminating the conductor.

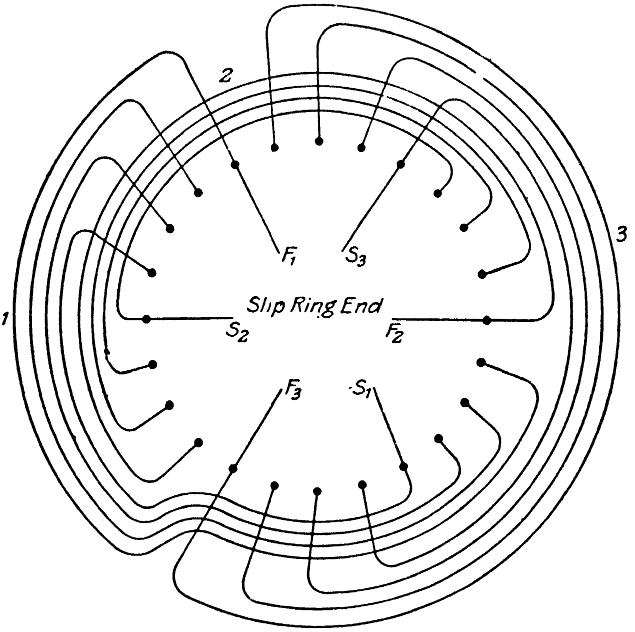


FIG. 14.7. CRANK COIL NO. 1

When open slots are used, the coils can be thoroughly insulated and impregnated, before being placed in the slots. The process of impregnation in vacuum is a very important one, for most insulation failures are due to pockets of air in the coil, which are subjected to dielectric stress. The varnish used is important. Great progress has been made in America in this field, and the introduction of silicone varnishes has allowed the use of still higher temperature rises. While the use of open slots has some advantages, there are also very serious drawbacks to their use. They certainly increase the iron losses greatly, due to flux pulsation in the teeth, and introduce undesirable harmonics into the flux wave, which may affect the torque-speed curve, introducing saddle-backs and also noise.

It is quite possible to use semi-enclosed slots with the double-layer winding; in which case, with coil windings, the conductors are dropped in through the mouth of the semi-enclosed slot. With bar windings, too, the coils are bent at one end, insulated, and pushed through the slot.

There are numerous advantages in the use of the double-layer winding, among which may be tabulated the following—

(1) It is possible by adjusting the span of the coil, i.e. the chording, to obtain the equivalent of a fractional number of turns per coil; thus, for example, in a given case, say, five turns per coil are too few, and six too many, and, say, 5·4 turns are required for the requisite overload capacity, then a coil with six turns and coil-span factor of 0·9 will give us what we want. This is a very useful characteristic, and gives flexibility, which is not present in the single-layer types.

(2) With chorded windings, a considerable saving in copper is effected, especially on two-pole machines; the amount and weight of ineffective end-connection copper is greatly reduced. With two-pole machines a span of coil of two-thirds of full pitch is usual.

(3) Chording reduces the inductance due to slot-leakage flux, and the end-connection inductance is also reduced. With chording, there are slots, which hold top and bottom coil sides of different phases. In the three-phase machine, the ampere-conductors per slot are reduced in the ratio of $\frac{\sqrt{3}}{2}$ for the slots which contain coil sides of two different phases. The ideal short-circuit current is increased, being inversely proportional to the (coil-span factor)^2, viz. $(\sin \beta/2)^2$, were β = actual span of the coil. This follows because the inductance is reduced in the ratio $\left(\frac{\sin \beta/2}{1}\right)^2$.

(4) The chording enables us to eliminate undesirable harmonics from the flux and torque; the coil-span factor for the nth harmonic $= \cos n\frac{\varepsilon}{2}$, where ε = deficiency of pitch from full pitch. When $\cos n\frac{\varepsilon}{2} = 0$, the nth harmonic is zero.

Plate XII shows a double-layer, wave-wound stator winding for an English Electric motor.

With the double-layer windings, the coils are usually of diamond shape, and the end connections lie on a cylindrical surface. With bar windings, one may have either lap or wave end connections. The lap connections are most usual. All the coils have constant span, so can be formed on one former. This is in contrast to the concentric winding, in which coils of different span are used. With the double-layer winding, the number of coils is either equal to the number of slots, or a multiple of the number of slots, depending on the number of coils per slot. There is a top layer of coil sides and a bottom layer. One side of a coil lies at the top of a slot, and its other side lies at the bottom of a slot, a pole pitch away, with full-pitch coils, and a fraction of a pole pitch away with a chorded winding. The connection of the coils in series, to form a single phase, proceeds clockwise, counter-clockwise, clockwise, and so on.

Thus, consider a *full-pitch* winding with p poles and N_s slots. The pole pitch in slots $= \frac{N_s}{p}$. Numbering the slots, clockwise around the machine 1, 2, 3, etc., we start phase 1, say, at the top coil side in slot 1, and bring the connection to the terminal. This coil side is connected at the back of the machine to a bottom coil side in slot $\left(1 + \frac{N_s}{p}\right)$. The front end of this coil side in the bottom of slot $\left(1 + \frac{N_s}{p}\right)$ is then connected to a coil side in slot $\left(1 + \frac{2N_s}{p}\right)$ at the bottom of this slot, and the back end of this coil side is connected to the top coil side in slot number $\left(1 + \frac{N_s}{p}\right)$.

If there are q slots per pole per phase, we first connect the coils in series in the q slots, joining the end of coil 1 to the beginning of coil 2 and the end of coil 2 to the beginning of coil 3, and so on till q slots are wound, and then the end of the q coil group is connected to the end of the next coil group, which is a full pole pitch away, and we proceed in this way around the machine, clockwise and counter-clockwise, as shown in Fig. 14.8. With this winding, we may have as many parallel circuits per phase as there are poles.

This winding is also illustrated for a three-phase rotor for 6 poles, 72 slots later. Now with a full pitch span of coil, the bottom coil sides of the first group of coils, of one phase, fall exactly below the top coil sides of the next coil group, which is connected in series with them. In a fractional pitch winding, however, this is not so. To illustrate this, as before we assume N_s slots and N_s coils. Let the span in slots be λ times the pole pitch, where

$$\lambda = \frac{\text{span of coil in slots}}{\text{number of slots per pole}}$$

We start phase 1 at the front end of coil 1 in slot number 1. The other side of the coil lies at the bottom of slot $\left(1 + \lambda \frac{N_s}{p}\right)$. These coil sides are connected at the back of the stator by an end connection. If there are q slots per pole per phase, we connect the q coils in series first, the end of coil 1 to beginning of coil 2, and the end of

2 to beginning of 3, and so on till q coils or the coils in q slots are so connected. We have proceeded around these coils, or rather the current flows around these coils, say, in a clockwise direction. Now the end of the q coil group is connected to the end of a coil group which is exactly a double pole pitch away from the *beginning* of the first coil group.

To illustrate, let take the case of a four-pole, three-phase winding with 36 slots, with one coil per slot.

The number of slots per pole per phase = 3.

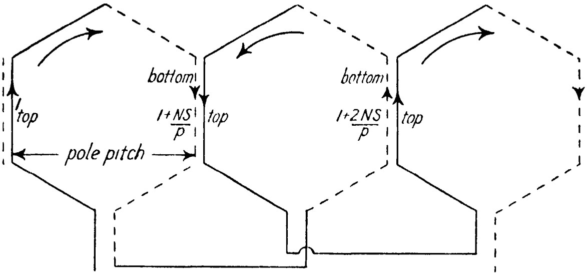
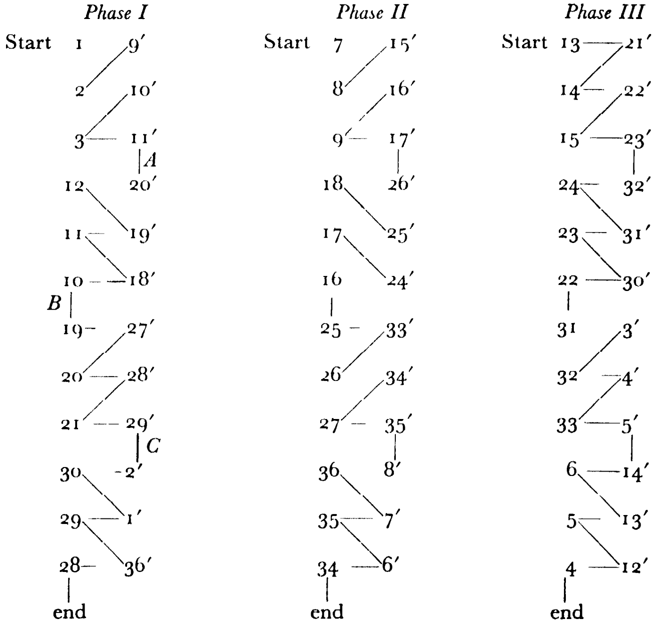


FIG. 14.8

Suppose the winding is chorded 1 slot, i.e. the span of the coil, in slots = 8. The pole pitch, in slots = 9.

Number the coils in the slots thus: in slot 1, top coil side = 1, bottom coil side = 1′; slot 2, top coil side = 2, bottom coil side = 2′; and so on. Our winding proceeds thus—



Note that we proceed, clockwise through 3 coils, then counter-clockwise through the next three, then clockwise through the next three, and counter-clockwise.

The forward pitch is 8 slots, and the back pitch is 7 slots, and at the end of every q slots, we go from bottom coil side to bottom coil side with a full pitch of q slots, or from top coil side to top coil side with full pitch. Thus, in phase 1, these steps are marked A, B, and C.

Fractional Slot Winding

Occasionally one uses a fractional slot winding. In low-speed induction motors, where the number of poles is large, one is sometimes forced to use a number of slots per pole per phase less than 2. It may be, say, 1·25 or 1·5. In this case there is no alternative. In other cases, it may be desired to take advantage of the reduction of harmonics due to the fact that the breadth factors of such windings for these harmonics is smaller than those for integral slot windings. Let the number of slots per pole per phase

$$q = \frac{P}{\alpha} = r + \frac{s}{\alpha}$$

where r is an integer.

The winding will repeat itself after every number of poles $= \alpha$, and each phase has P slots in α poles.

Each coil group will consist of a whole number of single coils. In ordinary windings there is one group per phase per pole. For three-phase windings, if $\alpha = 3$ or any multiple of 3, it is impossible to obtain a balanced winding; α equals the least number of poles in one circuit of the winding, and the maximum number of parallel circuits $= \dfrac{\text{poles in machine}}{\alpha} =$ number of groups which recur.

The breadth factor is the same as for an integral slot winding having P slots per pole per phase. Each phase has $\alpha - s$ coil groups with r single coils and s coil groups with $(r + 1)$ single coils in a number of poles $= \alpha$.

Thus, consider a 24-pole machine, three-phase, with 168 slots. Here

$$q = \tfrac{168}{72} = \tfrac{7}{3}$$

Number of recurrent groups $= \tfrac{24}{3} = 8$.

$r = 2$, $s = 1$, $\alpha = 3$, $P = 7$, but here $\dfrac{\alpha}{3} = 1$, and the winding is unbalanced, i.e. neither the phase voltages nor the phase angles are equal.

In studying these fractional slot windings, Dr. M. M. Liwschitz has introduced what he calls the "slot star."* The slot star is a vector diagram, in which equal vectors represent the e.m.f.s generated in the various slots. The angle between adjacent slots or adjacent vectors $= \frac{180}{mq}$, where m = number of phases, and q = slots per pole per phase.

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* Liwschitz: "Double Chorded Fractional Slot Windings" and "Balanced Fractional-Slot Wave Windings," *J. Amer. I.E.E.*, September (1948)

The slot star represents either the top or the bottom layer of the winding. The slot stars of both layers are identical but are shifted with respect to each other by the chording angle, i.e. by the angle of deficiency from full pitch.

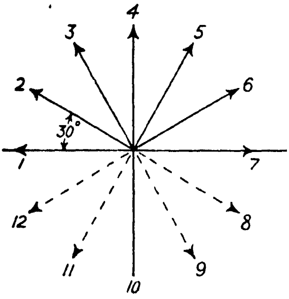


Fig. 14.9

The slot star is shown for a three-phase winding with $q = 2$, in Fig. 14.9, the angle of the vectors $= \frac{180}{6} = 30°$.

Considering the top layer, slots 1, 2, 7, and 8 belong to phase 1, slots 3, 4, 9, 10 to phase 3, and slots 5, 6, 11, 12 to phase 2. Vector 7 is shifted 180° with respect to vector 1. The slot star of the two-pole integral-slot winding is represented completely by half the circle. Since two poles are the unit of the winding, this applies to the integral slot winding in general. In the same way α poles are the unit of the fractional pitch winding, and the slot star of α poles of the fractional slot winding is represented by a semicircle. In this case the angle between consecutive slots

$$= \lambda_s = \frac{180}{mq}$$

where m = phases (number of)

Take $q = 1\frac{1}{4} = \frac{5}{4}$, then

$$\lambda_s = \frac{180 \times 4}{3 \times 5} = 48°$$

Here we have 5 slots in 4 poles per phase, or 15 slots for the 3 phases. The angle between the vectors 1 and 2, 2 and 3, etc. = 48°. We have the following relation between the number of the slot and the slot angle.

Slot	Angle	Slot	Angle
1	0	9	384
2	48	10	432
3	96	11	480
4	144	12	528
5	192	13	576
6	240	14	624
7	288	15	672
8	336	16	720

Now the connections between the coil groups take account of a shifting of 180°, so the real angles between the slots, i.e. the displacement of the slots in the field are as follows—

Slot	Angle	Slot	Angle
1	0	9	24
2	48	10	72
3	96	11	120
4	144	12	168
5	12	13	36
6	60	14	84
7	108	15	132
8	156	16	0

It is clear that the angle between consecutive vectors is not the angle between consecutive slots. The winding creeps in the field, and one must distinguish between the angle between the slots and the angle between consecutive vectors.

The angle in the field

$$= \theta_M = \frac{180}{Pm} = \frac{180}{5 \times 3} = 12^\circ$$

Note in the above, between the slots 1 and 2, with angles 0 and 48 we have the vector 5, 9, 13; between slots 2 and 3 and angles 48 and 96 we have vectors 6, 10, 14; between slots 3 and 4 and angles 96 and 144 we have vector 7, 11, 15; between slots 4 and 5 and angles 144 and 192 we have vectors 8, 12, 16; etc.

Thus, we have slot angle

$$= \frac{180}{qm} = \theta_s \quad . \quad . \quad . \quad . \quad . \quad (14.1)$$

and magnetic field angle

$$= \frac{180}{Pm} = \theta_M \quad . \quad . \quad . \quad . \quad . \quad (14.2)$$

$\theta_M = \theta_s$ for integral slot windings, but for fractional slot windings $\theta_s > \theta_M$.

From equations (14.1) and (14.2), we have

$$\frac{\theta_s}{\theta_M} = \frac{P}{q}$$

i.e. between two adjacent slots in the slot star there are

$$\frac{P}{q} - 1 = \alpha - 1 \quad . \quad . \quad . \quad . \quad . \quad (14.3)$$

other slots for $P = \alpha q$. In our example above, there are $\alpha - 1$, i.e. $4 - 1 = 3$ other slots vectors between adjacent slots. The sequence of slots in the slot star is 1, $1 + d$, $1 + 2d$, $1 + 3d$, and $1 + 4d$, etc., in our case $d = 4$, and we have—

1, 5, 9, 13, 2, 6, 10, 14, 3, 7, 11, 15, etc.

d is the number difference between two slots which correspond to two adjacent vectors in the slot star.

Now $$d \times \theta_s = \theta_M + 180x \quad . \quad . \quad . \quad (14.4)$$

where x = number of full pole pitches between two slots which correspond to two adjacent vectors in the slot star

$$\therefore \qquad d \times \frac{180}{qm} = \frac{180}{Pm} + 180x$$

$$\therefore \qquad d = \frac{1 + xPm}{\alpha} \quad . \quad . \quad . \quad . \quad (14.5)$$

since $\alpha = \frac{P}{q}$

x must be the smallest integer that will make d an integer.

As Dr. Liwschitz shows, it is simple to lay out the fractional slot winding when d is found. Thus, take an example: 20 poles, $q = 1\frac{1}{4} = \frac{5}{4} = 1 + \frac{1}{4}$; $\alpha = 4$; $P = 5$; $m = 3$ phases; $r = 1$; $s = 1$ for

$$q = r + \frac{s}{\alpha}$$

$$\theta_s = \frac{180 \times 4}{3 \times 5} = 48^\circ \quad . \quad . \quad . \quad (14.6)$$

$$\theta_M = \frac{180}{5 \times 3} = 12^\circ \quad . \quad . \quad . \quad (14.7)$$

Now 4 poles make a recurrent group, and there are $P \times m$, i.e. 15 slots in 4 poles for 3 phases. The total number of slots will be 75. $P = 5$ for each phase. Each phase will have, for each 4 poles, the unit of recurrence, 3 groups with single coils and 1 group with 2 coils.

$$d = \frac{1 + x5 \times 3}{4} \quad . \quad . \quad . \quad (14.8)$$

and $x = 1$, gives $d = 4$.

There are in each unit of 4 poles, 15 slots, and 5 slots in each unit of 4 poles per phase.

The sequence of slots for phase 1 for 1 unit of 4 poles

$$= 1,\ 1 + d,\ 1 + 2d,\ 1 + 3d,\ 1 + 4d - 3P \quad . \quad (14.9)$$
$$= 1, \quad 5 \quad , \quad 9 \quad , \quad 13 \quad , \qquad 2$$

or arranged corresponding to the sequence of slots in the machine

$$= 1, 2, 5, 9, 13 \quad . \quad . \quad . \quad (14.10)$$

The 5 slots which belong to phase 2 are

$$\begin{array}{ccccccccc} 1 + 5d - 3P, & 1 + 6d - 3P, & 1 + 7d - 3P, & 1 + 8d - 6P, & 1 + 9d - 6P \\ 6 \quad , & 10 \quad , & 14 \quad , & 3 \quad , & 7 \end{array}$$

or arranged in order for phase 2, slots 3, 6, 7, 10, 14; and for phase 3, slots 11, 15, 4, 8, 12.

There will be 5 such units, each corresponding to 4 poles.

The rule then is, to lay out a fractional-slot winding, first find the number d, and set out for phase 1 the slot series in α poles (as per (14.9)) and the slot sequence (14.10). The sequence of the coil groups for 4 poles (i.e. one recurrent group) for the 3 phases is as follows: 1, 1, 1, 2; and this sequence will be the same for each phase, and, of course, each group of 4 poles will be composed of coil groups having the same sequence.

It has been pointed out that if $\frac{\alpha}{m}$ is an integer there can be no balanced winding, for

$$d = \frac{1 + xPm}{\alpha}$$

and no value of x can be found to make d an integer.

Rotor Windings

For small induction motors of the slip-ring type, it is common practice to use mush rotor windings. These, up to approximately 50 h.p., have semi-enclosed slots. The coils are roughly formed outside the machine and dropped into the slots through the slot-mouth, turn by turn. It is also usual, as in the stator, to use several wires in parallel per turn, to keep the size of conductor small enough to go through the narrow slot mouth. As many as 8 conductors in parallel per turn are often used. This, of course, also reduces the eddy current loss in the conductors of the rotor. This winding has been described in the section on stator windings. It is similar, in every respect, to the mush stator winding. The rotor is invariably star connected, and the three leads from the rotor winding are brought—through the shaft—to three slip-rings. It is usual to use three-phase rotors. Plate X (facing p. 213) shows such a rotor in process of winding. It is usual, however, with rotor windings to bring out the three leads to the slip-rings from parts of the winding, which are disposed *as nearly as possible* 120 mechanical degrees apart, as well as being exactly 120 electrical degrees apart. This simply means that the start of phase 2 is not in the same pole pair as phase 1, and also the start of phase 3 is not in the same pole pair as phase 2. It is not possible, of course, with numbers of poles which are exactly divisible by 3, to bring out the starts of the phases exactly 120 mechanical degrees apart. For example, with six poles, one starts phase 2 with $2\frac{2}{3}$ pole pitches apart from phase 1, and in every case one must get 120 electrical degrees between the starts of phases for the three-phase machine. Occasionally two-phase rotors are used also, and in this case there must be 90° (electrical) phase difference between the phases.

For larger induction motors, of the slip-ring type (*see* Plate XII), a double-layer winding is generally used. This is usually of the two-bar-per-slot type, but with motors, whose outputs run to 1000 b.h.p. and over, one is sometimes forced to use four bars per slot.

The rotor current at full load

$$I_2 = \frac{\text{b.h.p.} \times 746}{3 \times V_p \times \cos\phi_r \times \eta_r}$$
$$\times \frac{\text{number of stator slots} \times \text{conductors per slot in series}}{\text{number of rotor slots} \times \text{conductors per rotor slot in series}}$$

where V_p = applied volts per phase

$\cos\phi_r$ = power factor of the rotor at full load

η_r = efficiency of rotor at full load

$\cos\phi_r \times \eta_r$ is usually about 0·9.

The rotor current per conductor or bar is given by the equation above.

The volts between rings at the start, for star connection,

$$= \frac{\text{b.h.p.} \times 746}{\sqrt{3} \times I_2 \times 0{\cdot}9}$$

It is clear that with large powers, the current to be collected per ring becomes large with two bars per slot rotors, and the transition is then made to four bars per slot, when the current per ring becomes unduly large.

Now this winding is of the barrel type, and is usually wave wound. In the ordinary wave winding, used for d.c. machines, the number of coils and slots must not be divisible by the pairs of poles, for the commutator pitch must be integral

$$y_c = \frac{C \pm a}{p'}$$

where C = the number of coils

a = 1 for a simplex wave

and p' = pairs of poles

In the ordinary d.c. wave winding, one has coils of constant span, and one proceeds with constant pitch of coil around the machine, until one automatically comes to the next conductor from which one started and so on till the winding closes.

With this winding, however, which has an integral number of slots per pole per phase, one proceeds with a normal step round the machine, but to prevent closing the winding after one tour, one increases or decreases the step of the winding by unity. Thus, at every tour or complete turn, one has these abnormal steps. After having completed one-half of the winding, it is necessary to reverse the travel around the machine. It is best to number the bars in a clockwise direction, looking at the rotor from the slip-ring end. The winding is usually left-handed, but this need not be so.

Let q = slots per pole per phase
p = number of poles.

Then with two bars per slot, the number of slots and coils for a three-phase rotor $= 3q \times p$.

The normal winding step $= 3q$.

The abnormal step $= 3q - 1$.

Then one splits the winding into six portions. The number of coils in each portion

$$= \frac{3pq}{6} = \frac{pq}{2}$$

With openings grouped together, the travel of the openings

$$= \frac{pq}{2} \div \frac{p}{2} = q$$

The openings, on the upper bars, will be

Phase	I,	III′,	II,	I′,	III,	II′
Bar	1,	$q+1$,	$2q+1$,	$3q+1$,	$4q+1$,	$5q+1$

The corresponding openings, on lower bars, will be

Bar	$4q$	$5q$	$6q$	$7q$	$8q$	$9q$

To obtain openings, which are distributed round the rotor, it is necessary to displace phases II and III by an integral number of pole pairs. In doing this, one should take note of the arms of the rotor centre carrying the leads to the slip-rings. For a symmetrical arrangement, displace phase II by the nearest whole number of pole pairs to $p/6$, and phase III by the nearest whole number of pole pairs to $p/3$.

Let phase II be displaced by m pole pairs, i.e. by $6mq$ bars and phase III be displaced by n pole pairs, i.e. by $6nq$ bars. Then the openings are—

Phase I

Upper bars: $1;\ 3q + 1$
Lower bars: $4q;\ 7q$

Phase II

Upper bars: $2q + 1 + 6mq;\ 5q + 1 + 6mq$
Lower bars: $6q + 6nq;\ 5q + 6nq$

Phase III

Upper bars: $4q + 1 + 6nq$; $q + 1 + 6nq$
Lower bars: $8q + 6nq$; $5q + 6nq$

The phases can be displaced a different number of pole pairs, if desired. The displacement must always be an integral number of pole pairs. The lower bars, with abnormal step, will be $6(q-1)$ in number, and will be on the bars preceding the lower bar at which the connection is made, i.e. $4q - (q-1)$ to $4q - 1$. It will be seen that, by opening up the winding into six equal portions, the first section and the fourth are 180° out of phase with each other. If one wants to connect these two sections in series, then the beginning of the first section is connected to a slip-ring, and the end of the first section is connected to the end of the fourth, and the beginning of the fourth section goes to the star-point. Similarly sections 3 and 6, and sections 5 and 2 are 180° out of phase with each other and are dealt with in a similar manner. If two circuits, in parallel per phase, are desired, then sections 1 and 4 are connected in parallel, and also 5 and 2, and 3 and 6 for each phase. Sections 1 and 4 are connected in parallel by connecting the end of section 1 to the beginning of section 4, and the beginning of section 1 to the end of section 4, and similarly for the other phases.

Example

Number of poles = 16.
Number of slots = 144.
Number of bars per slot = 2.
144 upper bars, numbered clockwise around the rotor, as follows: 1, 2, 3, etc.
144 lower bars numbered: 1′, 2′, 3′, etc.
Step of winding, front = 9 bars.
Step of winding, back = 9 bars.
Abnormal step = 8 bars.

Lower bars with abnormal step on the front—

Phase I	*Phase II*	*Phase III*
10′–11′	106′–107′	58′–59′
19′–20′	115′–116′	67′–68′

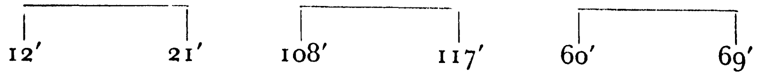
Total abnormal lower bars = 12 = $6(q-1)$; ($q = 3$).
Connections to slip-rings on upper bars—

Phase I	*Phase II*	*Phase III*
1	97	49

Connections to neutral on upper bars—

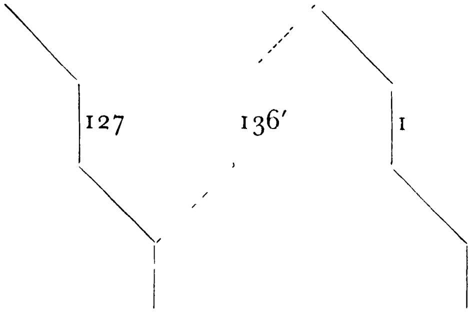
10 106 58

Cross-connections on lower bars—



The winding proceeds thus—

1—136′—127—118′—109—100′—91 and so on



It may be desirable in some cases to have a fractional number of slots per pole per phase, since in such cases the motor starts more smoothly, due to the absence of cogging points, and it gives a greater flexibility in the choice of slots. This winding is usually designed as a six-phase winding; opened out in six places, and each pair of opposite phases is connected in series or parallel.

Let a be any integer, 1, 2, 3, etc., and let it equal the number of similar parts or circuits in a phase of a symmetrical winding. With $a = 1$, it is only possible to get a symmetrical six-phase winding with machines having 2, 10, 14, and 22 poles. It is possible, however, to get a symmetrical three-phase winding for all numbers of poles, which are not a multiple of three. In this case, one can evolve a symmetrical three-phase winding from an unsymmetrical six-phase winding. The parallel connection is possible by making a wave winding with $a = 2$, which gives two similar parts or circuits per phase. Let us consider an ordinary d.c. wave winding, which is closed. We require to open it up and reconnect it to form a three-phase winding. Here again we make use of the vector diagram of e.m.f.s of consecutive coils, and we identify corresponding vectors and their segments or joints. Assume we have a wave winding with C coils and p' *pole-pairs.*

The commutator pitch

$$= y_c = \frac{C \pm a}{p'}$$

The resultant winding pitch

$$= y_r = y_b + y_f$$

where y_b = back pitch in coil sides

y_f = front pitch in coil sides

The openings may be found as follows—

The vectors of e.m.f. of consecutive coils are displaced in phase by a constant angle, assuming the coils to rotate in a pure sine wave of flux distribution. Then, corresponding to any vector number, we have a certain coil.

Thus, vector number: $1, \quad 2, \quad 3, \quad x$
Segments or joints: $1, \ 1 + y_c, \ 1 + 2y_c, \ 1 + (x - 1)y_c$
Top coil sides: $1, \ 1 + y_r, \ 1 + 2y_r, \ 1 + (x - 1)y_r$
$= 1, \ 1 + 2y_c, \ 1 + 4y_c, \ 1 + (x - 1)y_c$
Bottom coil sides: $1 - y_f, \ 1 + 2y_c - y_f, \ 1 + (x - 1)y_c - y_f$

Let the commutator bar (or joint)

$$1 + (x - 1)y_c = b$$

the corresponding top coil side

$$= 2b - 1$$

the corresponding bottom coil side

$$= (2b - 1) - y_f$$

The following example, taken from a paper by Dr. S. P. Smith, will prove helpful.

A wave winding with 8 poles, 105 slots, and 105 coils, has a simplex wave winding $a = 1$, it is required to obtain from it a three-phase winding, suitable for a three-phase rotor. The winding is closed, of course, and it is to be split in six portions, in the way we have indicated.

We have the commutator pitch

$$y_c = \frac{C \pm a}{p'} = \frac{105 \pm 1}{4} = 26 \text{ coils}$$

Resultant winding pitch in coil sides

$$= 2y_c = 52 = y_b + y_r$$

$$\left.\begin{array}{l}\text{Back pitch in coil sides } = 27 \\ \text{Front pitch in coil sides } = 25\end{array}\right\} \text{both must be odd}$$

Coil span y_b *in slots* = slots 1 to 14

Coils per phase in a six-phase winding $= \frac{105}{6} = 17\frac{1}{2}$.

Each of the six portions, into which we split the winding, will consist of seventeen and eighteen coils alternately.

The starts and finishes of the three phases

$$= \frac{2\pi}{3} \text{ radians apart}$$

$$= 2\pi \times \tfrac{35}{105}$$

The following table shows the joints to open, and the corresponding coil sides—

Vector No	*Joint* $b = 1 + (x - 1)y_c$	*Top Coil Side* $2b - 1$	*Bottom Coil Side* $(2b - 1) - y_f$
1	1	1 = top coil side in slot 1	186 = bottom coil side in slot 93
18	23	45 = ,, ,, ,, ,, ,, 23	20 = ,, ,, ,, ,, ,, 10
36	71	141 = ,, ,, ,, ,, ,, 71	116 ,, ,, ,, ,, ,, 58
53	93	185 = ,, ,, ,, ,, 93	160 ,, ,, ,, ,, ,, 80
71	36	71 = ,, ,, ,, ,, ,, 36	46 = ,, ,, ,, ,, ,, 23
88	58	115 = ,, ,, ,, ,, ,, 58	90 = ,, ,, ,, ,, ,, 45

Fig. 14.10 shows the connections for the three phases.

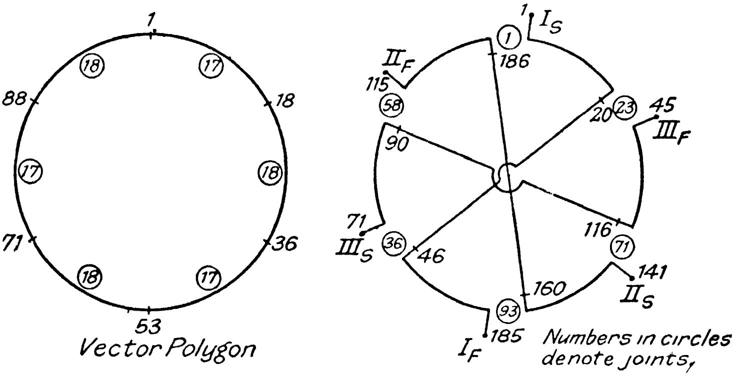


Fig. 14.10

The table on pages 316–17 shows the slottings which may be used for symmetrical wave windings for various numbers of phrases.

Dr. Liwschitz gives his method of dealing with balanced fractional-slot wave windings, which we have already explained. As before, the number of slots per pole per phase $q = \frac{P}{\alpha}$, where P and α have no common divisor; α is the number of poles in a repeatable group of the winding and P the number of slots per phase per repeatable group.

The number of repeatable groups $= \frac{p}{\alpha}$ where $p =$ poles.

We have also seen that the vectors of the slot star are arranged in order, thus: 1, $1 + d$, $1 + 2d$, $1 + 3d$, etc., where $d = \frac{1 + xPm}{\alpha}$ and x is the smallest integer which makes d an integer. If a number of the series of the slot star becomes larger than mP, this quantity is subtracted from the number (m = number of phases). In the lap winding adjacent consecutive upper conductors lie in adjacent slots, but in the wave, two consecutive upper conductors must lie y slot pitches apart, where y = winding pitch, i.e. the consecutive upper

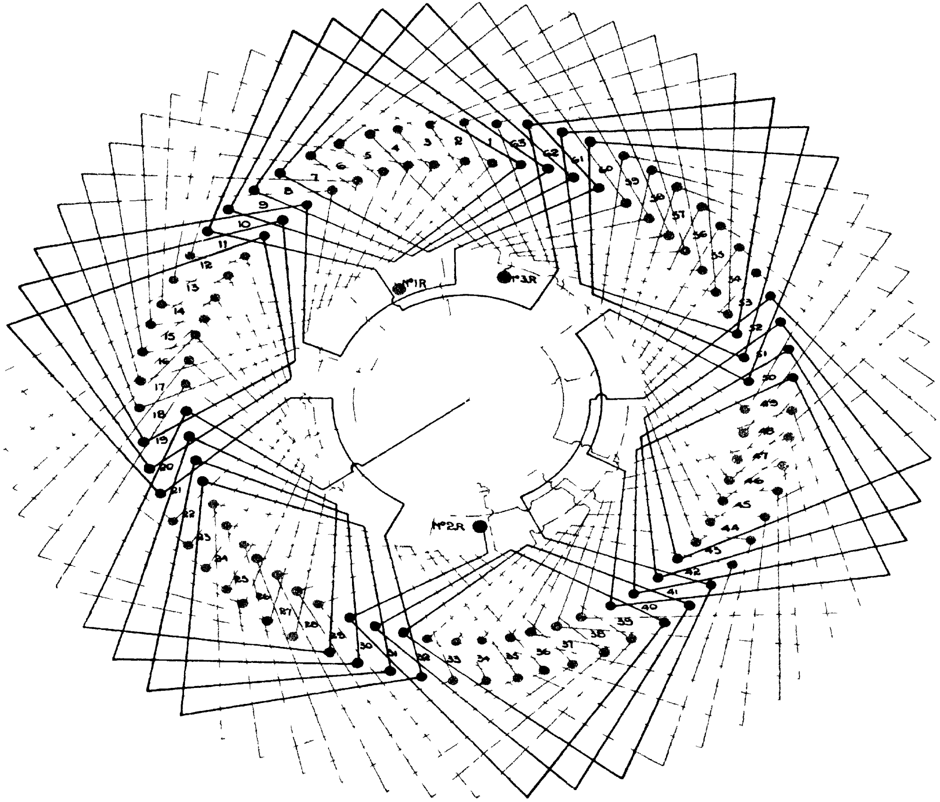
WAVE WINDINGS

Pole-pairs p'	Slots S'	Permissible Coil-sides per slot u	Number of Similar Parts in Winding a	Number of Pole-pairs in Machine $p = p'a$	Number of Slots in Machine $S = S'a$	Number of Slots, S, in Machine when		
						$N = 3$	$N = 4$	$N = 6$
1	n	any even number	1	1	n	$3n$	$4n$	$6n$
			p	p	pn	$3pn$	$4pn$	$6pn$
2	$2n \pm 1$	2, 6, 10, 14	1	2	$2n \pm 1$	$6n \pm 3$	—	—
			2	4	$4n \pm 2$	$12n \pm 6$	—	—
			3	6	$6n \pm 3$	$18n \pm 9$	—	—
			4	8	$8n \pm 4$	$24n \pm 12$	—	—
			5	10	$10n \pm 5$	$30n \pm 15$	—	—
			6	12	$12n \pm 6$	$26n \pm 18$	—	—
3	$3n \pm 1$	2, 4, 8, 10, 14, 16	1	3	$3n \pm 1$	—	$12n \pm 4$	—
			2	6	$6n \pm 2$	—	$24n \pm 8$	—
			3	9	$9n \pm 3$	—	$36n \pm 12$	—
			4	12	$12n \pm 4$	—	$48n \pm 16$	—
4	$4n \pm 1$	2, 6, 10. 14	1	4	$4n \pm 1$	$12n \pm 3$	—	—
			2	8	$8n \pm 2$	$24n \pm 6$	—	—
			3	12	$12n \pm 3$	$36n \pm 9$	—	—
5	$5n \pm 1$	2, 8, 12	1	5	$5n \pm 1$	$15n \pm 6$	$20n \pm 4$	$30n \pm 6$
			2	10	$10n \pm 2$	$30n \pm 12$	$40n \pm 8$	$66n \pm 12$
	$5n \pm 2$	4, 6, 14, 16	1	5	$5n \pm 2$	$15n \pm 3$	$20n \pm 8$	$30n \pm 12$
			2	10	$10n \pm 4$	$30n \pm 6$	$40n \pm 16$	$60n \pm 24$
6	$6n \pm 1$	2, 10, 14	1	6	$6n \pm 1$	—	—	—
			2	12	$12n \pm 2$	—	—	—

7	$7n \pm 1$	2, 12, 16	1	7	$7n \pm 1$	$21n \pm 6$	$28n \pm 8$	$42n \pm 6$
	$7n \pm 2$	6, 8	1	7	$7n \pm 2$	$21n \pm 9$	$28n \pm 12$	$42n \pm 12$
	$7n \pm 3$	4, 10	1	7	$7n \pm 3$	$21n \pm 3$	$28n \pm 4$	$42n \pm 18$
8	$8n \pm 1$	2, 14	1	8	$8n \pm 1$	$24n \pm 9$	—	—
	$8n \pm 3$	6, 10	1	8	$8n \pm 3$	$24n \pm 3$	—	—
9	$9n \pm 1$	2, 16	1	9	$9n \pm 1$	—	$36n \pm 8$	—
	$9n \pm 2$	8, 10	1	9	$9n \pm 2$	—	$36n \pm 16$	—
	$9n \pm 4$	4, 14	1	9	$9n \pm 4$	—	$36n \pm 4$	—
10	$10n \pm 1$	2	1	10	$10n \pm 1$	$30n \pm 9$	—	—
	$10n \pm 3$	6, 14	1	10	$10n \pm 3$	$30n \pm 3$	—	—
11	$11n \pm 1$	2	1	11	$11n \pm 1$	$33n \pm 12$	$44n \pm 12$	$66n \pm 12$
	$11n \pm 2$	10, 12	1	11	$11n \pm 2$	$33n \pm 9$	$44n \pm 20$	$66n \pm 24$
	$11n \pm 3$	8, 14	1	11	$11n \pm 3$	$33n \pm 3$	$44n \pm 8$	$66n \pm 30$
	$11n \pm 4$	6, 16	1	11	$11n \pm 4$	$33n \pm 15$	$44n \pm 4$	$66n \pm 18$
	$11n \pm 5$	4	1	11	$11n \pm 5$	$33n \pm 6$	$44n \pm 16$	$66n \pm 6$
12	$12n \pm 1$	2	1	12	$12n \pm 1$	—	—	—
	$12n \pm 5$	10, 14	1	12	$12n \pm 5$	—	—	—

Number of Coils in Winding: $C = uS/2$; Commutator Pitch: $y_c = (C \pm a)/p$; Potential Pitch: $y_c = C/a$
Phase Pitch: $y_{ph} = C/(aN)$: no idle coils permissible, n = any integer

N.B.—$a = 1$ denotes the common wave winding: i.e. $y_c = (C \pm 1)/p$



Slots numbered anti clockwise looking from slip-ring end
— Indicates Coil + Indicates Soldered Connection
B Indicates Bottom coil Side in Slot *T* Indicates Top coil Side in Slot

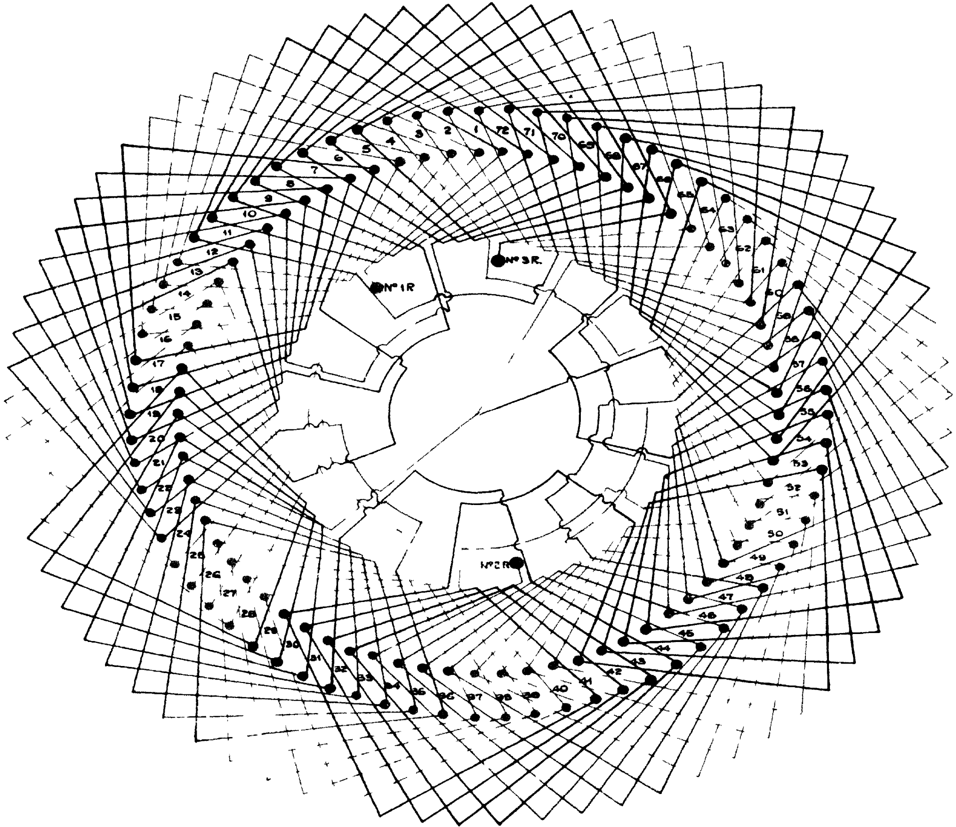
COIL CONNECTIONS

End 1*B* — 12*T* + 2*B* 13*T* + 3*B* 14*T* End
End 4*B* 15*T* + 5*B* 16*T* + 6*B* 17*T* + 7*B* 18*T* End
End 8*B* 19*T* + 9*B* 20*T* + 10*B* 21*T* End
End 11*B* — 22*T* + 12*B* 23*T* + 13*B* 24*T* + 14*B* 25*T* End
End 15*B* 26*T* + 16*B* 27*T* + 17*B* 28*T* End
End 18*B* 29*T* + 19*B* 30*T* 20*B* 31*T* + 21*B* 32*T* End
End 22*B* — 33*T* + 23*B* 34*T* + 24*B* 35*T* End
End 25*B* 36*T* + 26*B* — 37*T* + 27*B* 38*T* + 28*B* 39*T* End
End 29*B* 40*T* + 30*B* 41*T* + 31*B* 42*T* End
End 32*B* — 43*T* + 33*B* 44*T* + 34*B* 45*T* + 35*B* 46*T* End
End 36*B* — 47*T* + 37*B* 48*T* + 38*B* 49*T* End
End 39*B* — 50*T* + 40*B* 51*T* 41*B* 52*T* + 42*B* 53*T* End
End 43*B* 54*T* + 44*B* 55*T* + 45*B* 56*T* End
End 46*B* 57*T* + 47*B* 58*T* + 48*B* 59*T* + 49*B* 60*T* End
End 50*B* 61*T* + 51*B* 62*T* + 52*B* 63*T* End
End 53*B* 1*T* + 54*B* 2*T* + 55*B* 3*T* + 56*B* 4*T* End
End 57*B* 5*T* + 58*B* — 6*T* + 59*B* 7*T* End
End 60*B* 8*T* + 61*B* 9*T* + 62*B* 10*T* + 63*B* 11*T* End

INTER-COIL CONNECTIONS

No 1 Slip Ring 1*B* — 14*T* + 25*T* 11*B* + 22*B* — 35*T* + 46*T*
32*B* + 43*B* 56*T* + 4*T* 53*B* Neutral
No 2 Slip Ring 29*B* 42*T* + 53*T* 39*B* 50*B* — 63*T* + 11*T*
60*B* + 8*B* 21*T* + 32*T* 18*B* Neutral
No 3 Slip Ring 57*B* 7*T* + 18*T* 4*B* + 15*B* 28*T* + 39*T*
25*B* + 36*B* 49*T* + 60*T* 46*B* Neutral

FIG. 14 11. THREE-PHASE, LAP-TYPE, STAR-CONNECTED, ROTOR WINDING, SIX-POLE, 63 SLOTS
Outer circle represents Top-coil Side
Inner circle represents Bottom-coil Side



Slots numbered anti clockwise looking from slip ring end
— Indicates Coil   + Indicates Soldered Connection
B Indicates Bottom Coil Side in Slot   *T* Indicates Top Coil Side in Slot

COIL CONNECTIONS

End 61*B* — 1*T* + 62*B* — 2*T* + 63*B* 3*T* + 64*B* 4*T* End
End 65*B* — 5*T* + 66*B* 6*T* + 67*B* 7*T* + 68*B* 8*T* End
End 69*B* — 9*T* + 70*B* 10*T* + 71*B* 11*T* + 72*B* 12*T* End
End 1*B* 13*T* + 2*B* 14*T* + 3*B* 15*T* + 4*B* 16*T* End
End 5*B* — 17*T* + 6*B* 18*T* + 7*B* 19*T* + 8*B* 20*T* End
End 9*B* — 21*T* + 10*B* 22*T* + 11*B* 23*T* + 12*B* 24*T* End
End 13*B* — 25*T* + 14*B* 26*T* + 15*B* 27*T* + 16*B* 28*T* End
End 17*B* — 29*T* + 18*B* 30*T* + 19*B* 31*T* + 20*B* 32*T* End
End 21*B* 33*T* + 22*B* 34*T* + 23*B* 35*T* + 24*B* 36*T* End
End 25*B* 37*T* + 26*B* 38*T* + 27*B* 39*T* + 28*B* — 40*T* End
End 29*B* — 41*T* + 30*B* 42*T* + 31*B* — 43*T* + 32*B* 44*T* End
End 33*B* — 45*T* + 34*B* 46*T* + 35*B* 47*T* + 36*B* 48*T* End
End 37*B* 49*T* + 38*B* 50*T* + 39*B* 51*T* + 40*B* 52*T* End
End 41*B* 53*T* + 42*B* 54*T* + 43*B* 55*T* + 44*B* 56*T* End
End 45*B* 57*T* + 46*B* 58*T* + 47*B* 59*T* + 48*B* 60*T* End
End 49*B* 61*T* + 50*B* 62*T* + 51*B* — 63*T* + 52*B* 64*T* End
End 53*B* 65*T* + 54*B* 66*T* + 55*B* 67*T* + 56*B* 68*T* End
End 57*B* 69*T* + 58*B* 70*T* + 59*B* 71*T* + 60*B* 72*T* End

INTER COIL CONNECTIONS

No 1 Slip Ring 1*B* 16*T* + 28*T* 13*B* + 25*B* 40*T* + 52*T*
37*B* + 49*B* 64*T* + 4*T* 61*B* Neutral
No 3 Slip Ring 65*B* 8*T* + 20*T* 5*B* + 17*B* 32*T* + 44*T*
29*B* + 41*B* 56*T* + 68*T* 53*B* Neutral
No 2 Slip Ring 33*B* 48*T* + 60*T* 45*B* + 57*B* 72*T* + 12*T*
69*B* + 9*B* 24*T* + 36*T* — 21*B* Neutral

FIG. 14 12 THREE-PHASE, LAP-TYPE, STAR-CONNECTED, ROTOR WINDING, SIX-POLE, 72 SLOTS

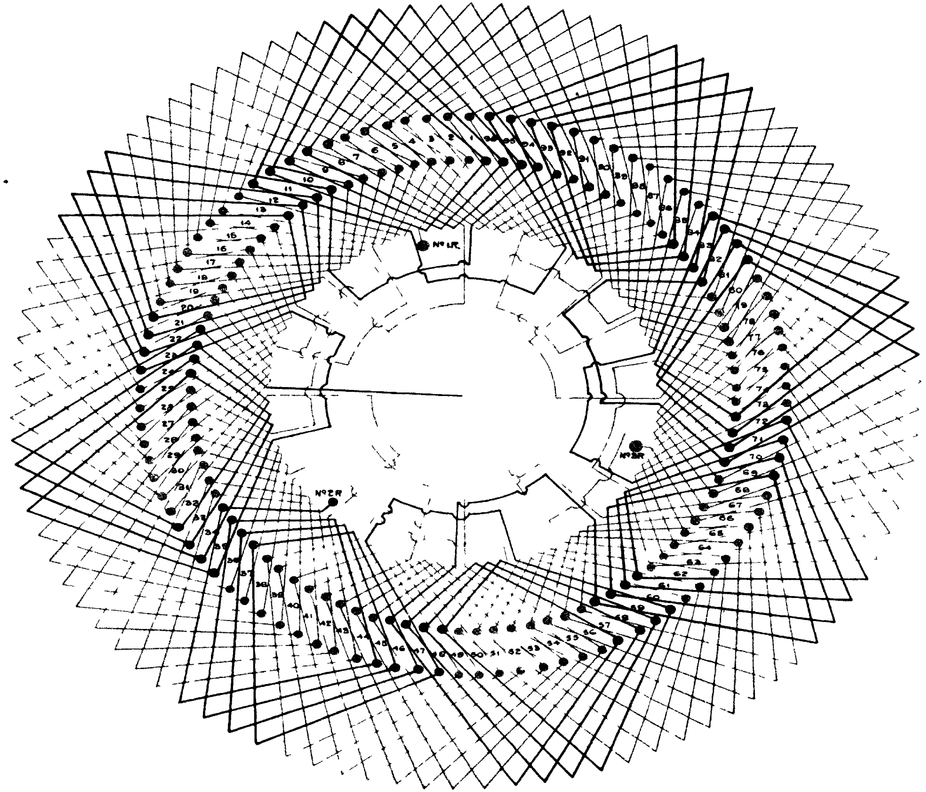


FIG. 14.13 THREE-PHASE LAP-TYPE, STAR-CONNECTED, ROTOR WINDING, EIGHT-POLE, 96 SLOTS
Outer circle represents Top-coil Side
Inner circle represents Bottom-coil Side

Slots numbered anti-clockwise looking from slip-ring end
— Indicates Coil + Indicates Soldered Connection
B Indicates Bottom-coil Side in Slot *T* Indicates Top-coil in Slot

COIL CONNECTION

End 1*B* − 13*T* + 2*B* 14*T* + 3*B* 15*T* + 4*B* − 16*T* End
End 5*B* 17*T* + 6*B* 18*T* + 7*B* 19*T* + 8*B* − 20*T* End
End 9*B* 21*T* + 10*B* 22*T* + 11*B* 23*T* + 12*B* 24*T* End
End 13*B* 25*T* + 14*B* 26*T* + 15*B* 27*T* + 16*B* − 28*T* End
End 17*B* 29*T* + 18*B* 30*T* + 19*B* 31*T* + 20*B* − 32*T* End
End 21*B* 33*T* + 22*B* 34*T* + 23*B* 35*T* + 24*B* 36*T* End
End 25*B* 37*T* + 26*B* 38*T* + 27*B* 39*T* + 28*B* 40*T* End
End 29*B* 41*T* + 30*B* 42*T* + 31*B* 43*T* + 32*B* 44*T* End
End 33*B* 45*T* + 34*B* 46*T* + 35*B* 47*T* + 36*B* 48*T* End
End 37*B* 49*T* + 38*B* 50*T* + 39*B* 51*T* + 40*B* 52*T* End
End 41*B* 53*T* + 42*B* 54*T* + 43*B* 55*T* + 44*B* 56*T* End
End 45*B* − 57*T* + 46*B* 58*T* + 47*B* 59*T* + 48*B* 60*T* End
End 49*B* 61*T* + 50*B* 62*T* + 51*B* 63*T* + 52*B* 64*T* End
End 53*B* − 65*T* 54*B* 66*T* + 55*B* − 67*T* + 56*B* − 68*T* End
End 57*B* 69*T* + 58*B* 70*T* + 59*B* 71*T* + 60*B* 72*T* End
End 61*B* 73*T* + 62*B* 74*T* + 63*B* 75*T* + 64*B* 76*T* End
End 65*B* 77*T* + 66*B* 78*T* + 67*B* 79*T* 68*B* 80*T* End
End 69*B* 81*T* + 70*B* 82*T* 71*B* 83*T* + 72*B* − 84*T* End
End 73*B* 85*T* + 74*B* 86*T* + 75*B* 87*T* + 76*B* 88*T* End
End 77*B* 89*T* + 78*B* 90*T* + 79*B* 91*T* + 80*B* 92*T* End
End 81*B* 93*T* + 82*B* 94*T* + 83*B* 95*T* + 84*B* 96*T* End
End 85*B* 1*T* + 86*B* 2*T* + 87*B* 3*T* + 88*B* 4*T* End
End 89*B* 5*T* + 90*B* 6*T* + 91*B* 7*T* + 92*B* 8*T* End
End 93*B* − 9*T* + 94*B* 10*T* + 95*B* 11*T* + 96*B* 12*T* End

INTER COIL CONNECTIONS

No 1 Slip Ring 1*B* 16*T* + 8*T* 13*B* + 25*B* − 40*T* + 52*T* 37*B*
+ 49*B* − 64*T* 76*T* 61*B* + 73*B* 88*T* + 4*T* 85*B* Neutral
No 2 Slip Ring 33*B* 48*T* + 7 45*B* 57*B* 72*T* 84*T* 69*B*
+ 81*B* − 96*T* 12*T* 93*B* + 9*B* 24*T* + 36*T* 21*B* Neutral
No 3 Slip Ring 65*B* 80*T* + [illegible]*T* 77*B* + 59*B* 8*T* + 20*T* 5*B*
+ 17*B* [illegible] 44*T* 9*B* 41*B* 56*T* 68*T* 53*B* Neutral

(T 591)

or lower conductors of the wave follows the series 1, $1 + y$, $1 + 2y$, $1 + 3y$, where

$$y = \frac{2mP + \delta}{\alpha}$$

in which δ is the smallest integer, which makes y integral. δ can be plus or minus. To lay out a balanced wave winding one must co-ordinate both the d series and the y series. One must consider, therefore, the whole of the recurrent groups, i.e. $\frac{p}{\alpha}$ groups, simultaneously. The vectors of the $\frac{p}{\alpha}$ slot stars coincide. Taking the example given by Dr. Liwschitz, a three-phase winding with $1\frac{4}{5}$ slots per pole per phase. Here $P = 9$, $\alpha = 5$. and the number of vectors in the slot star = 27.

The vectors of the slot star are given by 1, $1 + d$, $1 + 2d$, $1 + 3d$, . . . ,

where $$d = \frac{xmP + 1}{\alpha} = \frac{x \times 3 \times 9 + 1}{5}$$

$x = 2$ makes $d = 11$, and the vectors in the slot star are 1, 12, 23, 7, 18, 2, 13, 24, 8.

The slots occupied by phase I in the first recurrent group are given above.

The slots occupied by phase III are 19, 3, 14, 25, 9, 20, 4, 15, 26.

The y series of phase A, 1, $1 + y$, $1 + 2y$, . . . ,

where $$y = \frac{2mP + \delta}{\alpha} = \frac{2 \times 3 \times 9 + \delta}{5}$$

$\delta = 1$ makes $y = 11$, and it is obvious the vectors of the d series and y series correspond.

The magnitude of δ determines the number of winding parts in each phase. In the above $\delta = 1$ and there are two winding parts per phase. These windings have no abnormal steps. We give illustrations of rotor windings.

Figs. 14.11, 14.12, and 14.13 illustrate various three-phase rotor windings.

Fig. 14.11 illustrates a rotor winding, with lap end connections, for six poles, 63 slots, three-phase. Fig. 14.12 shows a six-pole, three-phase star connected winding with 72 slots, while Fig. 14.13 shows an eight-pole, three-phase rotor winding with 96 slots.

The student will, no doubt, follow these diagrams quite easily from the principles we have enunciated.

Windings for Small Motors

There are three general methods in use for winding very small motors, they are: (*a*) the skein winding; (*b*) the mould winding; and (*c*) hand winding.

In method (*a*), a skein of wires is looped several times through the slots to form one pole. The total turns per pole is a multiple of the turns in the skein. This depends on the number of times the skein is looped through the slots and the slot number. The skein length may be estimated, or determined experimentally. The skein is then placed in the two inner slots with the short-end of the skein pressed against the core, and the coil leads should be on the long end of the skein. A half twist is next made in the skein, and the long end of the loop is passed through the core and the sides threaded through the two slots adjacent to the original pair. The winding is pressed firmly against the side of the core. Then another half twist is made in the opposite direction to the first twist. These half-twists are in opposite directions alternately. The loop is then laid back through the same slots for the second time, and then another half-twist is made and the loop laid in the next two slots and so on. All the poles are wound in the same way.

Mould Windings. The coils are first wound on a mould and then placed in the slots. Usually the set of coils for one pole is wound together, so separate coils do not have to be connected together, when placed in the slots. Its general appearance is the same as the hand type of winding.

CHAPTER XV

Selsyns and Autosyns

THESE machines are classified as: (*a*) power selsyns; (*b*) instrument selsyns; and (*c*) indicator and control selsyns. The name is a contraction of self-synchronous.

Power Selsyns

Power selsyns are similar in construction to wound-rotor type induction motors. They are used, for example, to act as a tie between two or more drives to keep the speeds and angular correspondence of the different drives the same. Synchronism of the ends of a lift-bridge may be obtained by their use. They may also be used to tie together different units of printing press-drives, and the rotors of the two selsyns will always tend to remain in angular correspondence.

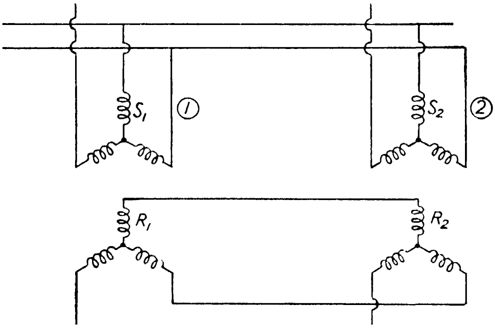


FIG. 15.1

In Fig. 15.1 are shown the stator windings S_1 and S_2 of the two selsyns, each connected to a three-phase supply. The two rotors R_1 and R_2 are connected together as shown. The rotor of selsyn 1 is mechanically coupled to one of the machines, whose speed is to be kept the same as the speed of the other. The rotor of selsyn 2 is mechanically coupled to the other unit. Thus, each selsyn is connected to two sources of power and, thus, both will act as synchronous machines.

It is clear that if one selsyn is caused, by the unit which it controls, to run fast, and the other selsyn is forced to run slow, then each selsyn will tend to force the other to run at its own speed. The speed of the units must be such that the rotor frequencies in the two selsyn rotors must be equal.

This then involves power transference, for one selsyn will act as generator and the other as motor. The speed of rotation may be either in the same direction as the rotating fields or in the opposite direction. Consider two d.c. motors, which are exactly alike, driving two loads, and it is required to keep the speed of both alike, and also the two rotors are to remain in angular correspondence. Two selsyns, connected as shown in Fig. 15.1, are mechanically coupled to the two d.c. machines; selsyn 1 to d.c. machine 1, and selsyn 2 to d.c. machine 2. The stators have three-phase windings and are both excited from the same a.c. source. The rotors have three-phase windings, which are connected through slip-rings, as shown in Fig. 15.1. The two d.c. machines and two selsyns may be situated far apart from each other. The size of the selsyns must be such as to deal adequately with the torque demanded from each. Suppose the two d.c. machines are equally and fully loaded, they will run at the same speed if they have identical torque-speed curves. In that case both d.c. machines will draw equal power from the d.c. supply lines, and the selsyn rotors will carry no current, for the e.m.f.s are equal and of the same frequency, and they are connected to oppose each other.

Now suppose one d.c. motor has its load dropped by 25 per cent and the other retains its full load as before. The lightly-loaded machine will experience a rise in speed, but the selsyns will force both machines to run at the same speed. Since the speed remains the same, it follows that one selsyn acts as motor and delivers to the shaft 12½ per cent of the load on the first d.c. machine.

It is clear that since the speed of each machine is the same, the load on the two is but 175 per cent of full load, and 87½ per cent is supplied to each from the d.c. mains. The difference of 12½ per cent must come from the selsyn as motor. The selsyn on the lightly-loaded d.c. motor must act as generator, since its load is but 75 per cent of full load, but it receives power equivalent to 87½ per cent from the d.c. mains. Therefore, the second selsyn is driven as generator by the lightly-loaded d.c. machine.

Let W be the full load of each d.c. machine; then, neglecting losses, the input to the stator of the first selsyn

$$= \frac{0{\cdot}125W}{1 - s}$$

where s = slip of the selsyn

The input to the stator of selsyn (2), neglecting losses,

$$= \frac{-0{\cdot}125W}{1 - s}$$

The power delivered, neglecting losses, by the rotor of selsyn (2)

$$= \frac{-0{\cdot}125\,Ws}{1 - s}$$

These machines behave as synchronous machines, but the induction motor equations apply to them.

The torque in synchronous watts

$$= \text{input to the rotor circuit}$$

$$= m_2\bar{E}_2 I_2 \cos\phi_2 \quad . \quad . \quad . \quad . \quad (15{\cdot}1)$$

The vector diagram for the selsyn *motor* receiver is given in Fig. 15.2.

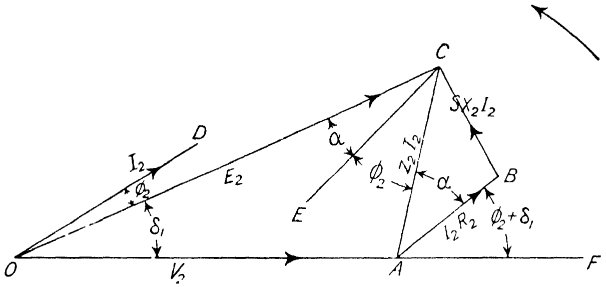


FIG. 15.2

OA = vector of rotor voltage, common to both

AB = resistance drop, due to $\bar{I}_2 = R_2\bar{I}_2$

$OD = \bar{I}_2$

$BC = sX_2\bar{I}_2$ = reactance drop in rotor at slip s

$AC = Z_2\bar{I}_2$ = impedance drop in the rotor

ϕ_2 = angle of lead of I_2 ahead of E_2

Draw CE making an angle of ϕ_2 with Z_2I_2.

δ_1 = angle of lead of E_2 on V_2

The angle $\qquad BAF = \phi_2 + \delta_1$

and $\qquad \angle CAF = \angle AOC + \angle OCA$

$$\angle CAF = \angle CAB + \angle BAF = \alpha + \phi_2 + \delta_1 = \delta_1 + \phi_2 + \angle ECO$$

$$\therefore \qquad \angle ECO = \alpha$$

Now the projection of AC on CE = difference of the projections of OC and OA on EC.

$$\therefore \qquad Z_2I_2\cos\phi_2 = E_2\cos\alpha - V_2\cos(\delta_1 + \alpha) \quad . \quad (15{\cdot}2)$$

Therefore, torque in synchronous watts = rotor input

$$= m_2\bar{E}_2\bar{I}_2\cos\phi_2 = \frac{m_2\bar{E}_2}{Z_2}\{\bar{E}_2\cos\alpha - V_2\cos(\delta_1 + \alpha)\} \quad (15{\cdot}3)$$

If E_2 is sensibly equal to V_2.

Therefore, torque in synchronous watts for the receiver

$$= \frac{m_2\bar{E}_2\bar{V}_2}{Z_2}[\cos\alpha - \cos(\delta_1 + \alpha)]$$

$$= \frac{m_2\bar{E}_2\bar{V}_2}{Z_2}\{\cos\alpha - \cos\delta_1\cos\alpha + \sin\delta_1\sin\alpha\}$$

$$= \frac{m_2\bar{E}_2\bar{V}_2}{Z_2}\{\cos\alpha(1 - \cos\delta_1) + \sin\delta_1\sin\alpha\}$$

$$= \frac{m_2\bar{E}_2\bar{V}_2}{Z_2}\left\{\sin\delta_1\sin\alpha + 2\cos\alpha\sin^2\frac{\delta_1}{2}\right\} \quad . \quad . \quad (15.4)$$

Thus, the torque of the selsyn depends on the displacement of its rotor angle δ_1, but also on the slip, since

$$\tan\alpha = \frac{sX_2}{R_2}$$

where s = slip

$$\sin\alpha = \frac{sX_2}{\sqrt{R_2 + s^2X_2^2}} = \frac{sX_2}{Z_2}$$

The first term in equation (15.4) is the synchronizing torque, viz.

$$\frac{m_2\bar{E}_2\bar{V}_2}{Z_2}\sin\delta_1\sin\alpha$$

The second term is a rotor loss torque. For the second selsyn (the transmitter) this second term subtracts from the synchronizing torque and appears with a negative sign in front of it. The above applies for rotation in the direction of the field. For reverse rotation, or rotation opposite to that of the field, the signs are reversed.

These selsyns for power duties are built for relatively large powers of over 100 h.p. They must be lined up before starting, for if the initial angle of displacement is large, when these machines are up to speed, one may be in such a phase relation to the other that it forms a short-circuit for it. It is best to use single-phase excitation for the stators, since the single-phase motor has no starting torque and is not able to come up to speed. With single-phase excitation in the stators, the rotor e.m.f.s will be in phase and no torque will be produced. The lining up of the two rotors is better accomplished with single-phase excitation of the stators. Should the rotors be not in juxtaposition, a torque will result, due to excitation flux produced.

Instrument Selsyns

These machines are very small and light, and their function is to indicate the position of some object, such as the depth of water in a reservoir, or the transmission of signals between remote points, or to show the position of the trim tabs or wheels in an aeroplane. They do not transmit torque, but simply move a pointer. The selsyn transmitter and receiver units are identical, and are interchangeable electrically and mechanically. The mechanical construction of these instrument selsyns is very similar to that of the salient-pole alternators. The rotor has a single-phase winding on an "H" type of armature, to which the supply is taken via special alloy brushes and solid silver slip-rings, which are used to reduce friction to a minimum. The stator has a normal type of three-phase winding which is designed to give accurately balanced phase voltages and correct flux distribution. The rotor has precision type ball-bearings, fitted in special housings to prevent the ingress of foreign matter. A connection diagram for a transmitter and receiver is given in Fig. 15.3. and a photograph of the component parts of a selsyn is given in Plate XIII.

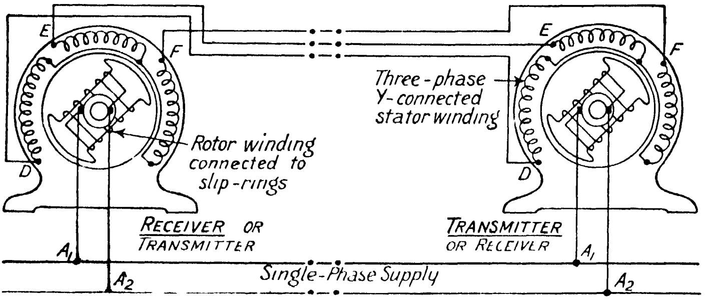


FIG. 15.3

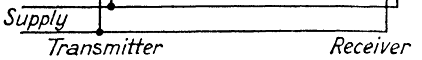


FIG. 15.4

Elementary Theory. In Fig. 15.4 the primary windings of the transmitter and receiver selsyns are indicated as A_1A_2 and are connected to a single-phase supply. The secondary windings DEF are usually wound for three phases.

The stator windings of each selsyn can be replaced by two windings, viz. D_1D_2 and Q_1Q_2 in Fig. 15.5.

The coil D_1D_2 has its magnetic axis in the same line as the axis of the rotor, and is the direct-axis component. The coil Q_1Q_2 has its magnetic axis at right angles to the rotor axis. It is the quadrature component of the stator turns. The rotor axis is the transformer axis, and currents induced in the coil D_1D_2 cannot produce torque, but simply result in a reduction of current of one selsyn and an increase in current of the other. The current in the secondary is due to unbalance of the two secondary voltages. With no angular displacement between the rotors of the transmitter and receiver, voltages will be induced in coils D_1D_2 only, but there will be no secondary current.

Now assume that the transmitter rotor is displaced by an angle δ from its original position and let $\bar{V}$ be the initial induced voltage in the direct axis when $\delta = 0$, the voltage induced in the coils D_1D_2 will be $\bar{V} \cos \delta$ and in Q_1Q_2 will be $\bar{V} \sin \delta$. In the D_1D_2 circuit we have the resultant voltage $(\bar{V} - \bar{V} \cos \delta)$. This will produce current in the D_1D_2 coils, but will not produce torque. In the quadrature coils the resultant voltage is $V \sin \delta$ and this will produce current in these coils. This current in the quadrature axis of the receiver will interact with the field to produce the torque which tends to reduce the angular displacement between the rotors to zero. The voltage which produces the current in the quadrature coils is proportional to sin δ, and the torque itself, for a constant flux, will be proportional to sin δ. The torque angle characteristic is practically linear up to displacement of about 30°.

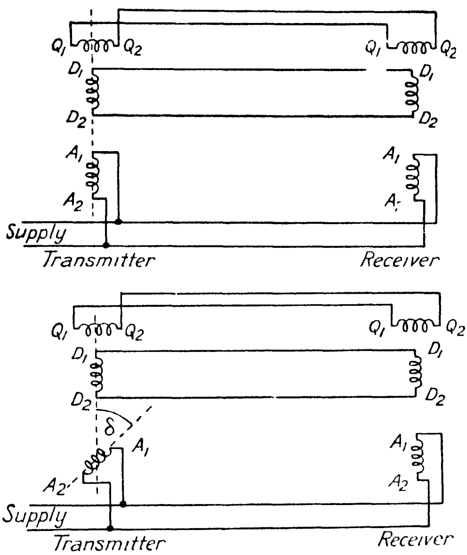


FIG. 15.5

The current in the Q_1Q_2 coils will be equal to

$$\frac{\bar{V} \sin \delta}{2\sqrt{R_q^2 + L_q^2\omega^2}}$$

where R_q = resistance of one of the Q coils
$L_q\omega$ = reactance of one of the Q coils

The phase angle of the current in the Q coils will be

$$\tan^{-1} \frac{L_q\omega}{R_q}$$

behind the voltage $\bar{V} \sin \delta$.

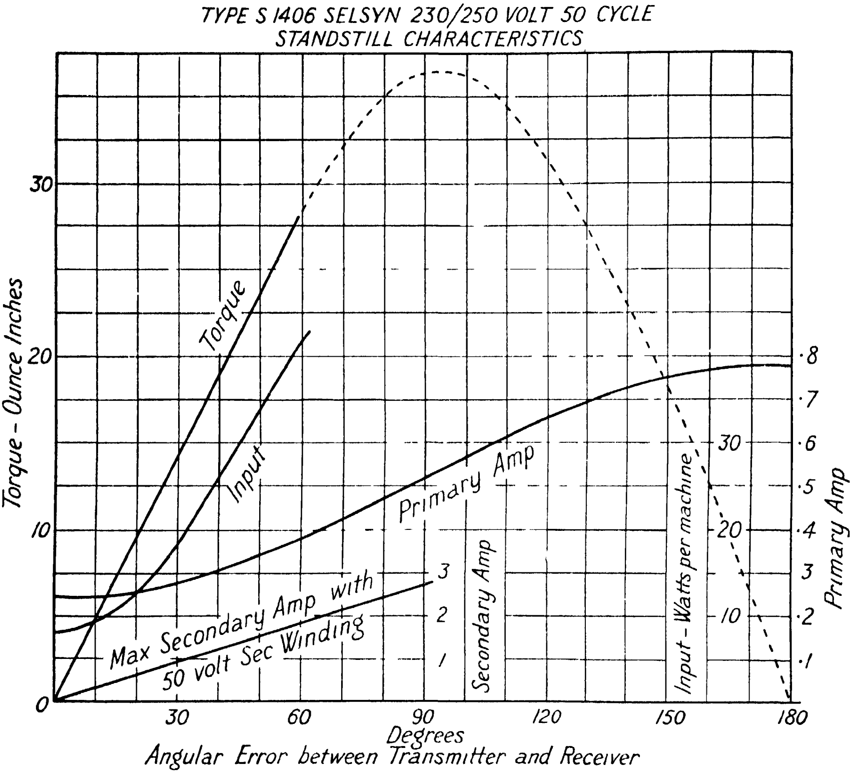


FIG. 15.6

The reactance of these coils will depend on the relative position of the salient pole rotor with respect to the stator windings. Hence both the magnitude and phase of the quadrature secondary current will vary with rotor position.

The torque is proportional to the product of the quadrature axis current, the field strength and the cosine of the time-phase angle between the two. Since the time-phase angle of the quadrature axis current depends on the ratio of reactance to resistance, it will vary with different sizes of machines for any angular displacement, and will vary in the same machine for different angular displacements, due to changes in the position of the salient-pole rotor with respect to the stator windings. The magnitude of the working flux, and its angular lag behind the applied voltage, will vary for different angular displacements between the transmitter and receiver. This is due to the different voltage drops in resistance and leakage reactance of the primary winding, because of the varying magnitude and power factor of the primary current. With constant exciting voltage, the torque-angle characteristic will depart from the sine law and will be different for different sizes of machine. Machines with a high ratio of resistance to reactance (i.e. small machines) will have steeper torque-angle characteristics than those with a lower ratio, but this steepness is obtained at the expense of pull-out torque. Should the transmitter selsyn be turned through 180 electrical degrees, the voltages in the direct axis (D_1D_2) coils in transmitter and receiver are in opposition and their sum will produce circulating circuit in these coils. The condition is analogous to that of a transformer with the secondary short-circuited, and heavy primary currents will flow. No voltages are produced in the Q coils and there is no force tending to restore angular correspondence. This condition is highly unstable, and the slightest movement of either unit would cause the rotors to pull into correspondence. Typical

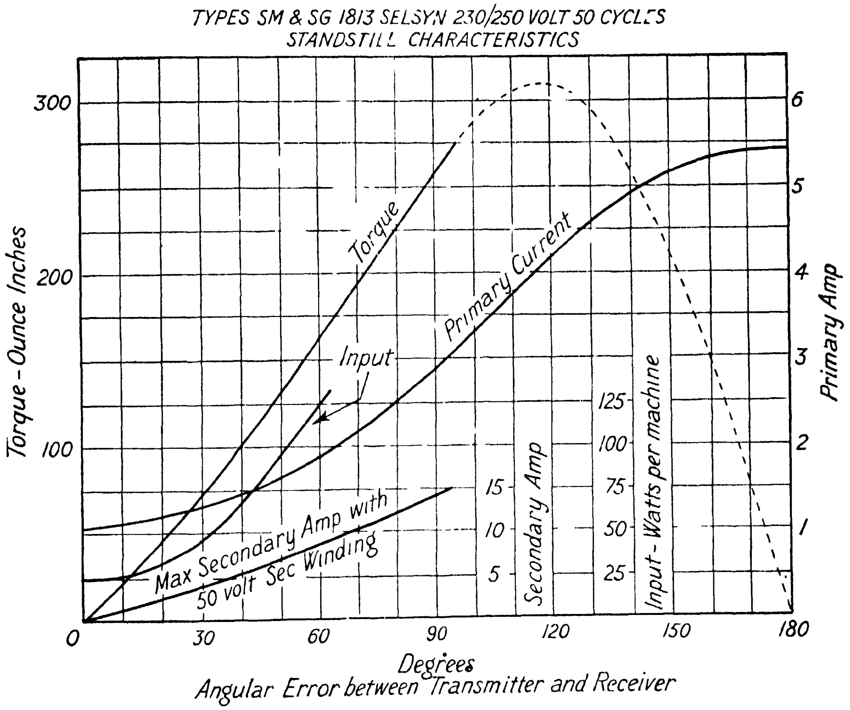


FIG. 15.7

characteristic curves, supplied by the B.T.H. Co., for a selsyn, manufactured by them, are appended in Figs. 15.6 and 15.7. Fig. 15.6 is for a small selsyn unit, operating on 230-250 V, 50 c/s. It will be noticed that for small angles, the torque is about $\frac{1}{2}$ in.-oz per degree of displacement of transmitter and receiver. The variation of primary amperes over the whole range of δ from 0 to 180° and the shape of the input curve is given.

Fig. 15.7 gives the characteristics of a larger selsyn (also built by the B.T.H. Co., of Rugby). This gives approximately $2\frac{1}{2}$ in.-oz torque per degree of displacement.

The American companies manufacture various models for different applications with torque-angle, i.e. torque per degree varying from 0·07 to 3·4 in.-oz.

Selsyn Generator and Control Transformer System (Voltage Indication)

In this arrangement two selsyns are connected as shown in Fig. 15.8. The rotor of the second selsyn is not connected to the a.c. supply,

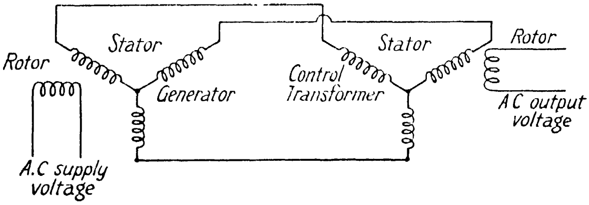


FIG. 15.8

but an output voltage is taken from it, which may be used for purposes of indication, but more usually acts as a control voltage for a servo system. The magnitude of the output voltage is a function of the displacement between the shafts of the two units.

The selsyn control transformer is similar to the generator but it has a stator with a winding of high impedance. The currents in the control transformer are proportional to the voltages in the generator windings, and produce a magnetic field in the control transformer corresponding to that in the generator. The voltage induced in the single-phase winding of the control transformer is a sine function of δ.

Differential Selsyns

The voltages applied to the three-phase primary of the differential generator, from the selsyn generator, induce a new set of voltages in the secondary three-phase windings, which appear as if the generator had been turned additionally by the amount of shaft rotation of the differential. An exciting capacity is placed in parallel with the primary of the differential system to reduce the heating of the rest of the system. (*See* Fig. 15.9.)

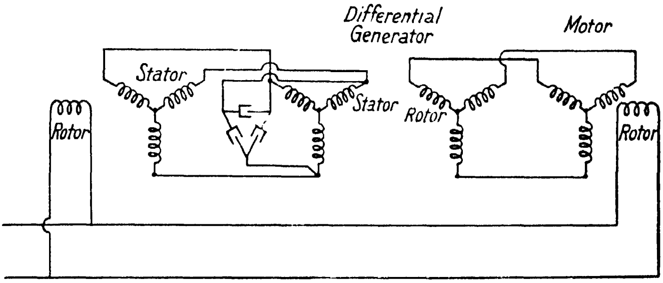


FIG. 15.9

Selsyn Generator-Control Transformer System, Controlling a Servo System

It is often desirable to control large amounts of power by means of small power. High accuracy is made possible by using a selsyn generator and control transformer system, which controls a servo system.

The output voltage of a control transformer, which is proportional to the relative position of generator and control transformer, is delivered to an electronic control device, which controls the flow of power to the driving motor. The driving motor turns the load and control transformer, until the output voltage of the control transformer becomes zero.

Any movement of the generator results in a corresponding motion of the load, and this is independent of the size of load.

Errors of Selsyns

The errors encountered in selsyn transmitters and receivers are: (*a*) errors due to friction in bearings; and (*b*) electrical error of the magnetic field. Static errors vary with different sizes, but values of 0·6 to 1·5 degrees are guaranteed as maximum errors. There are dynamic errors, which may be two to three times the static errors. Selsyn motors have dampers to reduce undesirable oscillations. There is a large and growing field of application for these machines, particularly in conjunction with modern electronic circuits.

CHAPTER XVI

Unbalanced Conditions in Induction Motors

FREQUENTLY it is desirable to estimate the effects of various forms of unbalance on the performance of induction motors. To the late Dr. C. L. Fortescue we are indebted for the solutions of the problems of unbalance. His work on symmetrical components, followed by the excellent work of Prof. Waldo Lyon and Messrs. Wagner and Evans, has enabled us to assess the influence of the various forms of unbalance on the performance of the machines.

In giving the following account of these problems and their solution, I have been guided by the work of these authors, to whom the results are due. We shall assume that the machine is provided with symmetrical windings. This is almost invariably true, but occasionally one finds dissymmetry in rotor windings. With symmetrical windings, the various phase sequence quantities do not react on each other. We can, therefore, deal with the positive- and negative-sequence quantities in independent circuits and investigate the effects occurring in each.

Voltage Unbalance

The first type of unbalance, which we shall investigate, is that of unbalanced voltages applied to the machine. This is a very common type of unbalance, and results in reduction of torque and increased heating. However, the unbalanced voltages, which occur in practice, have usually small effect in torque reduction, but do result in greatly increased heating, if the unbalance is pronounced.

Let the three line or phase voltages be $\bar{V}_1$, $\bar{V}_2$, and $\bar{V}_3$ and let them be unequal, and also let the phase angles between them differ from 120 electrical degrees. Our first procedure is to resolve these three voltages into their symmetrical components. Thus, the positive-sequence components of the circuit have the value

$$V_{p1} = \tfrac{1}{3}(\mathbf{V}_1 + \alpha \mathbf{V}_2 + \alpha^2 \mathbf{V}_3) \qquad (16.1)$$

where

$$\alpha = e^{j120} = -0{\cdot}5 + j0{\cdot}866 \qquad (16.2)$$

$$\alpha^2 = e^{j240} = -0{\cdot}5 - j0{\cdot}866 \qquad (16.3)$$

In equation (16.1), vector addition is required.

The negative-sequence components of the applied voltage have the value

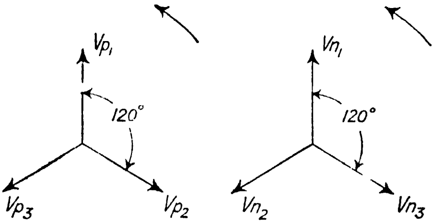
$$V_{n1} = \tfrac{1}{3}[\mathbf{V}_1 + \alpha^2\mathbf{V}_2 + \alpha\mathbf{V}_3] \quad . \quad . \quad (16.4)$$

The zero-sequence components have the value

$$V_{z1} = \tfrac{1}{3}[\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3] \quad . \quad . \quad . \quad (16.5)$$

and are zero for $V_1 + V_2 + V_3 = 0$.

Once again, in equations 16.4 and 16.5 the bold symbols used for $\mathbf{V}_1$, etc., indicate that these are vector quantities and vector addition must, of course, be employed.



positive sequence voltages *negative sequence voltages*

FIG. 16.1

We will refer all quantities to the stator, this is effected by multiplying rotor resistances and reactances by

$$\left\{\frac{\text{Stator turns per phase}}{\text{rotor turns per phase}} \times \frac{bf_s \times c_s f_s}{bf_r \times c_s f_r}\right\}^2$$

where bf_s = breadth factor of stator winding for fundamental

$c_s f_s$ = coil-span factor of stator winding for fundamental

bf_r = breadth factor of rotor winding for fundamental

$c_s f_r$ = coil-span factor of rotor winding for fundamental

The postive- and negative-sequence voltages are given by equations (16.1) and (16.4). They each form symmetrical three-phase systems, which, since the windings are symmetrical, produce positive- and negative-sequence currents, and, furthermore, these two sequencies do not react on each other. They may, therefore, be considered as acting independently. We are concerned only with the positive- and negative-sequence components of applied potential.

The positive- and negative-sequence system of voltages is shown in Fig. 16.1. Since the positive-sequence system consists of three equal voltages, each making an angle of 120° with the other, they will, when applied to the symmetrical stator winding, produce a rotating field rotating in the positive direction. This field—at least the fundamental—will have a constant amplitude. The negative-sequence system of voltages, when applied to the symmetrical stator windings, will also produce a rotating field—rotating in the opposite direction to the field due to the positive sequence—and the fundamental of this field will also have constant amplitude. The two systems, positive and negative, are considered quite independently, and since they do not react on each other, the results can be superimposed.

The equivalent circuit for the positive-sequence system is shown in Fig. 16.2.

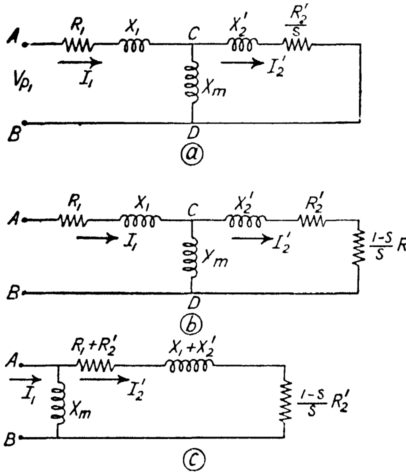


FIG. 16.2

Let R_1 = resistance per phase of the stator

R_2' = resistance per phase of the rotor, referred to the stator

X_1 = *leakage* reactance per phase of the stator

X_2' = leakage reactance per phase of the rotor, referred to the stator

X_m = mutual reactance

s = slip

Then

$$\mathbf{V}_{p1} = \bar{\mathbf{I}}_1(R_1 + jX_1) + jX_m(\bar{\mathbf{I}}_1 - \bar{\mathbf{I}}_2') \quad . \quad . \quad (16.6)$$

where $\bar{\mathbf{I}}_1$ = stator current per phase (positive sequence)

$\bar{\mathbf{I}}_2'$ = rotor current per phase, referred to stator (positive sequence)

Now $\mathbf{E}_1$ = back e.m.f. in the stator $= -jX_m(\bar{\mathbf{I}}_1 - \bar{\mathbf{I}}_2')$.

The voltage induced in the rotor per phase at standstill $= \bar{\mathbf{E}}_2$ and the rotor voltage per phase, at slip s, $= s\bar{\mathbf{E}}_2$, the rotor current per phase $= \mathbf{I}_2$

and
$$I_2 = \frac{sE_2}{\sqrt{R_2^2 + s^2X_2^2}} \quad . \quad . \quad . \quad (16.7)$$

and
$$\mathbf{I}_2 = \frac{s\mathbf{E}_2}{R_2 + jsX_2} \quad . \quad . \quad . \quad (16.8)$$

i.e.
$$s\mathbf{E}_2 = \mathbf{I}_2(R_2 + jsX_2) \quad . \quad . \quad . \quad (16.9)$$

or
$$\mathbf{E}_2 = I_2\left(\frac{R_2}{s} + jX_2\right) \quad . \quad . \quad . \quad (16.10)$$

Now X_2 = reactance per phase at *standstill*, therefore, the rotor current at slip s is the same as the rotor current at standstill, provided the rotor resistance is made $\frac{R_2}{s}$.

We may refer all these rotor quantities to the stator, and $\mathbf{E}_2$ becomes, when referred,

$$\mathbf{E}_2' = \mathbf{E}_2 \times \frac{T_1}{T_2} \times \frac{k_1 \times k_3}{k_2 \times k_4} = \mathbf{E}_1$$

where T_1 = turns in series per stator phase
$k_1 \times k_3$ = breadth factor for the stator fundamental × coil-span factor
T_2 = turns in series per rotor phase
$k_2 \times k_4$ = product of breadth factor and coil-span factor for the rotor

$$\therefore \mathbf{E}_2' = \mathbf{I}_2\left\{\frac{T_2 \times k_2 \times k_4}{T_1 \times k_1 \times k_3}\right\}\left\{\frac{R_2}{s} \times \left(\frac{T_1 \times k_1 \times k_3}{T_2 \times k_2 \times k_4}\right)^2 + jX_2 \times \left(\frac{T_1 \times k_1 \times k_3}{T_2 \times k_2 \times k_4}\right)^2\right\} \quad . \quad (16.11)$$

i.e.
$$\mathbf{E}_2' = \mathbf{E}_1 = X_m(\mathbf{I}_1 - \mathbf{I}_2') = \mathbf{I}_2'\left(\frac{R_2'}{s} + jX_2'\right) \quad . \quad (16.12)$$

where
$$R_2' = R_2 \times \left(\frac{T_1 \times k_1 \times k_3}{T_2 \times k_2 \times k_4}\right)^2 \quad . \quad . \quad (16.13)$$

and
$$X_2' = X_2 \times \left(\frac{T_1 \times k_1 \times k_3}{T_2 \times k_2 \times k_4}\right)^2 \quad . \quad . \quad (16.14)$$

Looking at Fig. 16.2, we see that the voltage across CD
$$= X_m(\mathbf{I}_1 - \mathbf{I}_2')$$

The well-known artifice of bringing the motor to rest by making the referred resistance $\frac{R_2'}{s}$ instead of R_2' enables us to treat the motor as a stationary transformer. Our diagram, Fig. 16.2 (*a*), clearly illustrates our equations. We have the applied positive-sequence p.d., V_{p1}, at the terminals A and B, and the stator current I_1 flows through the stator resistance and reactance, giving us a voltage drop $= I_1(R_1 + jX_1)$. The voltage across CD = the component of applied volts per phase to overcome the back e.m.f. The current through the mutual reactance $X_m = I_1 - I_2'$. The p.d. across $CD = X_m(I_1 - I_2')$, but this is the p.d. across the rotor circuit in which all quantities are referred to the stator winding.

Also $$\frac{R_2'}{s} = R_2' + R_2'\frac{(1-s)}{s} \qquad . \quad . \quad . \ (16.15)$$

and $$(I_2')^2\frac{R_2'}{s} = (I_2')^2R_2' + (I_2')^2R_2'\frac{(1-s)}{s} \qquad . \ (16.16)$$

but $(I_2')^2\dfrac{R_2'}{s}$ = power input to the rotor per phase . (16.17)

and $(I_2')^2R_2'$ = copper loss in the rotor per phase . (16.18)

$\therefore\ (I_2')^2R_2'\dfrac{(1-s)}{s}$ = loss in a fictitious resistance, which must be equal to the mechanical power (gross) developed per phase

the fictitious resistance $$= R_2'\frac{(1-s)}{s} \qquad . \quad . \quad . \quad . \ (16.19)$$

The gross mechanical output for the three phases

$$= 3(I_2')^2R_2'\frac{(1-s)}{s}\ \text{watts} \qquad . \quad . \ (16.20)$$

The gross mechanical output for three phases

$$= \frac{\tau \times 2\pi\ \text{r.p.m.}}{33\ 000}\ (\text{h.p.}) \qquad . \quad . \quad . \ (16.21)$$

and r.p.m. = r.p.m.$_{(\text{syn})}$ $(1 - s)$, and τ is the torque in lb-ft, i.e. the actual r.p.m. = synchronous r.p.m. $(1 - s)$,

$$\therefore \quad \frac{\tau \times 2\pi(\text{r.p.m.}_{(\text{syn})})(1-s)}{33\ 000} = \frac{3(I_2')^2R_2'}{746}\frac{(1-s)}{s} \qquad . \ (16.22)$$

and $$\text{r.p.m.}_{(\text{syn})} = \frac{120 \times f}{p} = \frac{120 \times \text{frequency}}{\text{number of poles}}$$

$$\therefore \quad \tau\ \text{lb-ft} = \frac{3 \times 33\ 000}{2\pi \times 746 \times 120} \times \frac{p}{f} \times \frac{(I_2')^2R_2'}{s} \qquad . \ (16.23)$$

$$= 0{\cdot}1765 \times \frac{p}{f} \times \frac{(I_2')^2R_2'}{s}$$

Fig. 16.2 (*c*) is the usual approximation made in transferring the mutual reactance from C and D to A and B.

Negative-Sequence Equivalent Circuit

Now since the rotor is rotating in the direction of the positive-sequence field with slip s, and since the negative-sequence field is rotating at synchronous speed in the opposite direction, it is clear that the slip of the rotor will be $s' = 2 - s$, for we have the speed of the field, in the negative direction $= +\omega_0$, and the speed of the rotor in the positive direction $= \omega_0(1 - s)$, slip with regard to the negative-sequence field

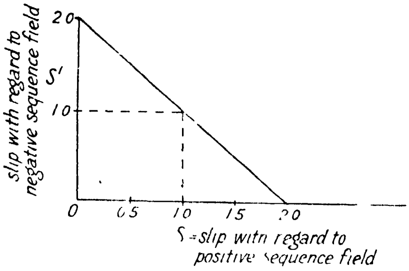


FIG. 16.3

$$s' = \frac{+\omega_0 + \omega_0(1 - s)}{+\omega_0} = 2 - s \quad . \quad . \quad (16.24)$$

Fig. 16.3 shows the slips of the rotor with regard to the negative-sequence field as ordinates, and the slips with regard to the positive-sequence field as abscissae.

Note. When $s = 0$, $s' = 2$.

When the motor is running under normal full load, its slip is of the order of 2 or 3 per cent, i.e. 0·02 to 0·03. It is clear, therefore, at normal load s' is nearly $= 2$. At standstill $s = 1$; $s' = 1$, and $s = 2$, $s' = 0$.

Negative-Sequence Circuit (Fig. 16.4)

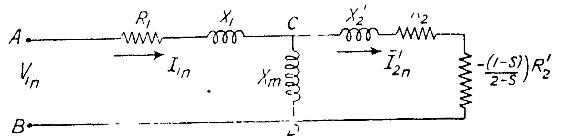


FIG. 16.4

Our negative-sequence diagram is similar to Fig. 16.2, but in place of $\left(\frac{1 - s}{s}\right) R_2'$, we have $\left(\frac{1 - s'}{s'}\right) R_2'$

$$= \frac{1 - (2 - s)}{2 - s} R_2'$$

$$= -\frac{(1 - s)}{2 - s} R_2' \quad . \quad . \quad . \quad (16.25)$$

Since s is small, under normal load, and of the order of 0·02 to 0·03,

$$\left(\frac{1 - s'}{s'}\right) R_2' \simeq -\frac{R_2'}{2}$$

and the equivalent circuit becomes as in Fig. 16.5.

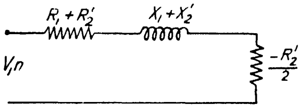


FIG. 16.5

The gross mechanical power, due to the negative-sequence voltages, per phase

$$= \text{loss in the resistance} - \frac{(1 - s)}{2 - s} R_2'$$

$$= -3\frac{(1 - s)}{2 - s} R_2' (I_{2n}')^2 \text{ for the three phases} \quad . \quad (16.26)$$

where I_{2n}' = rotor current per phase, referred to the stator, due to negative-sequence voltages

The gross mechanical power of the machine due to the positive- and negative-sequence voltage

$$= 3\frac{(1 - s)}{s} R_2'(I_2')^2 - 3\frac{(1 - s)R_2'(I_{2n}')^2}{2 - s} \quad . \quad (16.27)$$

The gross torque in lb-ft., due to three phases

$$= 0{\cdot}1765 \times \frac{pR_2'}{f}\left\{\frac{(I_2')^2}{s} - \frac{(I_{2n}')^2}{2 - s}\right\} \quad . \quad . \quad (16.28)$$

where I_2' = r.m.s. value of rotor current, referred to the stator, due to positive sequence

I_{2n}' = r.m.s. value of rotor current, referred to the stator, due to negative-sequence voltages

It is seen that, in equation (16.27) and also in equation (16.28), the gross power and gross torque, *due to the negative-sequence voltages*, is of negative sign. The torque and power are, therefore, reduced by the presence of negative-sequence voltages, but it is obvious that, in the normal range, this reduction is relatively small. In one case, given by Messrs. Wagner and Evans, with a 10 per cent negative-sequence voltage and a normal value of $s = 0{\cdot}015$, the torque due to the negative-sequence voltage is but 0·4 per cent of the normal positive-sequence torque, but the I^2R losses are given as 2·8 per cent for the positive-sequence current and 1·55 per cent for the negative-sequence currents.

It is clear, therefore, that moderate voltage unbalance does not affect the torque, given by the motor, unduly, but it *does increase greatly the copper losses*, and in every case it results in *smaller* output from the machine.

Machine with Symmetrical Windings on Both Stator and Rotor, but Working with One Stator Phase Opened

This may be due to a fuse blowing in one line.

Assume the machine is connected in star. Let the three stator phases be A, B, and C, and let A be opened. The current in phase A is zero and

$$I_B = I,\ I_A = 0, \text{ and } I_C = -I \tag{16.29}$$

The positive-sequence current in phase A is

$$I_{ap} = \tfrac{1}{3}[I_A + \alpha I_B + \alpha^2 I_C] \tag{16.30}$$

$$= \tfrac{1}{3}[0 + \alpha I - \alpha^2 I] \tag{16.31}$$

$$= \frac{(\alpha - \alpha^2)I}{3} \tag{16.32}$$

$$\alpha - \alpha^2 = -0{\cdot}5 + j\frac{\sqrt{3}}{2} + 0{\cdot}5 + j\frac{\sqrt{3}}{2} = j\sqrt{3} \tag{16.33}$$

$$\therefore \qquad I_{ap} = j\frac{I}{\sqrt{3}} \tag{16.34}$$

The negative-sequence current in phase A is

$$I_{an} = \tfrac{1}{3}[I_A + \alpha^2 I_B + \alpha I_C] \tag{16.35}$$

$$= \tfrac{1}{3}[0 + \alpha^2 I - \alpha I] \tag{16.36}$$

$$= \tfrac{1}{3}I[\alpha^2 - \alpha] = -j\frac{I}{\sqrt{3}} \tag{16.37}$$

and

$$I_A = I_{a0} + I_{ap} + I_{an}$$

$$= 0 + j\frac{I}{\sqrt{3}} - j\frac{I}{\sqrt{3}} = 0 \tag{16.38}$$

$$V_{bc} = V_{cn} - V_{bn} \tag{16.39}$$

but

$$V_{cn} = V_{cnp} + V_{cnn} \tag{16.40}$$

$$V_{bn} = V_{bnp} + V_{bnn} \tag{16.41}$$

$$V_{cnp} = \alpha V_{anp} \tag{16.42}$$

$$V_{bnp} = \alpha^2 V_{anp} \tag{16.43}$$

$$V_{cnn} = \alpha^2 V_{ann} \tag{16.44}$$

$$V_{bnn} = \alpha V_{ann} \tag{16.45}$$

$$\therefore \qquad V_{bc} = \alpha V_{anp} + \alpha^2 V_{ann} - \alpha^2 V_{anp} - \alpha V_{ann} \tag{16.46}$$

$$= (\alpha - \alpha^2)(V_{anp} - V_{ann}) \tag{16.47}$$

$$= j\sqrt{3}\,(V_{anp} - V_{ann}) \tag{16.48}$$

where V_{an} = potential to neutral of phase a
V_{bn} = potential to neutral of phase b
V_{cn} = potential to neutral of phase c
V_{anp} = positive-sequence component of V_{an}
V_{ann} = negative-sequence component of V_{an}
V_{bnp} = positive-sequence component of V_{bn}
V_{bnn} = negative-sequence component of V_{bn}
V_{cnp} = positive-sequence component of V_{cn}
V_{cnn} = negative-sequence component of V_{cn}

Our equivalent circuit is, therefore, as shown in Fig. 16.7.

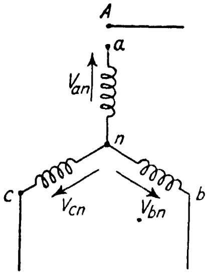


FIG. 16.6

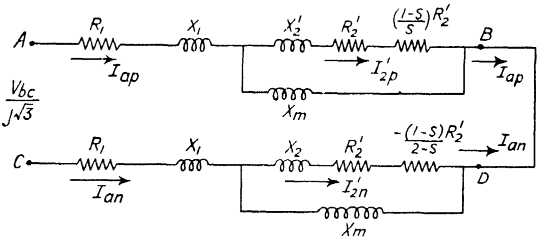


FIG. 16.7

In Fig. 16.7 is shown the positive-sequence circuit between A and B. Across AB, we have applied the voltage V_{anp}. Between C and D, we have the negative-sequence circuit and across CD is applied the voltage V_{ann}. The voltage across AC is the difference and

$$= V_{anp} - V_{ann} = \frac{V_{bc}}{j\sqrt{3}}$$

It will be noticed that the performance of the machine is calculated in terms of positive- and negative-sequence components of the disconnected phase.

The approximate diagram, in which the magnetizing reactances X_m are transferred to the terminals AC, may be used for the calculation of the gross torque. The fictitious resistance, corresponding to the output

$$= \frac{(1-s)}{s}R_2' - \left(\frac{1-s}{2-s}\right)R_2' = \frac{R_2'2(1-s)^2}{s(2-s)} \quad . \quad (16.49)$$

The gross mechanical power

$$= 6(I_{2p}')^2\frac{R_2'(1-s)^2}{s(2-s)} \quad . \quad . \quad . \quad (16.50)$$

$$I_{2p}' = \frac{I_{ap} \times jX_m}{\frac{R_2'}{s} + j(X_2' + X_m)} \quad . \quad . \quad (16.51)$$

The gross torque in lb-ft

$$= 0{\cdot}353 \times \frac{p}{f} \times \frac{1 - s}{s(2 - s)} R_2' \times (I_{2p}')^2 \qquad . \quad (16.52)$$

For small values of s the torque is positive, and the machine will run as a single-phase machine, but the maximum torque is smaller than when running as a three-phase machine. When $s = 1$, i.e. standstill, equation (16.52) shows the torque is zero, and the machine will fail to start.

From equation (16.34), we have

$$I = \frac{\sqrt{3}\, I_{ap}}{j} = -j\sqrt{3}\, I_{ap} . \qquad . \quad . \quad (16.53)$$

If Z_1 and Z_2 are the impedances *per phase* to the positive- and negative-sequence currents, we have, for the voltage drop in phase A

$$V_{anp} - V_{ann} = \frac{V_{bc}}{j\sqrt{3}} \qquad . \quad . \quad . \quad (16.54)$$

i.e.

$$I_{ap}Z_1 - I_{an}Z_2 = \frac{V_{bc}}{j\sqrt{3}} \qquad . \quad . \quad . \quad (16.55)$$

but

$$I_{ap} = j\frac{I}{\sqrt{3}} \text{ and } I_{an} = -j\frac{I}{\sqrt{3}} \qquad . \quad . \quad (16.56)$$

$$\therefore \quad j\frac{I}{\sqrt{3}} Z_1 + j\frac{I}{\sqrt{3}} Z_2 = \frac{V_{bc}}{j\sqrt{3}} \qquad . \quad . \quad (16.57)$$

$$\therefore \quad I = -\frac{V_{bc}}{(Z_1 + Z_2)} \qquad . \quad . \quad . \quad (16.58)$$

The line current I equals the line potential divided by the sum of the impedances of the motor to positive- and negative-sequence currents per phase.

The p.d. between terminal a and the line conductor, from which it is disconnected, is

$$V_{Ab} - V_{bn} = V_{bc}e^{j120} - V_{bn} \qquad . \quad . \quad (16.59)$$

$$= j\sqrt{3}\,(V_{anp} - V_{ann})\left(-0{\cdot}5 + j\frac{\sqrt{3}}{2}\right) - \alpha^2 V_{anp} - \alpha V_{ann} \quad (16.60)$$

$$= \left(-j\frac{\sqrt{3}}{2} - \frac{3}{2}\right)(V_{anp} - V_{ann})$$

$$- \left(-0{\cdot}5 - j\frac{\sqrt{3}}{2}\right) V_{anp} - \left(-0{\cdot}5 + j\frac{\sqrt{3}}{2}\right) V_{ann} \quad (16.61)$$

i.e. $-\frac{3}{2}V_{anp} + \frac{3}{2}V_{ann} + \frac{1}{2}V_{anp} + \frac{1}{2}V_{ann} =$ p.d. between A and a (16.62)

Therefore, p.d. between the line terminal A and the disconnected phase a

$$2V_{ann} - V_{anp} = 3V_{ann} \quad . \quad . \quad . \quad (16.63)$$

but

$$V_{anp} + V_{ann} = 0$$

$$\therefore \quad V_{anp} = -V_{ann}$$

Therefore, the p.d. between the terminal A and the end a of phase A

$$= 3V_{ann}$$

This is true only if the line voltages are balanced.

Now

$$I_{ann} = \frac{V_{ann}}{Z_2} \quad . \quad . \quad . \quad . \quad (16.64)$$

or

$$Z_2 = \frac{V_{ann}}{I_{ann}} \quad . \quad . \quad . \quad . \quad (16.65)$$

but $I_{ann} = -j\dfrac{I}{\sqrt{3}}$ (equation (16.37))

$$\therefore \quad Z_2 = \frac{V_{ann}\sqrt{3}}{-jI} = j\frac{V_{ann}\sqrt{3}}{I} \quad . \quad . \quad (16.66)$$

The unbalance factor

$$= \frac{V_{ann}}{V_{anp}} = \frac{I_{ann}Z_2}{I_{anp}Z_1} = \frac{-Z_2}{Z_1} \quad . \quad . \quad . \quad (16.67)$$

for

$$I_{anp} + I_{ann} = 0$$

or

$$I_{anp} = -I_{ann}$$

UNBALANCED STATOR OR ROTOR CIRCUITS

If the stator circuits are not symmetrical, and are connected to a symmetrical voltage system, the stator currents are unbalanced and the rotor will be subjected to two fields, forward and reverse, and there will be induced in it e.m.f.s having frequencies sf and $(2 - s)f$. Thus, there are positive- and negative-sequence currents in the rotor, which, if the rotor is symmetrical, will produce forward and reverse rotating fields running synchronously with the stator fields.

Unsymmetrical Rotor Circuits

Unbalance is more usual in rotor circuits than in the stator. It will be assumed that the stator circuits are symmetrical and are supplied with positive-sequence voltages. There will be produced in the rotor unbalanced currents of slip frequency sf. These unbalanced currents are resolved into balanced positive and negative components. These symmetrical components of current will produce positive- and negative-rotating fields. The forward field will rotate at a speed, with

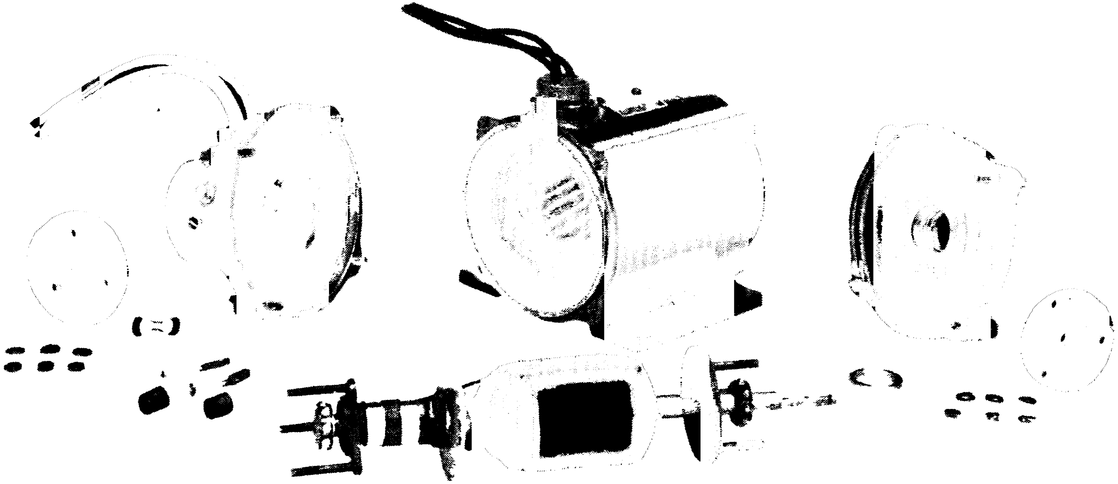


PLATE XIII THE COMPONENT PARTS OF A SELSYN RECEIVER OR TRANSMITTER

Courtesy B T H Co Ltd

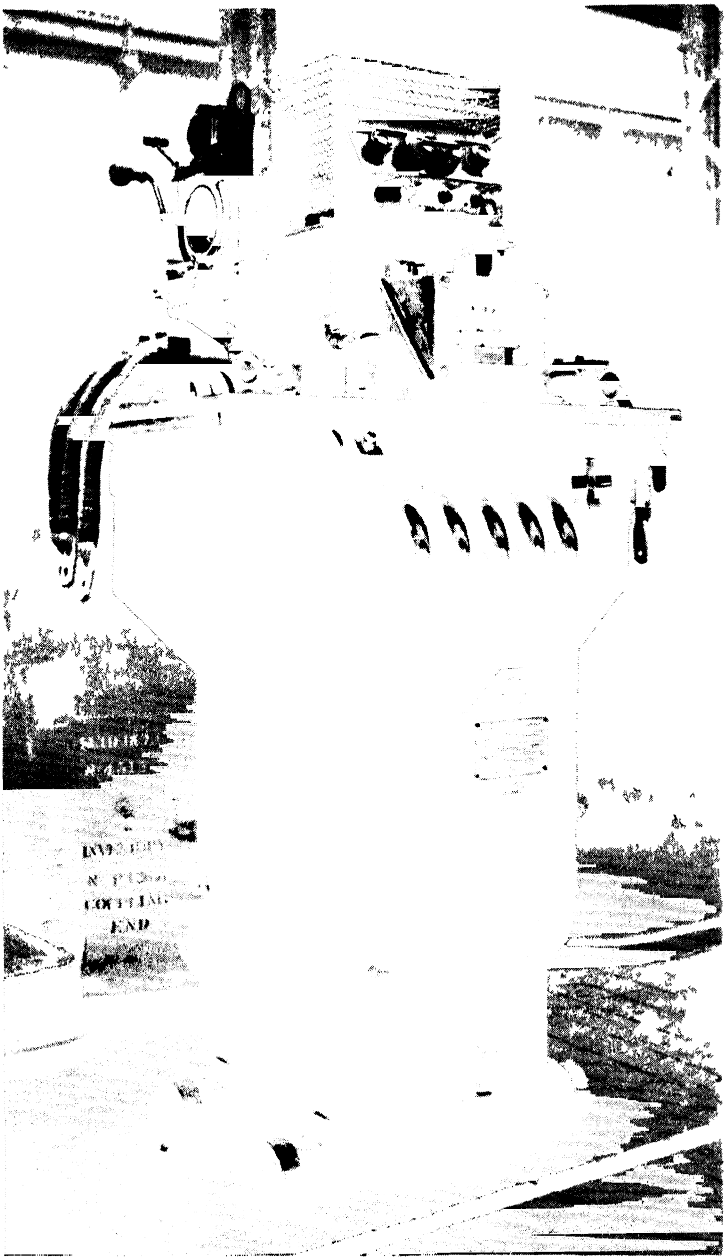


PLATE XIV. THREE-PHASE AUTOMATIC VOLTAGE REGULATOR FOR 10 PER CENT VARIATION ON A 250 KVA, 400 V FEEDER

(Courtesy English Electric Co. Ltd.)

respect to the rotor, corresponding to the slip frequency. The reverse field will generate e.m.f.s in the stator having a frequency $(1 - 2s)f$, and so, in the stator, we have currents of frequencies f and $(1 - 2s)f$. The currents of frequency $(1 - 2s)f$, are called negative-sequence currents, but it should be noted they are only called negative-sequence currents because they are due to negative-sequence currents in the rotor. When s is less than $\frac{1}{2}$, the phase order of these stator currents is like the phase order of the positive-sequence currents, but for values of s greater than 0·5 the phase order is reversed.

Let the impedances of the three rotor phases, referred to the stator, be Z_a', Z_b', and Z_c'. These are assumed to be external unequal impedances connected to otherwise three-phase symmetrical windings of the rotor.

Then $$Z_0' = \tfrac{1}{3}[Z_a' + Z_b' + Z_c'] \quad . \quad . \quad (16.68)$$

$$Z_1' = \tfrac{1}{3}[Z_a' + \alpha Z_b' + \alpha^2 Z_c'] \quad . \quad . \quad (16.69)$$

and $$Z_2' = \tfrac{1}{3}[Z_a' + \alpha^2 Z_b' + \alpha Z_c'] \quad . \quad . \quad (16.70)$$

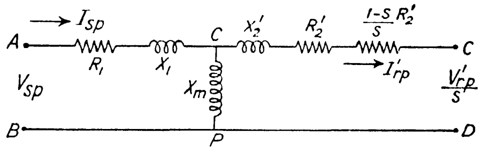


FIG. 16.8

Z_0', Z_1', and Z_2' are the zero, positive- and negative-sequence components of the impedances Z_a, Z_b, and Z_c, all referred to stator.

If V_{rp}' = positive-sequence component of the rotor terminal phase voltage, referred to the stator

I_{rp}' = positive-sequence component of the rotor phase current, referred to stator

V_{rn}' = negative-sequence component of the phase terminal voltage of the rotor, referred to stator

I_{rn}' = negative-sequence component of rotor current, referred to stator

Then

$$V_{rp}' = I_{rp}'\{\tfrac{1}{3}(Z_a' + Z_b' + Z_c')\} + I_{rn}'\{\tfrac{1}{3}(Z_a' + \alpha^2 Z_b' + \alpha Z_c')\} \quad (16.71)$$

and

$$V_{rn}' = I_{rp}'\{\tfrac{1}{3}(Z_a' + \alpha Z_b' + \alpha^2 Z_c')\} + I_{rn}'\{\tfrac{1}{3}(Z_a' + Z_b' + Z_c')\} \quad (16.72)$$

The equivalent circuit diagram for the positive-sequence network is shown in Fig. 16.8.

In Fig. 16.8 V_{sp} = positive-sequence voltage per stator phase

R_1 and X_1 = stator resistance and *leakage* reactance per phase

X_m = mutual reactance

X_2' = rotor leakage reactance per phase, referred to the stator

R_2' = rotor resistance per phase, referred to the stator

V_{rp}' = rotor positive-sequence terminal voltage, referred to the stator

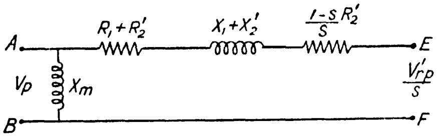


FIG. 16.9

We have
$$V_{sp} = jX_m(I_{sp} - I_{rp}') + I_{sp}(R_1 + jX_1) \quad . \quad . \quad (16.73)$$
$$= E_{g1} + I_{sp}(R_1 + jX_1)$$
$$sE_{g1} = I_{rp}'[R_2' + jsX_2'] + V_{rp}' \quad . \quad . \quad . \quad (16.74)$$
$$\therefore \quad E_{g1} = I_{rp}'\left\{\frac{R_2'}{s} + jX_2'\right\} + \frac{V_{rp}'}{s} \quad . \quad . \quad . \quad (16.75)$$

The approximate equivalent circuit for the positive sequence, obtained by transferring X_m from CD to AB, is given by Fig. 16.9.

Negative-Sequence Diagram

The negative-sequence p.d. across the rotor, due to dissymmetry of the connected impedances, acts like a primary supply p.d. to the rotor, with the stator as the secondary. The frequency of this rotor p.d. is that of the slip, viz. sf. The magnetizing reactance is sX_m. The negative-sequence currents in the rotor will produce a field, travelling at a speed corresponding to the frequency sf, in the opposite direction to the speed of the rotor. The slip, with respect to the stator, of the negative-sequence field

$$= \frac{s - (1 - s)}{s} \text{ or } \frac{-(1 - 2s)}{s}$$

The frequency of the negative-sequence stator e.m.f. and also the negative-sequence current will be $(1 - 2s)f$.

Now we have to consider the resistance and reactance of the stator windings and also those of the supply system to these negative-sequence currents of frequency $(1 - 2s)f$. Usually the motor is connected to a large supply system, and if so, we may consider the resistance and reactance of the stator windings only, for the resistance and reactance of the supply system will be negligible in this case. If this is not so, one must take due account of the external system. The negative-sequence terminal potential at the rotor is V_{rn}. Our negative-sequence diagram assumes the form shown in Fig. 16.10.

The reactance of the stator, at frequency $(1 - 2s)f = (1 - 2s)X_1$. The reactance appearing in the equivalent negative-sequence circuit

$$= \frac{(1 - 2s) X_1 s}{-(1 - 2s)}$$

We have shown that the slip with respect to the stator is $-\dfrac{(1 - 2s)}{s}$ therefore, the reactance of the stator, in the equivalent negative-sequence circuit

$$= \frac{(1 - 2s) X_1 s}{+(1 - 2s)} = + sX_1$$

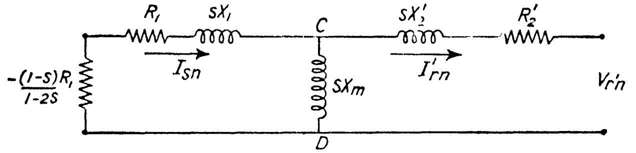


FIG 16.10

Likewise, the resistance of the stator circuit in the equivalent circuit

$$= \frac{R_1 s}{-(1 - 2s)} = - \frac{R_1 s}{1 - 2s}$$

but
$$\frac{-R_1 s}{1 - 2s} = R_1 - \frac{(1 - s)R_1}{1 - 2s}$$

We must remember that the stator, for this negative-sequence voltage originating in the rotor, and due to rotor dissymmetry, is the secondary, and the rotor the primary.

Also all our resistances and reactances are referred to the stator, so our negative-sequence rotor voltage, appearing in the equivalent circuit, will be the actual negative-sequence rotor voltage, referred to the stator. If the actual negative-sequence rotor voltage $= V_{rn}$, the referred voltage we will designate as V_{rn}'. Our negative-sequence equivalent circuit is shown in Fig. 16.10.

Negative-Sequence Equivalent Circuit

The voltage across $CD = E_{gn}$.

$$E_{gn} = jsX_m(I_{sn} - I_{rn}') \quad . \quad . \quad . \quad (16.76)$$

and
$$V_{rn}' = E_{gn} - (R_2' + jsX_2')I_{rn}' \quad . \quad . \quad (16.77)$$

Also $$I_{sn} = \frac{\mathbf{E}_{gn}}{\dfrac{-R_1 s}{1-2s} - jsX_1} = \frac{\mathbf{E}_{gn}}{R_1 - \dfrac{(1-s)R_1}{1-2s} - jsX_1}$$

$$= \frac{I_{rn}' jsX_m}{R_1 - \dfrac{(1-s)R_1}{1-2s} + j(sX_m - sX_1)} \qquad (16.78)$$

Also $$V_{rn}' = I_{rn}'\left[R_2' + jsX_2' + \frac{jsX_m\left\{\dfrac{-R_1 s}{1-2s} - jsX_1\right\}}{\dfrac{-R_1 s}{1-2s} + js(X_m - X_1)}\right] \qquad (16.79)$$

From the positive-sequence circuit, we have

$$\frac{V_{rp}'}{s} = V_{sp} - \left[R_1 + \frac{R_2'}{s} + j(X_1 + X_2')\right] I_{rp}' \quad . \quad (16.80)$$

The last relation is obtained from the approximate circuit, Fig. 16.9.

From equations (16.71), (16.72), (16.80), and (16.79), we may eliminate V_{rp}' and V_{rn}' and these equations can then be solved for the sequence currents. Then the sequence voltages are determined.

The gross mechanical power developed for three phases

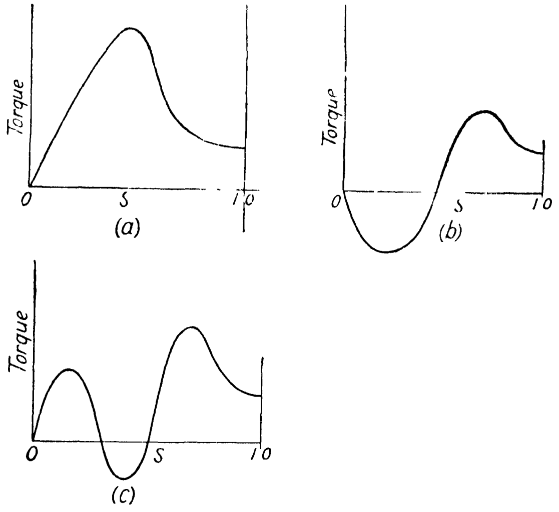
$$= 3\,\frac{(1-s)}{s}\,R_2'(I_{rp}')^2 - 3\,\frac{(1-s)}{(1-2s)}\,(I_{sn}')^2 R_2' \quad . \quad (16.81)$$

Polyphase Motor Operated with Single-phase Rotor

The late B. G. Lamme gave a simple account of this matter. He assumed, for purposes of clearer exposition, the rotor to have the primary polyphase winding, and the stator the secondary or single-phase winding. At standstill, the field, due to the polyphase winding, is assumed to rotate counter-clockwise; and generates in the secondary a single-phase e.m.f. and current. This secondary current sets up a pulsating wave of flux, which is analysed into two fields, each of half the amplitude of the pulsating field, and rotating at synchronous speed in opposite directions. One of these two components runs in the same direction as the polyphase field, and at the same speed. This field produces a torque similar to that produced in the polyphase motor. Now the other component, which travels in the opposite direction to the main field, may be regarded as a primary field.

During rotation, it will generate e.m.f.s and currents in the polyphase winding on the rotor. At standstill this field will tend to make the rotor travel *clockwise.* But the other component of the secondary current will also tend to turn the rotor clockwise, so the two components produce torque together at the start. Note, when the primary is on the rotor, the rotor will run in the opposite direction to its rotating field. As the rotor gets up to speed, the slip with respect to the stator is reduced, and the secondary or stator frequency is reduced. The frequency of the stator current is reduced, and since the speed of the rotating fields $= \frac{120 \times sf}{\text{poles}}$, the two secondary fields rotate more slowly. But the component which rotates in the same direction as the main field, will rotate at the same speed in space as the main field.

The other component of the secondary fields still rotates in the opposite direction to the main field, but more slowly. Up to half synchronous speed, the torque due to it assists the other component. At half-speed, the secondary frequency is one-half the primary frequency. The component secondary field, travelling in the same direction as the main field, is still travelling at the same speed in space as the main field; but at half-speed, the other component is moving at the same speed as the rotor and in the same direction. It is, therefore, stationary with respect to the rotor and its torque vanishes. Above half-speed, the rotor is now travelling faster than the secondary component travelling in the same direction. The torque developed by this field now becomes negative or a generator torque. This negative torque characteristic is somewhat similar to the positive torque below half speed. As the rotor speed approaches synchronous speed, the frequency of the stator currents approaches zero, and the stator field and currents fall to zero at synchronism, and the torque due to both fields vanishes. Stable torque conditions exist at approximately half-speed. Starting torque is good, and the



(a) (b) (c)

FIG. 16.11

rotor will continue to run at full speed, if brought to this speed and will carry a fair load without dropping to the lower stable speed.

Summing up, we have the torques due to the two secondary fields positive up to half-speed. Above half-speed, the torque due to one field is positive, and that due to the other negative. This is illustrated in Fig. 16.11.

In Fig. 16.11, (*a*) represents the torque due to the secondary field rotating in the same direction as the main field, (*b*) represents the torque due to the secondary field rotating in the opposite direction to the main field, and (*c*) the resultant of (*a*) and (*b*), i.e. the resultant torque. The method of symmetrical components can be applied to this case. Assume the rotor connected in star, with phases *A*, *B*, and *C*, and *A* is opened. The stator has a polyphase winding. The positive-sequence network is as shown in Fig. 16.12.

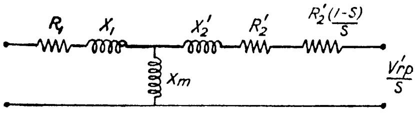


FIG. 16.12

Let the rotor current in phase $A = I_A = 0$

$$B = I_B = I$$

$$C = I_C = -I$$

Then the three symmetrical components of the current in phase A

$$= \tfrac{1}{3}[I_A + I_B + I_C] = 0 = I_0 \tag{16.82}$$

$$I_{pr}' = \tfrac{1}{3}[I_A + \alpha I - \alpha^2 I] = \frac{jI}{\sqrt{3}} \tag{16.83}$$

$$I_{nr}' = \tfrac{1}{3}[0 + \alpha^2 I - \alpha I] = -\frac{jI}{\sqrt{3}} \tag{16.84}$$

$$\therefore \qquad I_{pr}' = -I_{nr}' \tag{16.85}$$

where I_0 = zero sequence

I_{pr}' = positive sequence referred to stator

I_{nr}' = negative sequence referred to stator

If V_b = potential of point *b*

and V_c = potential of point *c*

and V_a = potential of point *a*

then $$V_{pr} = \tfrac{1}{3}[V_a + \alpha V_b + \alpha^2 V_c]$$

but $$V_b = V_c$$

$$\therefore \qquad V_{pr} = \tfrac{1}{3}[V_a + \alpha V_c + \alpha^2 V_c] \tag{16.86}$$

Also $$V_{nr} = \tfrac{1}{3}[V_a + \alpha^2 V_b + \alpha V_c]$$
$$= \tfrac{1}{3}[V_a + \alpha^2 V_c + \alpha V_c] \quad . \quad . \quad . \ (16.87)$$

$$\therefore \quad V_{pr} = V_{nr} \quad . \quad . \quad . \quad . \quad . \quad . \ (16.88)$$

The negative-sequence network is given in Fig. 16.13.

Here we regard the sequence voltages in the rotor, which are of slip frequency, as applied to the rotor as primary, and as in the section on unbalanced impedances in the rotor we saw that the slip of the negative-sequence field, with respect to the stator

$$= \frac{s - (1 - s)}{s} = -\frac{(1 - 2s)}{s}$$

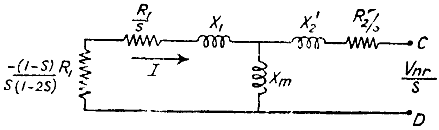


Fig. 16.13

The stator equivalent current resistance for the negative sequence

$$= \frac{-R_1 s}{1 - 2s}$$

and the stator equivalent circuit reactance for the negative sequence

$$= \frac{+X_1(1 - 2s)s}{1 - 2s} = +sX_1$$

Also $$\frac{-R_1 s}{1 - 2s} \quad R_1 - \frac{(1 - s)}{1 - 2s} R_1$$

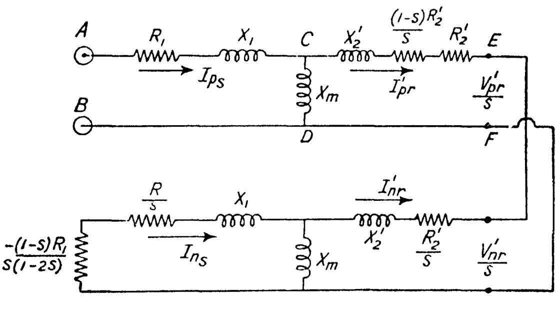


Fig. 16.14

The quantities on the right are all of slip frequency, i.e. sX_2' and R_2' and $X_m s$.

Dividing through by s, we get the quantities in Fig. 16.13.

But since $$I_{pr}' = -I_{nr}'$$
and $$V_{pr}' = V_{nr}'$$

our positive and negative equivalent circuit can be connected together, as shown in Fig. 16.14.

The gross mechanical power

$$= \frac{6(1-s)}{s} R_2'(\bar{I}_{pr}')^2 - \frac{6(1-s)}{1-2s} R_1(\bar{I}_{ns})^2 \text{ watts}$$

where $\bar{I}_{ns}$ = negative-sequence current in the stator

The copper loss

$$= 3(\bar{I}_{ps})^2 R_1 + 3R_2'(\bar{I}_{pr}')^2 + 3R_2'(I_{nr}')^2 + 3R_1(\bar{I}_{ns})^2$$

The torque (lb-ft)

$$= \left\{\frac{6R_2'(\bar{I}_{pr}')^2}{s} - \frac{6R_1(\bar{I}_{ns})^2}{1-2s}\right\} \times \frac{33\,000}{746 \times 2\pi \times \text{r.p.m.}_{\text{syn}}}$$

CHAPTER XVII

The Induction Regulator, Phase Converter, and Generator

VOLTAGE regulation of out-going lines from a central station or of ring mains, or low-voltage distribution systems, or the ring voltage of a rotary converter may be carried out by the induction regulator.

The induction motor may also be used for transformation of single-phase to polyphase supply. The motor can be wound for several different phases, and if supplied with two- or three-phase current a rotating flux wave is produced and this generates e.m.f.s in the several phases. It is thus possible to convert single-phase to polyphase.

If the rotor of an induction motor be driven above synchronous speed, the stator being connected to an a.c. supply, the machine will act as a generator, and supply power to the mains.

VOLTAGE REGULATION BY THE INDUCTION REGULATOR

The machine is simply an induction motor, in which the rotor does not rotate continuously, but can be turned through 180 electrical degrees. The rotor is turned by a worm-gear, with the gear on the shaft of the rotor, and the worm is turned by a small motor, which is automatically controlled by voltage relays. The smaller regulators are operated by hand, and may be used for special purposes.

The machine resembles, in physical structure, the induction motor, but since the rotor is stationary, it can be oil cooled and is usually built as a vertical two-pole machine, but for larger sizes as a four-pole machine. The number of poles is chosen small for purposes of easier voltage adjustment, for there are $\frac{p}{2}$ electrical degrees in one mechanical degree, where p = number of poles, and thus the mechanical angle of movement is larger the smaller the number of poles. The machine is a rather leaky transformer, but the ratio of transformation is determined, not by the ratio of the number of turns alone, but also by the position of the rotor with respect to the stator.

The regulator may be either single-phase or multi-phase.

Single-phase Regulators

Since we require to buck or boost the voltage of the line in which the regulator is connected, the secondary is connected in series with the line, and the primary, usually the rotor, is connected across the lines (*see* Fig. 17.1).

Both stator and rotor have single-phase windings.

If V_1 = voltage across the primary

V_2 = voltage across the secondary

Then $\mathbf{V}$, the voltage when boosted or bucked,

$$= \mathbf{V}_1 \pm \mathbf{V}_2$$

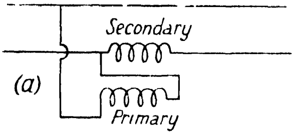


FIG. 17.1

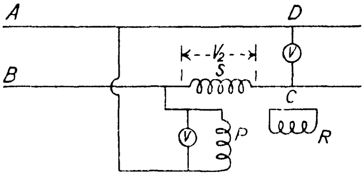


FIG. 17.2

Now the value of the secondary voltage V_2 depends on the angle the magnetic axis of the rotor makes with the magnetic axis of the stator.

In Fig. 17.1 the magnetic axes are parallel, and one has the maximum boost or maximum buck, depending on whether the secondary volts add or subtract from the volts V_1.

In Fig. 17.2 the magnetic axes are at right angles, and the secondary volts V_2 are zero. The angle between the positive and negative maxima of V_2 = 180 electrical degrees. In the position, given by Fig. 17.2, although V_2 is 0, there is a voltage drop in S, due to reactance. To overcome this reactive drop, it is usual to place, on the rotor, another winding whose magnetic axis is at right angles to the primary magnetic axis. This winding, shown as R in Fig. 17.2. is short-circuited on itself. Thus, in the position of Fig. 17.2, the coils R and S behave as the secondary and primary of a short-circuited transformer, and the flux linking S is reduced, and the reactive drop partially nullified.

Three-phase Regulators

The rotor is usually the primary and can be connected in star or delta, and the secondary is in series with the line, whose voltage is to be regulated. Since the secondary currents are usually large, it is desirable to make the stator the secondary, and then, of course, we have smaller currents to deal with at the rotor slip-rings. In the three-phase induction regulator, however, the voltage induced in the secondary winding is constant, for a given value of applied p.d. to the primary, i.e. V_2 is constant, but the relative position of the windings determines the phase of the voltage V_2 with respect to V_1. We have a constant rotating field generating the constant voltage V_2. So now the voltage relations are shown by Fig. 17.3.

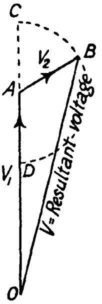


FIG. 17.3

The resultant voltage $= \vec{V_1} + \vec{V_2} = V$ and is the *vector* sum of V_1 and V_2 as B moves around the semicircle. When it reaches C, we have

$$V = V_1 + V_2$$

When it coincides with D, we have

$$V = V - V_2$$

No compensating winding is required in this regulator.

There are cases where phase-displacement, produced by the regulator is of no importance, and in such cases a single three-phase regulator is usual, but on distribution systems, supplied from a high-voltage ring main through transformers and regulators, it is necessary to avoid voltage-phase displacement and twin three-phase regulators may be employed. It may be wished to regulate the phase voltages independently in the case of three-phase, four-wire systems, where much out-of-balance load exists, and then banks of independently-operated regulators may be used.

In four-wire systems, it is usual to connect the primary windings in star, as shown in Fig. 17.4, and it is important to connect the star-point of the regulator to the neutral of the system.

It will be appreciated that, with the primaries connected across the lines, a phase displacement of 30° exists between the line and phase voltages. Various connections are possible to avoid this condition, among which may be mentioned the device of splitting each primary phase into two parts, which are fed from two adjacent phases and having the same magnetic axis; or two single-phase regulators in open-delta may be used, but here the danger of a shifted neutral may arise, which may cause trouble with earth faults. The operation of the protective switchgear may be interfered with due to this. With the primaries connected across the lines, instead of in star, the phase displacement of 30° results in an increase of 15·5 per cent in regulator size.

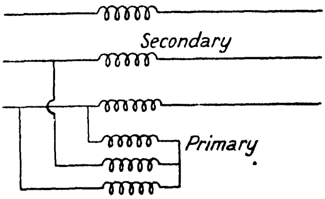


FIG. 17.4

Automatic voltage regulation can be obtained quite smoothly and easily by relay control. This control consists essentially of a sensitive-voltage control relay, a pair of interlocked motor contactors, a motor, limit switches, and magnetic brake. Thus, the control of a feeder, compensated for line voltage drop, is effected by use of a differential relay, which is supplied from two potential transformers, and a line drop compensator as shown in Fig. 17.5.

For parallel operation of single-phase or twin three-phase regulators, the potentiometer method may be used. In this, a potentiometer resistance is mounted on each regulator and has its sliding contact geared to the regulator motor. A sensitive polarized relay is connected between the slider of the master regulator and each of the trailers. The contacts of this relay close the "raise" or "lower" voltage motor contactors, thus restoring the trailing regulator to the same angular position as the master. The reader is referred to a very interesting paper by Mr. W. E. M. Ayres, M.I.E.E., read before the Institution of the Electrical Engineers, in 1931. In this paper, Mr. Ayres discusses various schemes for load balancing on parallel feeders and deals with various applications of these regulators, with a discussion on special applications.

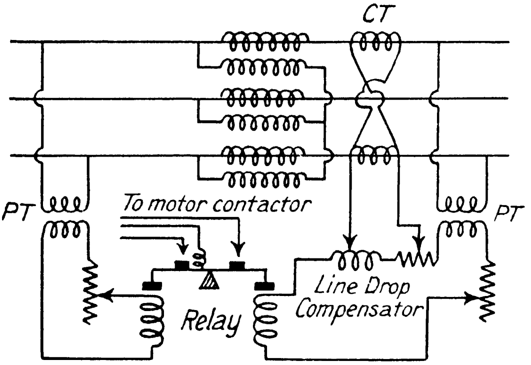


FIG. 17.5

INDUCTION MOTOR AS PHASE CONVERTER

This is of great service where motors are required for driving purposes, where only a single-phase supply is available. Single-phase motors are more costly, less efficient, and in every way inferior to polyphase motors. Three-phase motors, might, in such cases, be installed, and a phase splitting device used to start one motor. When operating near synchronous speed, the voltages between lines are caused to assume the correct phase relations, and all the other motors can be operated three phase. Phase converters are used on the Norfolk and Western Railway, U.S.A., and the locomotives are driven by three-phase induction motors, which are supplied from these converters. The supply is a single-phase one. The converter is virtually a two-phase induction motor with a squirrel-cage rotor, and is started by a single-phase commutator motor.

Phase Converter for Conversion from Single-phase to Balanced Three-phase

Fig. 17.6 illustrates the scheme. AB represents the transformer connected across the single-phase supply circuit. The phase splitter has the phase D connected across the transformer, while the phase E, which is in quadrature with D, is connected to some intermediate point of the transformer.

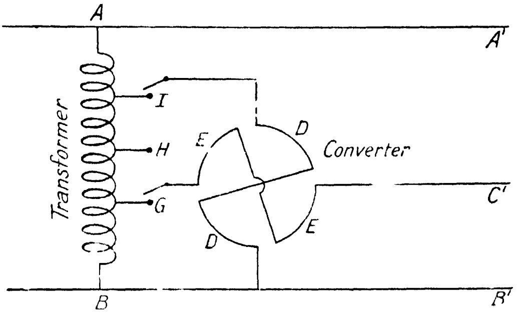


FIG. 17.6

AB = single-phase e.m.f.

CD = e.m.f. generated in phase E of the phase splitter

$$= \frac{\sqrt{3}}{2} \text{ of } AB$$

Therefore, with CD at right angles to AB, AD BD and a balanced three-phase circuit is obtained at the three-phase terminals. Under load, CD is reduced and shifted in phase to CD'. The three-phase voltages are then AB, AD', and BD' (*see* Fig. 17.7).

To restore the balanced-voltage condition under load, increase the voltage of phase D so that voltage of phase E is increased from CD' to CE', where DE' is parallel to AB, then move the connection at G along to point H, so that $CH = DE'$. Then E' coincides with point D and the balance is restored.

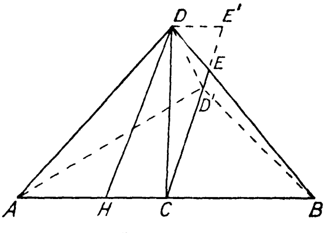


FIG. 17.7

Features must be incorporated into the design of the phase converter to annul the inductive effects, due to the load current in phase E, and, of course, it is necessary that phase E be connected to a point of the transformer winding which can be varied in position. The special feature of the split-phase locomotive is that regenerative braking can be obtained in the same way as in three-phase locomotives. Series connection of the phase converter to the induction motor supplied by it, tends, automatically, to regulate for equality of the voltages. Thus, consider a two-phase converter, supplying a two-phase induction motor. One phase of the converter is connected in series with one phase of the induction motor, driving the locomotive, into the circuit of the single-phase supply, and the second phase of the converter is connected to the second phase of the induction motor. At no-load, the voltage on the second phase is a maximum. Under load this voltage drops, and also the voltage across the other phase of the motor drops. This raises the voltage on the primary phase of the converter, and thus raises the voltage on the second phase, and tends to maintain equality of the two voltages impressed on the motor, much more so than if the phase converter were in shunt to the induction motor.

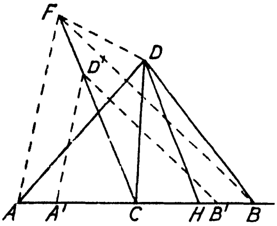


FIG. 17.8

The same arrangement, shown in Fig. 17.6, can be used in the transfer of power from three-phase to single phase. Under this condition, the voltage *CD* is shifted in phase with regard to *AB*, but in the opposite direction, and *AB* is shortened with respect to *CD*. The voltages under load are *A'*, *B' D'* (*see* Fig. 17.8).

To increase *A'B'* to the value *AB*, it is necessary to increase *CD'* to *CF*, and the triangle *AFB* has to be corrected to the value *ADB*. This can be done by moving the point *C* to *H* and reducing the e.m.f. of phase *E* of the converter, and bringing *F* to coincide with *D*. What we need is to add an e.m.f. having the magnitude and phase *FD*. This e.m.f. is obtained by combining two e.m.f.s, which are in quadrature.

Three-phase Phase Converter*

A is a three-phase induction motor, supplied from a single-phase supply (*see* Fig. 17.9). When running, there exists a three-phase system

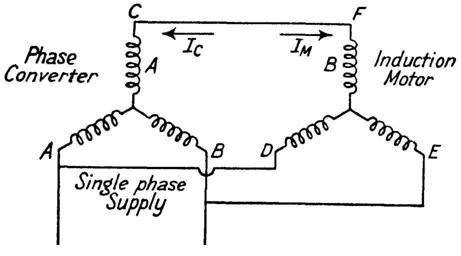


FIG. 17.9

of potentials, which are only a little out of balance. These potentials are supplied to a three-phase induction motor *B*. *A* is a phase converter, and *B* the motor.

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* See *Application of the Methods of Symmetrical Components*, by Prof. Waldo Lyon

Since C and F are connected, they are obviously at the same potential,

$$\therefore \qquad V_{cp} + V_{cn} = V_{mp} + V_{mn} \qquad (17.1)$$

and

$$V_{cp} = V_{mp} \qquad (17.2)$$

$$V_{cn} = V_{mn} \qquad (17.3)$$

where V_{cp} = positive-sequence component of voltage between C and the neutral

V_{cn} = negative-sequence component of voltage between C and the neutral

V_{mp} = positive-sequence component of phase voltage F to neutral

V_{mn} = negative-sequence component of phase voltage F to neutral

The unbalance factor

$$= \frac{V_{cn}}{V_{cp}} \qquad (17.4)$$

The current in phase A of the converter

$$\mathbf{I}_c = \mathbf{I}_{cp} + \mathbf{I}_{cn}$$

$$= \mathbf{V}_{cp}Y_{cp} + \mathbf{V}_{cn}Y_{cn} \qquad (17.5)$$

where Y_{cp} = positive-sequence admittance

Y_{cn} = negative-sequence admittance

$$\mathbf{I}_m = \mathbf{V}_{cp}Y_{mp} + \mathbf{V}_{cn}Y_{mn} \qquad (17.6)$$

where Y_{mp} = positive-sequence motor admittance

Y_{mn} = negative-sequence motor admittance

but

$$I_c + I_m = 0 \qquad (17.7)$$

$$\therefore \qquad V_{cp}Y_{cp} + V_{cn}Y_{cn} + V_{cp}Y_{mp} + V_{cn}Y_{mn} = 0 \qquad (17.8)$$

i.e.

$$V_{cp}(Y_{cp} + Y_{mp}) + V_{cn}(Y_{cn} + Y_{mn}) = 0 \qquad (17.9)$$

The unbalance factor

$$= \frac{V_{cn}}{V_{cp}} = -\frac{(Y_{cp} + Y_{mp})}{Y_{cn} + Y_{mn}} \qquad (17.10)$$

Since

$$V = -j\sqrt{3}\,V_{cp} + j\sqrt{3}\,V_{cn} \qquad (17.11)$$

V = single-phase voltage across AB.

$$V_{cp} = \frac{jV}{\sqrt{3}\left(1 - \frac{V_{cn}}{V_{cp}}\right)} \qquad (17.12)$$

$$= \frac{jV}{\sqrt{3}}\left\{\frac{Y_{cn} + Y_{mn}}{Y_{cp} + Y_{mp} + Y_{cn} + Y_{mn}}\right\} \qquad (17.13)$$

and

$$V_{cn} = -\frac{jV}{\sqrt{3}}\left\{\frac{Y_{cp} + Y_{cn}}{Y_{cp} + Y_{cn} + Y_{mp} + Y_{mn}}\right\} \qquad (17.14)$$

Thus, the unbalance factor is determined in terms of the various admittances of converter and motor.

Unbalance, in three-phase systems, is due to the presence of negative-sequence components, and the more these components are reduced, the more the condition of balance is approached. An induction motor, connected to a circuit, unbalanced by a single-phase load, corrects the unbalance to a large extent, in virtue of the fact that its impedance to negative-sequence potentials is much less than its impedance to postive-sequence potentials.

The negative-sequence current, required by the load, is transferred from the line, and is circulated between the motor and the load. The induction motor then serves as a phase balancer.

The phase balancer may be of the synchronous type, and, if so, can serve both as phase converter and as corrector of power factor. By using a synchronous booster, connected in series with one phase of the balancer, which may be a synchronous motor, with provision made to adjust the magnitude and phase of its injected e.m.f., we can balance a three-phase system, heavily unbalanced by a single-phase load of low power factor. The phase of the booster e.m.f. may be varied in diverse ways, for example, by having a field system movable with respect to the armature, or by adjustment of the excitation of two fields, displaced by 90° with respect to each other.

There are other methods of balancing line voltages, by using an unsymmetrical arrangement of static impedances between the phase converter and the motor, as suggested by Mr. R. R. Lawrence. An account of this method is given by Prof. Waldo Lyon, and the following closely follows his analysis.

Assume, in the first place, an unsymmetrical system of line impedances between the converter and the induction motor, and let these impedances have the symmetrical components Z_0, Z_1, and Z_2, and let the admittances of the converter and motor to positive- and negative-sequence currents be

$$Y_{cp} \text{ and } Y_{cn};\ Y_{mp} \text{ and } Y_{mn}$$

where Y_{cp} = positive-sequence component of the admittance of the converter

Y_{cn} = negative-sequence component of the admittance of the converter

Y_{mp} = positive-sequence component of the admittance of the motor

Y_{mn} = negative-sequence component of the admittance of the motor

V_{cp} = positive-sequence voltage of the converter of phase a

V_{cn} = negative-sequence voltage of the converter of phase a

V_{mp} = positive-sequence component of motor voltage of phase a'

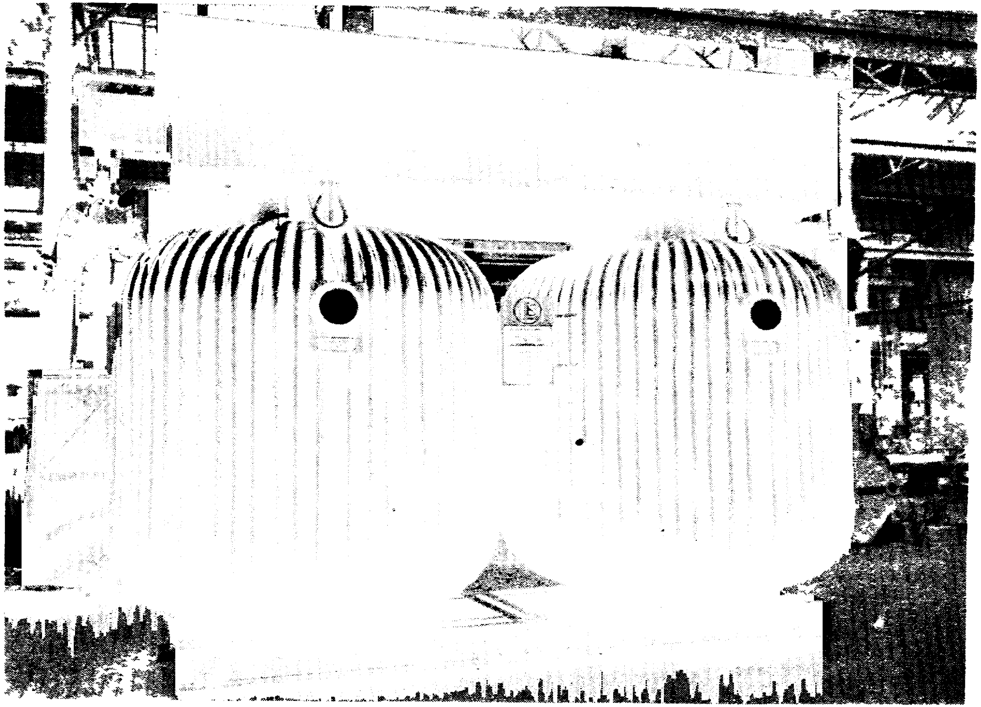
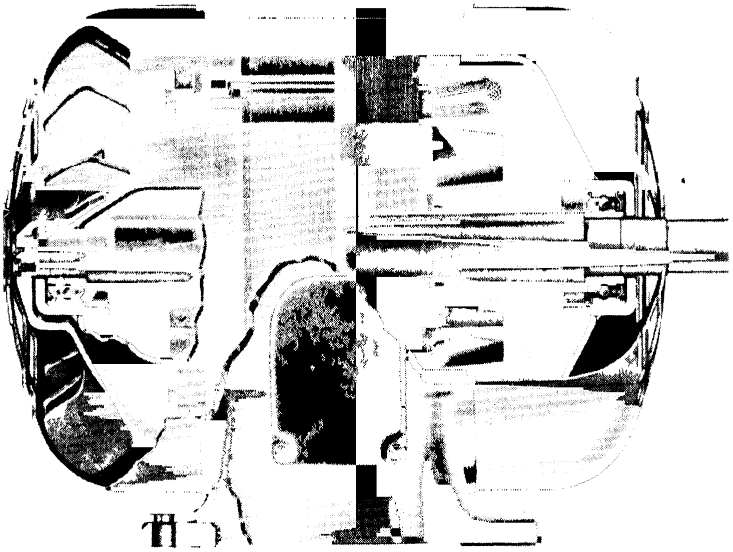


PLATE XV TWIN INDUCTION REGULATOR FOR A 20 000 kVA INTERCONNECTOR

(Courtesy, English Electric Co Ltd)



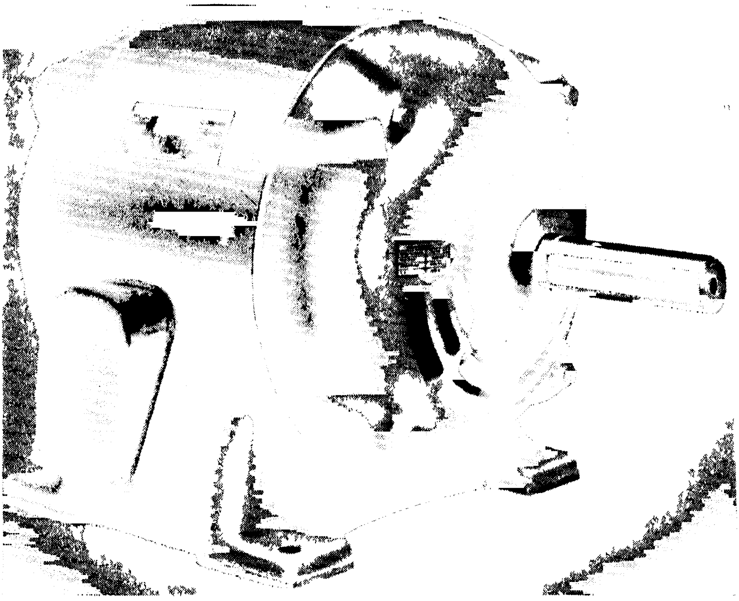


Plate XVI

(*Upper*) Totally-enclosed Fan-cooled Life-line Motor (Cut-away View)

(*Courtesy Westinghouse Electric Corp. of America*)

(*Lower*) Drip-proof Motor

(*Courtesy Westinghouse Electric Corp. of America*)

V_{mn} = negative-sequence component of motor voltage of phase a'

Then we have the following equations—

$$I_{cp} + I_{cn} + I_{mp} + I_{mn} = 0 \qquad (17.15)$$

$$V_{cp} - V_{mp} = I_{mp}Z_0 + I_{mn}Z_2 \qquad (17.16)$$

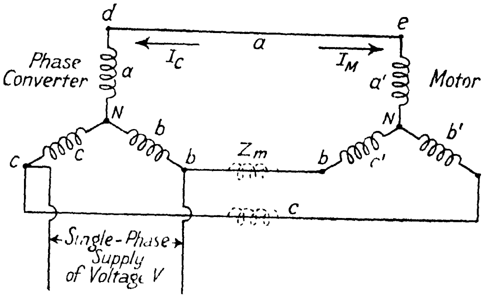


FIG. 17.10

Equations (17.15) and (17.16) refer to current in the line de, between phases a and a', and the voltages refer to voltages to the neutral from d and e.

The single-phase supply voltage to b and c = V and

$$V = -j\sqrt{3}\,V_{cp} + j\sqrt{3}\,V_{nc} \qquad (17.17)$$

$$V_{cn} - V_{mn} = I_{mn}Z_0 + I_{mp}Z_1 \qquad (17.18)$$

Now

$$\left.\begin{aligned} I_{cp} &= V_{cp}Y_{cp}; \; I_{mp} = V_{mp}Y_{mp} \\ I_{cn} &= V_{cn}Y_{cn}; \; I_{mn} = V_{mn}Y_{mn} \end{aligned}\right\} \qquad (17.19)$$

From equations (17.15) to (17.19), we may eliminate V_{cp} and V_{cn} and we get for the unbalance factor

$$\frac{V_{mn}}{V_{mp}} = -\left\{\frac{Y_{cp} + Y_{mp} + Y_{mp}(Y_{cp}Z_0 + Y_{cn}Z_1)}{Y_{cn} + Y_{mn} + Y_{mn}(Y_{cn}Z_0 + Y_{cp}Z_2)}\right\} \qquad (17.20)$$

and

$$\frac{jV}{\sqrt{3}} = V_{mp}\{1 + Y_{mp}(Z_0 - Z_1)\} - V_{mn}\{1 + Y_{mn}(Z_0 - Z_2)\} \qquad (17.21)$$

Now consider the unbalance under the action of the mutual impedance Z_m between lines b and c in Fig. 17.10. For simplicity Z_m is assumed to consist of inductance only, and the self and mutual inductances are assumed equal; then

$$Z_m = jX$$

The zero-, positive-, and negative-sequence components of *self* impedance for line a, are

zero: $$Z_0 = \tfrac{1}{3}[0 + jX + jX] = \tfrac{2}{3}jX \qquad (17.22)$$

positive: $$Z_1 = \tfrac{1}{3}[0 + \alpha jX + \alpha^2 jX] = -\frac{jX}{3} \quad . \quad . \quad (17.23)$$

negative: $$Z_2 = \tfrac{1}{3}[0 + \alpha^2 jX + \alpha jX] = -\frac{jX}{3} \quad . \quad . \quad (17.24)$$

Now we have to consider the voltage drops due to self-inductance, and mutual inductance in line *a*. The positive-sequence component of the self-inductance drop

$$= I_{mp}\frac{j2X}{3} - \frac{jX}{3}I_{mn} \quad . \quad . \quad . \quad (17.25)$$

The positive-sequence component of the mutual inductance drop in line *a*

$$= I_{mp}\frac{jX}{3} - \frac{j2X}{3}I_{mn} \quad . \quad . \quad . \quad (17.26)$$

The total positive-sequence voltage drop, due to self and mutual inductance, in line *a*, is the sum of equations (17.25) and (17.26), viz.

$$jXI_{mp} - jXI_{mn}$$

Similarly the negative-sequence component of the voltage drop in line *a* is—

$$jXI_{mn} - jXI_{mp}$$

The ratio of

$$\frac{V_{mn}}{V_{mp}} = \text{the unbalanced factor}$$

$$= -\left\{\frac{Y_{cp} + Y_{mp} + Y_{mp}(jXY_{cp} - jXY_{mn})}{Y_{cn} + Y_{mn} + Y_{mn}(jXY_{cn} - jXY_{cp})}\right\} \quad . \quad (17.27)$$

NOTE ON CALCULATION OF MUTUAL INDUCTIVE DROP

The positive, negative, and zero components of the mutual reactance between lines *b* and *a*, and *c* and *a* if the mutual reactance in lines *b* and *c* is arranged to be negative, are

(*a*) zero sequence $Z_0 = -\tfrac{1}{3}jX$ . . . . (17.28)
(*b*) positive sequence $Z_1 = -\tfrac{1}{3}jX$ . . . . (17.29)
(*c*) negative sequence $Z_2 = -\tfrac{1}{3}jX$ . . . . (17.30)

Voltage, due to mutual inductance, in line *a*, due to lines *b* and *c*

$$= jX_{ab}I_b + jX_{ac}I_c \quad . \quad . \quad . \quad (17.31)$$

jX_{bc}, which is the mutual reactance present, has symmetrical components Z_0, Z_1, and Z_2 and

$$jX_{bc} = Z_0 + Z_1 + Z_2 = -jX \quad . \quad . \quad (17.32)$$

$$jX_{ab} = Z_0 + \alpha Z_1 + \alpha^2 Z_2 \quad . \quad . \quad . \quad (17.33)$$

$$jX_{ac} = Z_0 + \alpha^2 Z_1 + \alpha Z_2 \quad . \quad . \quad . \quad (17.34)$$

$$Z_0 = \tfrac{1}{3}[jX_{bc} + jX_{ab} + jX_{ac}] = \tfrac{1}{3}[-jX + 0 + 0] = -\frac{jX}{3} \tag{17.35}$$

$$Z_1 = \tfrac{1}{3}[jX_{bc} + \alpha jX_{ab} + \alpha^2 jX_{ac}] = \tfrac{1}{3}[-jX + 0 + 0] = -\frac{jX}{3} \tag{17.36}$$

$$Z_2 = \tfrac{1}{3}[jX_{bc} + \alpha^2 jX_{ab} + \alpha jX_{ac}] = -\frac{jX}{3} \tag{17.37}$$

These results for Z_0, Z_1, and Z_2 are also given above in equations (17.28), (17.29), and (17.30).

Also

$$I_a = I_0 + I_p + I_n \tag{17.38}$$

$$I_b = I_0 + \alpha^2 I_p + \alpha I_n \tag{17.39}$$

$$I_c = I_0 + \alpha I_p + \alpha^2 I_a \tag{17.40}$$

Now by the sequence rule, we can form the components of the voltage drop, in line a, given by equation (17.31) and we have zero-sequence voltage

$$= V_{a0} = 2Z_0 I_0 - Z_1 I_n - Z_2 I_p \tag{17.41}$$

positive-sequence voltage

$$= V_{ap} = -Z_0 I_p - Z_1 I_0 + 2Z_2 I_n \tag{17.42}$$

and negative-sequence voltage

$$= V_{an} = -Z_0 I_n + 2Z_1 I_p - Z_2 I_0 \tag{17.43}$$

Since $I_0 =$ zero, we have, for the positive-sequence voltage, due to the mutual reactance

$$V_{ap} = -Z_0 I_p + 2Z_2 I_n \tag{17.44}$$

$$= +I_{mp}\frac{jX}{3} - \frac{2jX}{3} I_{mn} \tag{17.45}$$

since $Z_0 = -\dfrac{jX}{3}$ and $Z_2 = -\dfrac{jX}{3}$

This shows how equation (17.26) is obtained.

The reader will find numerical examples, given by Prof. Lyon, and also further cases of unbalance corrected by two-phase converters, in which the two phases of the converter are connected across auto-transformers and the motor connected across the auto-transformers. For this analysis and also for a very exhaustive treatment of symmetrical component applications, the reader is referred to his excellent book.

THE INDUCTION GENERATOR

If the rotor of an induction motor be driven above synchronous speed, the stator being connected to an a.c. supply, the machine will act as a generator and supply power to the mains.

It is obvious that the direction of motion of the conductors relative to the field is reversed, and, therefore, the e.m.f. and the current in the conductors are reversed. A component of current in the stator, in the reverse direction, is needed to offset the magnetizing action of the rotor currents as in the transformer. Since the current is reversed, the torque is reversed, and the machine acts as generator.

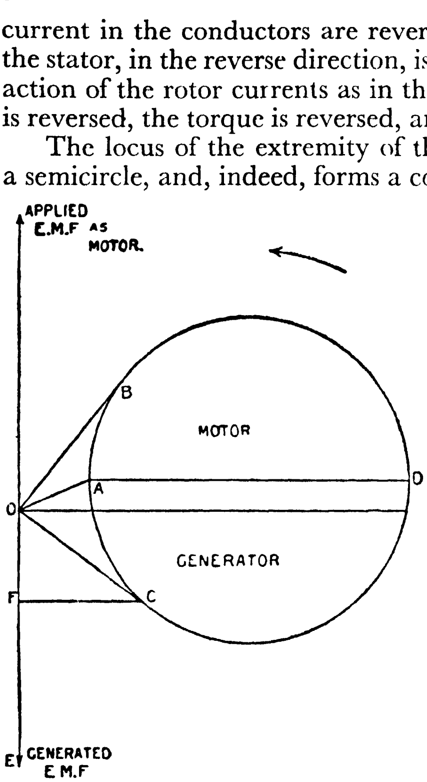


Fig. 17.11

The locus of the extremity of the primary current vector is still a semicircle, and, indeed, forms a complete circle with the semicircle of the motor. If f_0 is the frequency corresponding to the speed, and f_2 the secondary frequency, and f_1 the frequency of the primary, then $f_0 = f_1 + f_2$.

The three powers, viz. primary output, secondary output, and mechanical input are proportional respectively to f_1, f_2, and f_0.

The machine thus converts mechanical power into electric power at two frequencies f_1 and f_2, and is called a synchronous-induction generator, and the machine revolves in synchronism with the sum of the frequencies generated by it.

It is clear from our diagram that this machine requires a reactive lagging current for excitation —in other words, it is not self-exciting—and this excitation must be supplied from synchronous apparatus, or a static condenser may be used. The latter, however, is seldom practicable. Moreover the induction generator can deliver only *leading* current. If a load requires a lagging current, a synchronous machine in parallel with the induction generator must supply this.

It will be seen from Fig. 17.11 that the induction generator can furnish current at one power factor only for each value of the load, and this power factor is leading. It follows, therefore, that considerable kVA in synchronous apparatus is required for cases where loads of lagging power-factor exist, such as induction motors, etc. This is the principal objection to its use.

The advantages of the induction generator are that it does not hunt and its absence of synchronizing troubles, and the simplicity of switch-gear. It is simple and rugged, and, when short-circuited, delivers little or no power, because its excitation becomes zero. Its principal use seems to be in the development of small water-powers. It needs no synchronizing, requires no d.c. excitation, and does not fall out of synchronism. It delivers power if there is sufficient water;

if not, it merely runs idle as an induction motor. It is evident also, from the diagram, that the maximum output as generator will not be so large as the maximum output as motor. If this output is exceeded, and the torque applied to the machine is maintained, the machine will speed up indefinitely unless the governor of the prime mover acts to limit the speed. Also the maximum power factor is less as a generator than as motor.

It will be seen that by connecting the stator to a circuit of frequency f_1, the rotor generates a frequency $f_2 = f_0 - f_1$, or by connecting the rotor to a circuit of frequency f_2, the stator generates a frequency of $f_1 = f_0 - f_2$.

The powers generated in stator and rotor are proportional to their respective frequencies. Either circuit can be primary or secondary, and the magnetizing current, required for excitation, can be supplied to the stator circuit at frequency f_1 or to the rotor circuit at frequency f_2. Since the voltage needed for the exciting current is proportional to the frequency, it follows that the volt-amperes required for excitation are small when the low-frequency circuit is used for excitation. Such a machine as two-frequency generator may connect two systems of different frequency, and supply power to both in the ratio of the frequencies.

For example, the rotor may connect to a 25 c/s system and the stator to a 50 c/s system, the frequency of rotation being 75 c/s. If the two frequencies of stator and rotor are equal, i.e. $f_1 = f_2 = f_0/2$, the stator and rotor may be connected in parallel or in series, and the machine generates power at half the frequency of its speed. Such a double synchronous alternator may be useful for prime movers of very high speed, as steam turbines. Such a machine may also be used as a frequency converter. The rotor may be connected to a low-frequency exciter of frequency f_2, say, 5 c/s, and driven mechanically at a speed corresponding to 45 c/s. The stator then generates power at 50 c/s, or nine times the impressed frequency, and the electric power input is only one-ninth of the electric power output, the eight-ninths remaining being given as mechanical power input. If the rotor winding is connected to a commutator and has a frequency f_2, and is driven at a speed corresponding to frequency f_0, the effective frequency supplied by the commutator brushes to the external circuit is $f_0 - f_2 = f_1$ or the stator frequency. Hence the rotor may be connected in parallel with the stator. The rotor e.m.f. then equals the stator e.m.f. Since e.m.f. is proportional to the product of frequency and turns,

$$\therefore \quad t_2 f_2 = t_1 f_1 \quad . \quad . \quad . \quad . \quad (17.46)$$

t_2 = number of effective rotor turns

t_1 = number of effective stator turns

$$\therefore \quad \frac{f_1}{f_2} = \frac{t_2}{t_1} \quad . \quad . \quad . \quad . \quad (17.47)$$

i.e. the frequencies are inversely proportional to the number of effective turns in stator and rotor; and, since

$$f_0 = f_1 + f_2 \qquad . \quad . \quad . \ (17.48)$$

$$1 + \frac{f_2}{f_1} = 1 + \frac{t_1}{t_2}$$

i.e.
$$\frac{f_1 + f_2}{f_1} = \frac{t_2 + t_1}{t_2} \qquad . \quad . \quad . \ (17.49)$$

i.e.
$$\frac{f_0}{f_1} = \frac{t_2 + t_1}{t_2} \qquad . \quad . \quad . \ (17.50)$$

$$\therefore \qquad f_1 = f_0 \frac{t_2}{t_2 + t_1} \qquad . \quad . \quad . \ (17.51)$$

That is, the frequency generated in the stator

$$= \text{frequency of rotation} \times \frac{t_2}{t_1 + t_2} \qquad . \quad . \ (17.52)$$

This can be given any value by adjusting the turns in stator and rotor, or by interposing a transformer between stator and rotor.

The powers generated are proportional to the frequencies; and if t_1 and t_2 are very different, one generates much less power than the other.

Thus, by the use of the commutator any desired value for f_1 may be obtained, but it is obvious that economical use of material is possible only when f_1 is approximately $\frac{f_0}{2}$.

By shifting the commutator brushes, a component of the rotor current can be made to magnetize, and the machine is a self-exciting a.c. generator.

In the double synchronous alternator in which the stator and rotor have the same number of effective turns and phases, and are connected in series or in parallel with each other, the power components of the stator and rotor currents neutralize each other, neglecting the small hysteresis power current; but the reactive component of the rotor current in its reaction on the stator is reversed by the reversed direction of relative motion. Thus the synchronous exciter of the machine has to supply the total reactive component of the load in addition to the magnetizing current. It is suitable, therefore, for a load requiring leading current.

CHAPTER XVIII

Regulation and Stability of the Induction Motor

THE characteristic curves of an induction motor are usually derived for constant impressed voltage. If the voltage at the primary terminals of the transformer is constant, and such as to give the rated voltage at full load, it is clear that if there is considerable impedance in the line and transformer supplying the motor, the voltage at no-load will be higher, and the voltage at overload lower than the normal rated voltage.

At no-load, therefore, the flux and exciting current will be increased. At overload, due to fall of the voltage, the short-circuit current will be reduced, and also the maximum torque and maximum output, and also the starting torque of the motor. The magnitude of these effects will obviously depend on the variation of voltage which takes place; and it will be clear, from the expressions for the various quantities mentioned above, that at overloads and in starting it is very important to have as low impedance as possible between the primary terminals of the transformer and the motor terminals.

Variation of Frequency

If the supply frequency pulsates so rapidly that the motor speed cannot follow the frequency pulsations, then the actual slip obviously pulsates, and the motor current and torque pulsate between the values corresponding to maximum and minimum slips. The pulsation of current is moderate until synchronism is approached, but becomes large near synchronism.

The average torque drops below the torque corresponding to constant frequency. These effects are shown clearly in the following curve (Fig. 18.1) for a pulsation of frequency of supply of 2·5 per cent from the average.

Stability of the Induction Motor

The general character of the torque-slip curve for an induction motor will be seen from the curves already given. For small values of the slip, the torque increases proportionately to the slip, and the relation between torque and slip near synchronism is a linear one. The torque curve, therefore, rises in a straight line near synchronous speed and gradually bends over, increasing to a maximum value and then decreases with decrease of speed to a minimum at standstill. The slip at which maximum torque occurs varies with the rotor resistance. The higher the rotor resistance, the lower the speed or the greater the slip at which maximum torque occurs. Any value of the torque between the starting torque and maximum torque is possible at two different speeds, as is evident by drawing a horizontal line intersecting the slip-torque curve. An induction motor working against a constant torque, as, for example, driving a reciprocating pump against a constant head of water, could run at two speeds. Of these two speeds, however, one is unstable, and this speed is obviously the lower one of the two, for a momentary decrease of

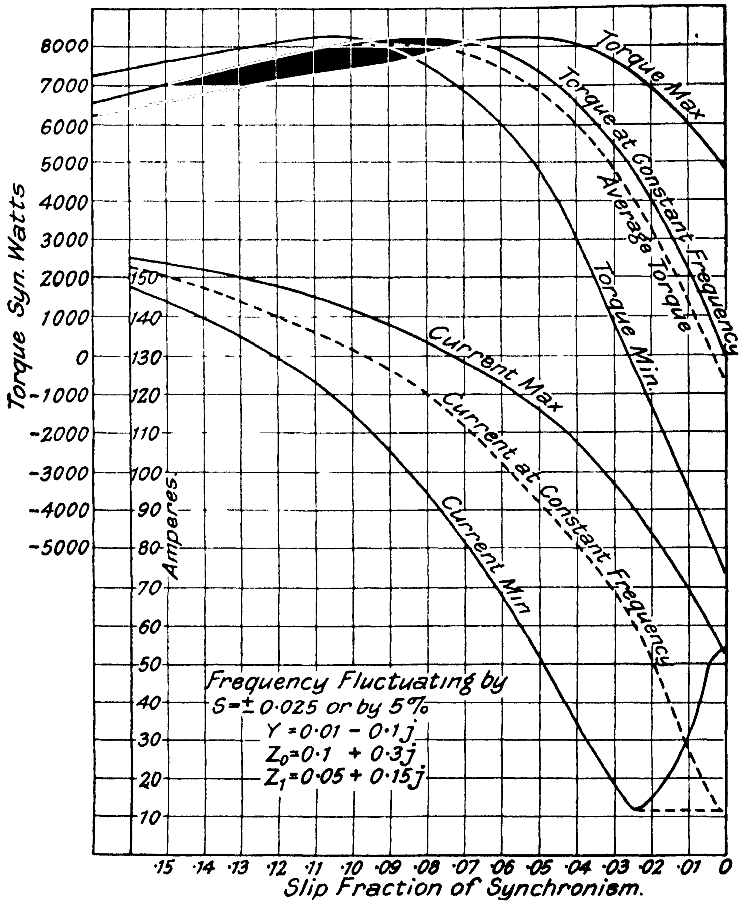


FIG. 18.1

speed, due to a variation in load or voltage, will result in a decrease of torque and speed, and the motor will come to rest. Similarly, an increase in speed, from whatever cause, results in an increase of torque and an increase of speed till the other point is reached on the curve between the maximum torque and synchronous speed. One point, therefore, is stable only, and that point is on the part of the curve between synchronous speed and the speed of maximum torque.

We may say, therefore, that, for loads requiring constant torque, the induction motor is unstable at speeds below the maximum torque point, but stable at speeds above it. This instability is, however, due to the nature of the load. Now, in a motor driving a ship's propeller or a centrifugal pump, the torque will consist of a constant part representing friction, and a part which is proportional to the square of the speed.

The load torque will consist of a parabolic curve whose origin is on the ordinate through zero speed. This parabola may, according to the magnitude of the load, intersect the speed-torque curve on any part of the curve, and in this case no unstable branch of the curve exists; but stability obtains over the entire range, and the motor will run at any such speed within the torque range. Again in the case of a load requiring a torque which consists of a constant portion and a part proportional to the speed, e.g. when driving a d.c. generator with constant excitation connected to a load of constant resistance, several cases may arise. With a moderate load, the motor accelerates up to a speed near synchronous speed, viz. the point on the speed-torque curve which the straight line representing the load torque intersects.

With a very heavy load, the motor runs up to a low speed only, for the straight line will intersect the torque curve at a low value of the speed. The straight line, representing the load torque, may intersect the torque-slip curve in three points, as will be obvious from an inspection of the curve; but of these three points, two only are stable, viz. the low speed and the high speed. The intermediate point is unstable. At the two stable points, a momentary speed increase results in a decrease of torque below that required by the load and thus limits itself, and inversely a decrease of speed increases the torque beyond that required by the load and thus restores the speed.

At the intermediate speed, the motor is unstable, and a momentary speed increase causes the motor to accelerate up to the higher speed stable point; a momentary decrease of speed causes the motor to slow down to the lower stable point. Stability is a function of the character of the load in its relation to the speed-torque curve. If the change of motor torque with change of speed is less than the change of torque required by the load, the condition is stable, otherwise it is unstable. It is thus clear that under certain classes of load, the motor may fail to accelerate the load to full speed.

This is especially the case with single-phase induction motors, in which there may be a marked depression in the torque-slip curve between standstill and maximum torque.

Again, the stability of the motor is affected by the generator regulation, and by the rapidity with which the voltage of the generator is changed with varying load, and the rapidity with which the motor speed can change as affected by the mechanical momentum of the motor and its load.

The stability coefficient of the motor, as defined by Steinmetz,

$$= \frac{1}{D}\frac{dD}{di} = K_s$$

where D = torque of the motor

i = current

If K_s is positive, an increase of current, caused by an increase of slip or decrease of speed, increases the torque D and, therefore, checks the decrease of speed and the motor is stable. Inversely, if K_s is negative, the motor is unstable. If the regulation of the generator to constancy of voltage requires a finite time, however short, the maximum output of the motor is thereby reduced, the more so the more rapidly the motor speed can change. At constant slip s, the motor torque is proportional to the square of the impressed e.m.f. e^2. If by a variation of slip, caused by a fluctuation of the load, the motor current varies by di, if the terminal voltage e remains constant, the motor torque D varies by the fraction $K_s = \frac{1}{D}\frac{dD}{di}$. If, however, the variation of current causes a variation of the impressed e.m.f. e, the motor torque, being proportional to e^2, changes still further, i.e. by the fraction

$$K_r = \frac{1}{e^2}\frac{de^2}{di} = \frac{2}{e}\frac{de}{di}$$

and the total change of motor torque resulting from a change of current $di = K_s + K_r$.

Since K_r is negative, the voltage decreasing with increasing current, the stability coefficient of the motor is reduced.

$$K_r = \frac{2}{e}\frac{de}{di}$$

represents the torque change due to momentary voltage change, and is a characteristic of the supply system. It depends on the motor only in so far as $\frac{de}{di}$ depends upon the power factor of the load.

CHAPTER XIX

Design Principles and Design

THE determination of the dimensions of the machine for a given output and the principles which determine the relation of the parts are now to be undertaken. For this purpose the output equation must be developed in which the following symbols will be used to denote the various quantities used in the exposition.

OUTPUT EQUATION

$\bar{E}$ = counter e.m.f. per phase in volts
$\bar{V}$ = applied p.d. per phase in volts
Z = conductors in series per phase
f = supply frequency
$\bar{\phi}$ = maximum value of flux per pole in c.g.s. lines
$\bar{I}$ = current per phase in amperes
p = number of poles
L = gross core length in centimetres
B = average flux density in the air-gap in c.g.s. lines per square centimetre corresponding to maximum flux
q = ampere-conductors per centimetre of stator periphery
= specific electric loading
η = efficiency
D = diameter of stator core (internal) in centimetres
θ = angle of lag of the current behind the applied p.d.
m = number of phases

Three-phase Machines

Input in watts $= 3\bar{V}\bar{I}\cos\theta$ . . . . . (19.1)
Assume $\bar{V} = 1{\cdot}05\bar{E}$ . . . . . (19.2)
Input in watts $= 3 \times 1{\cdot}05\bar{E}\bar{I}\cos\theta$ . . . . (19.3)

Now $$\bar{E} = 2{\cdot}22 \times K_1 \times K_3 \times \hat{\phi} \times Z_1 \times f \times 10^{-8} \quad (19.4)$$

K_1 = breadth factor for the fundamental wave

$$= \frac{\sin q_1 \lambda/2}{q_1 \sin \lambda/2}$$

where q_1 = slots per pole per phase in the stator
and λ = electrical slot pitch angle

$$3q_1\lambda = 180 = \pi \quad (19.5)$$

$$\therefore \quad q_1 \lambda/2 = \pi/6 \quad (19.6)$$

so $$K_1 = \frac{\sin \pi/6}{q_1 \sin (\pi/6)/q_1} \quad (19.7)$$

q_1	2	3	4	5	6	7	8	9
K_1	0·966	0·960	0·958	0·957	0·957	0·957	0·956	0·955

K_3 = coil-span factor for the fundamental $= \cos \frac{\varepsilon}{2}$. (19.8)
where ε = deficiency of pitch from full pitch

$$\cos \frac{\varepsilon}{2} = \sin \beta/2$$

where β = actual span of coil

$$\hat{\phi} = \frac{\pi D}{p} \times L \times B, \text{ and } f = \frac{p}{2} \times \frac{\text{revolutions per minute}}{60} \quad (19.9)$$

$$\text{b.h.p.} = \frac{\text{input} \times \eta}{746} = \frac{3 \times 1{\cdot}05 \times \bar{E}I \cos \theta \times \eta}{746} \quad (19.10)$$

$$\therefore \text{ b.h.p.} = 3 \times 1{\cdot}05 \times 2{\cdot}12 \times \frac{\pi D}{p} \times \frac{L \times B \times Z_1}{10^8}$$
$$\times \frac{p}{2} \times \frac{\text{r.p.m.} \times \cos \theta \times \eta \times I}{60 \times 746} \quad (19.11)$$

Now $$3Z_1 I = \pi D q \text{ and } Z_1 \bar{I} = \frac{\pi D q}{3} \quad (19.12)$$

$$\therefore \text{ b.h.p.} = \frac{1{\cdot}05 \times 2{\cdot}12 \times \pi^2 D^2 L \times B \times q \times \text{r.p.m.} \times \cos \theta \times \eta}{2 \times 60 \times 746 \times 10^8} \quad (19.13)$$

$$\therefore \quad D^2L \,(\text{cm}^3) = \frac{4{\cdot}06 \times 10^{11} \times \text{b.h.p.}}{B \times q \times \text{r.p.m.} \times \cos \theta \times \eta} \quad (19.14)$$

$B = \hat{B}\,\pi/2$
= average flux density in the air-gap

This may also be expressed as

$$\text{b.h.p.} = KD^2L \times \text{r.p.m.} \qquad . \qquad . \ (19.15)$$

where K = output coefficient

$$= \frac{1}{4{\cdot}06 \times 10''} \times B \times q \times \cos 0 \times \eta \qquad . \qquad . \ (19.16)$$

The larger the value of the output coefficient, the smaller the D^2L required, and the larger the number of revolutions per minute, the smaller the machine. The art of design consists in obtaining the maximum output per pound of material. This maximum output is limited by various factors, chief of which is the maximum permissible temperature rise in service. Failure of the insulation is caused chiefly by it being subjected to too high a temperature. Thus, cotton and fibrous insulations generally, are near the danger point at a temperature of 90°C. Thus, we usually find that, for continuous service the rise of temperature by thermometer is limited to 40°C, where the ambient temperature is of the order of 25°C. It is the "hot spots" which we are concerned with, and these are much higher in temperature than recorded by thermometer.

Much progress has been made in slot insulations in recent years, and the introduction of silicone varnishes has raised the permissible operating temperatures very greatly. Our value of B is very largely determined by the condition that, for high power factor, there must not be saturation in any part of the magnetic circuit, for the value of B in the air-gap is directly related to the value of B in the teeth and core, and then again we shall find that the value of B determines the overload capacity. Most induction motors have an overload capacity of twice in horse-power, but as the normal speed gets lower and lower —i.e. in machines with a large number of poles—it is very difficult to get this overload capacity and still get reasonably good power factor. One then has to be satisfied with a compromise. Then again the iron losses in teeth and core are determined by the value of B used, for the hysteresis loss varies as the value of $B_t^{1\cdot7}$ and the eddy-current loss varies as B_t^2, where B_t = density in teeth, and the same applies to the core also. Thus, the choice of our value of B in the output equation is a very important one, for we have to consider overload capacity, iron losses, power factor, and temperature rise in its choice.

Now q, the specific electric loading, also affects the heating and the overload capacity of the motor. We shall see later that the ideal short-circuit current is equal to the $\dfrac{\text{working flux per pole}}{\text{leakage flux per pole per ampere}}$ and q enters into the denominator of this ratio. It also enters into the copper loss, and so its choice is also important.

The other factors in our output coefficient are not so much at our choice, for $\cos \theta$ is usually fixed for us and also the efficiency. We can, by excellent ventilation and by the use of insulating materials which will withstand high temperature, obtain the maximum output per pound of material but we must not deceive ourselves by supposing that these have no limits. Our temperature rise may be limited readily by a good ventilation scheme, but if we increase our B and our q in this way, we soon run up against our power-factor requirement. At low speeds, our problem gets more difficult, for our ventilation is not so good, our power factor is naturally low, and our overload capacity difficult to obtain. It is, therefore, wise in selecting our motor to choose as high a speed as possible for our job, and use a coupling of some sort to get the low speed. The performance of the high-speed induction motor is incomparably better than the low-speed machine. Again also, it is also desirable to see that the average load is the *full load.* In many factories, induction motors are chosen much larger in output than they need be. The consulting engineer wants to make sure that each machine is more than equal to its job, and so sometimes a 20 h.p. machine is used where only 10 or 15 h.p. is required. If there are many such machines in a factory, and they are all running lightly loaded, the power factor of the whole load is very low, and this is rather hard on the supply authority, and also costs the factory in losses far more than is generally realized.

Two-phase Induction Motors

$$\text{Input} = 2\bar{V}\bar{I}\cos\theta \qquad (19.17)$$

$$= 2 \times 1{\cdot}05\bar{E}I\cos\theta \qquad (19.18)$$

$$\text{b.h.p.} = \frac{2 \times 1{\cdot}05\bar{E} \times \bar{I}\cos\theta \times \eta}{746} \qquad (19.19)$$

$$\bar{E} = 2 \times \hat{\phi} \times Z \times f \times 10^{-8}$$

$$= 2 \times \frac{\pi D}{p} \times L \times B \times Z \times \frac{p}{2} \times \frac{\text{r.p.m.}}{60} \times \frac{1}{10^8} \qquad (19.20)$$

$$\therefore \text{b.h.p.} = \frac{2 \times 1{\cdot}05 \times 2 \times \pi D \times L \times B \times \text{r.p.m.} \times ZI\cos\theta \times \eta}{120 \times 10^8 \times 746} \qquad (19.21)$$

$$2ZI = \pi Dq$$

$$D^2L = \frac{4{\cdot}3 \times 10^{11} \times \text{b.h.p.}}{q \times B \times \text{r.p.m.} \times \cos\theta \times \eta} \qquad (19.22)$$

It will be noticed that the D^2L for two-phase machines is about 6 per cent greater than for three-phase machines for the same output.

Single-phase Motors

$$\text{Input} = \bar{V}\bar{I}\cos\theta \qquad . \quad . \quad . \ (19.23)$$

$$\text{b.h.p.} = 1{\cdot}05\bar{E}\bar{I}\cos\theta \times \eta/746 \qquad . \quad . \ (19.24)$$

$$\bar{E} = 1{\cdot}41 \times \hat{\phi} \times Z \times f \times 10^{-8} \qquad . \quad . \quad . \ (19.25)$$

$$= 1{\cdot}41 \times \frac{\pi D}{p} \times L \times B \times Z \times f \times 10^{-8} \qquad . \ (19.26)$$

$$\text{b.h.p.} = 1{\cdot}05 \times 1{\cdot}41 \times \frac{\pi D}{p} \times L \times B \times \frac{Z\bar{I}}{10^8} \times \frac{\cos\theta \times \eta}{746} \times \frac{p}{2} \times \frac{\text{r.p.m.}}{60} \qquad . \quad . \ (19.27)$$

$$Z\bar{I} = \pi D q \qquad . \quad . \quad . \quad . \ (19.28)$$

$$\text{b.h.p.} = \frac{1{\cdot}05 \times 1{\cdot}41 \times \pi^2 D^2 L \times q \times B \times \text{r.p.m.} \times \cos\theta \times \eta}{2 \times 60 \times 10^8 \times 746} \qquad (19.29)$$

$$\therefore \qquad \underset{(\text{cm}^3)}{D^2L} = \frac{6{\cdot}1 \times 10^{11} \times \text{b.h.p.}}{q \times B \times \text{r.p.m.} \times \cos\theta \times \eta} \qquad . \ (19.30)$$

Single- and Triple-phase Motors

It will be observed that the D^2L for single-phase machines is 50 per cent greater than for three-phase, for the same output with all the slots wound; or in other words, the output for the single-phase machine is two-thirds of that for the three-phase machine for the same D^2L.

In practice, only two-thirds of the slots are utilized for the main winding, and the remaining slots are used for the auxiliary starting winding. In this case the output is only 50 per cent of the three-phase output for the same D^2L.

We have already observed that an overload capacity of twice in horse-power is usually required. This means that the ideal and actual short-circuit currents must have minimum values, determined by the above requirements. We have shown in Chapter I, equation (1.92), that the maximum h.p.

$$= \frac{m\bar{V} \times JF}{2(1 + \cos\phi_{sc}) \times 746}$$

where m = number of stator phases

$\bar{V}$ = applied volts per phase

JF = actual rotor short-circuit current, referred to the stator (*see* Fig. 1.12)

$\cos\phi_{sc}$ = power factor of rotor current at short-circuit

or maximum h.p.

$$= \frac{m\bar{V} \times (OB - OA)\sin\phi_{sc}}{2(1 + \cos\phi_{sc}) \times 746} \qquad . \quad . \ (19.31)$$

where OB = ideal short-circuit current per phase
OA = magnetizing current per phase

Now since it is desirable that the maximum h.p. shall be twice the normal b.h.p., we have

$$2 \text{ b.h.p.} = \frac{m\bar{V} \times (OB - OA) \sin \phi_{sc}}{2(1 + \cos \phi) \times 746} \quad . \quad . \quad (19.32)$$

for three-phase machines

$$\text{b.h.p.} = \frac{3\bar{V} \times (OB - OA) \sin \phi_{sc}}{4(1 + \cos \phi) \times 746} \quad . \quad . \quad (19.33)$$

for two-phase machines

$$\text{b.h.p.} = \frac{\bar{V} \times (OB - OA) \sin \phi_{sc}}{2(1 + \cos \phi) \times 746} \quad . \quad . \quad (19.34)$$

For three-phase machines

$$OB - OA = \frac{4 \times \text{b.h.p.}(1 + \cos \phi_{sc}) \times 746}{3V \sin \phi_{sc}} \quad . \quad (19.35)$$

For two-phase machines

$$OB - OA = \frac{2 \times \text{b.h.p.}(1 + \cos \phi_{sc}) \times 746}{V \sin \phi_{sc}} \quad . \quad (19.36)$$

For machines, whose outputs range to 50 h.p.,

$\cos \phi_{sc}$ at short-circuit = 0·5 approximately
and $\phi_{sc} = 60^\circ$
and $\sin \phi_{sc} = 0{\cdot}866$

Therefore, for three-phase machines of this range

$$OB - OA = \frac{1725 \times \text{b.h.p.}}{V} \text{ approximately} \quad . \quad (19.37)$$

and for two-phase machines up to about 50 h.p.

$$OB - OA = \frac{2580 \times \text{b.h.p.}}{V} \quad . \quad (19.38)$$

For large machines

$\cos \phi_{sc} = 0{\cdot}25$ approximately
and $\sin \phi = 0{\cdot}967$

therefore, for large three-phase machines

$$OB - OA = \frac{1285 \times \text{b.h.p.}}{V} \quad . \quad . \quad (19.39)$$

and for large two-phase motors

$$OB - OA = \frac{1930 \times \text{b.h.p.}}{V} \quad . \quad . \quad (19.40)$$

It is clear that the ideal short-circuit current is definitely fixed, if the overload condition is to be satisfied.

The ideal short-circuit current per phase

$$OB = \frac{\hat{E}}{L \times 2\pi f} \quad . \quad . \quad . \quad (19.41)$$

where L = equivalent coefficient of self-inductance per phase in henrys

$$\hat{E} = \sqrt{2}\pi \times \hat{\phi} \times \frac{Z}{2} \times f \times 10^{-8} \times K \quad . \quad . \quad . \quad (19.42)$$

where K = winding factor
= breadth factor × coil-span factor
Z = conductors in series per phase
= $p \times q_1 \times z_1$
where p = number of poles
q_1 = slots per pole per phase
z_1 = conductors in series per slot

$$\frac{\hat{E}}{f} = \hat{\phi} \times \sqrt{2}\pi \times \frac{p}{2} \times q_1 \times z_1 \times \frac{1}{10^8} \times K \quad . \quad (19.43)$$

$$L = \frac{4\pi \times (q_1 z_1)^2 \times K^2 \times \lambda \times p}{10^9} \text{ henrys} \quad . \quad (19.44)$$

K is about 0·96 for three-phase with full pitch coils = $K_1 \times K_3$.

λ = magnetic conductivity of the leakage flux path encircled by the ampere-turns.

$$OB = \frac{\hat{\phi} \times \sqrt{2}\pi \times p \times q_1 \times z_1 \times 10^9 \times K}{4\pi \times 2p \times 2\pi \times \lambda \times (q_1 \times z_1)^2 \times 10^8 \times K^2} \quad . \quad (19.45)$$

$$= \frac{\hat{\phi}}{0{\cdot}4\pi \times 2\sqrt{2} \times \lambda \times q_1 \times z_1 \times K}$$

$$= \frac{\hat{\phi}}{3{\cdot}55 \times K \times \lambda \times q_1 \times z_1} \quad . \quad . \quad . \quad (19.46)$$

i.e. the ideal short-circuit current per phase

$$= \frac{\text{working flux per pole}}{\text{leakage flux per pole per ampere}} \quad . \quad . \quad (19.47)$$

Now $$\lambda = \lambda_s + \lambda_z + \lambda_T \quad . \quad . \quad . \quad (19.48)$$

where λ_s = magnetic permeance of the *slot* stray flux path
λ_z = magnetic permeance of the zigzag leakage path
λ_T = magnetic permeance of the overhang leakage path

$$OB - OA = \frac{\text{constant} \times \text{b.h.p}}{V}.$$

$$\therefore 1 - \frac{OA}{OB} = \frac{\text{constant} \times \text{b.h.p.} \times 3{\cdot}55 \times K \times \lambda \times q_1 \times z_1}{V \times \hat{\phi}} \qquad (19{\cdot}49)$$

$$\text{b.h.p.} = m\bar{V}\bar{I} \cos \theta \times \eta$$

$$\therefore 1 - \frac{OA}{OB} = \frac{\text{constant} \times m\bar{I} \cos \theta \times \eta \times 3{\cdot}55 \times K q_1 z_1 \lambda}{\hat{\phi}} \quad . \quad (19{\cdot}50)$$

$$\therefore 1 - \frac{OA}{OB} = \frac{\text{constant} \times m q_1 z_1 \times p \times 3{\cdot}55 K \times \lambda \times I \cos \theta \times \eta}{p\hat{\phi}} \quad . \quad . \quad (19{\cdot}51)$$

$$m q_1 z_1 \times p \times I = \text{total ampere-conductors} \quad . \quad (19{\cdot}52)$$

and

$$p\hat{\phi} = \text{total flux} \quad . \quad . \quad . \quad . \quad (19{\cdot}53)$$

$$\therefore 1 - \frac{OA}{OB} = 1 - \sigma$$

$$= \frac{\text{constant} \times 3{\cdot}55 K \times \lambda \cos \theta \times \eta \times \text{total ampere-conductors}}{\text{total flux}} \quad . \quad . \quad (19{\cdot}54)$$

$$\frac{OA}{OB} = \sigma = \text{dispersion coefficient}$$

$$\therefore \frac{\text{total ampere-conductors}}{\text{total flux}}$$

$$= \frac{1 - \sigma}{\text{constant} \times 3{\cdot}55 \times K \times \lambda \times \cos \theta \times \eta} \quad . \quad (19{\cdot}55)$$

Loading and Overloading

The constant in the denominator is given in equations (19·37) to (19·40). This is the relation between the electric and magnetic loadings in order that a given overload capacity may be obtained. Now generally σ increases as λ increases, and we see that, for machines having large values of σ, i.e. machines of many poles, the ratio of electric to magnetic loading becomes smaller and smaller. In such machines, as a general rule, overload capacity must be sacrificed if reasonably high power factor is to be obtained, and the working flux must be reduced below the value required for an overload capacity of twice the normal. Now the greater the specific electric and magnetic loadings, the smaller the D^2L of the machine. From the commercial standpoint, it is desirable to push these two quantities to their limits. The question arises as to what are the limits, and how are they determined? It may be taken as axiomatic that a high power factor is the first desideratum, and, secondly, that the efficiency shall be as high as is consistent with the machine reaching its permissible temperature rise in all its parts. The temperature rise is fixed by the magnetic and electric loading, for the copper loss is proportional to the ampere-conductors per centimetre with constant current density, and the iron loss is a function of the magnetic loading.

Power factor considerations dictate that the specific magnetic loading *B*, shall be as small as possible. There should be no saturation in teeth or in core, if high power factor is to be obtained, and, indeed, at 50 c/s, this more or less agrees with the heating limit. The maximum tooth density, at minimum tooth section, should not exceed 16 000 lines/cm^2 at 50 c/s. This corresponds to 10 200 lines/cm^2 average at minimum section. The corresponding average apparent flux density in the air-gap will be about 4000 to 5000 lines/cm^2. Of course, in cases where power factor is not important, but overload capacity is the characteristic required, saturation in the teeth and core is usual, and higher densities are used than those given above. It will quickly be realized that one can get all the overload capacity one wishes with smaller values of *B* in the gap than 4000 lines/cm^2 on high-speed machines, i.e. with small numbers of poles and high power factor is easily obtainable on two- and four-pole machines, but, as the number of poles increases, it is difficult to get *both* high power factor and *high overload* capacity. The two things are incompatible, and one perforce has to compromise and sacrifice one or the other. Higher values of *B* average in the gap must be resorted to, when the number of poles is large, and then the power factor comes down.

The values of *B* in the gap, teeth, and core given above are for 50 c/s machines. For aircraft work, one needs to design machines for 400 c/s, and, of course, one has to reduce the densities in the teeth to 5000 lines/cm^2 or so, with a value of B_{av} in the gap of about 2500 lines/cm^2. Advantage is taken also of the great reduction of iron loss by using the silicon steel stampings—Stalloy and Super-Stalloy—for these high frequencies. An even better iron is that produced by the Westinghouse Corporation, known as "Hypersil." Where it is necessary to keep the iron losses down to reasonable values, especially at 400 c/s, the use of Stalloy and Super-Stalloy is indicated. Of course, for totally enclosed machines of continuous rating, one must also lower the densities in the magnetic circuit.

Again, other considerations than power factor and overload capacity may be more important. Occasionally one has to design machines which are free from noise of all kinds. This invariably means low flux densities, and large machines for their output. So in choosing our *B* one must keep in mind (*a*) power factor, (*b*) overload capacity, (*c*) efficiency, (*d*) temperature rise, and (*e*) noise.

We have now to investigate how the heating limit affects the electric loading of the machine.

The permissible watts per square centimetre of barrel surface at the gap of the stator for protected-type machines, fitted with fans, varies with peripheral speed, according to the curve shown in Fig. 19.1.

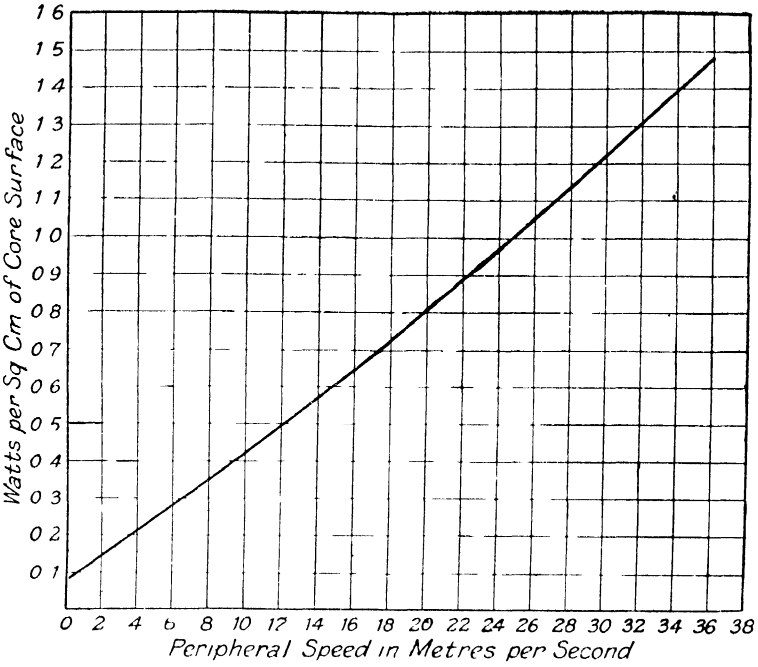


FIG 19 1

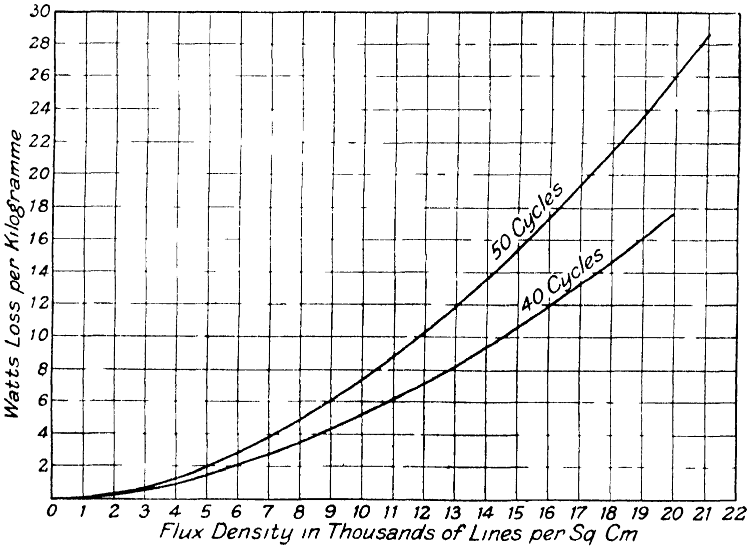


FIG 19 2

Analysis of the curves for iron losses, given in Fig. 19.2 shows that the iron loss per kilogramme for lohys

$$= 0{\cdot}000227B^{1{\cdot}8} \times f^{1{\cdot}6} \text{ watts} \quad . \quad . \quad (19.56)$$

where B = maximum flux density in thousands of lines per square centimetre

f = supply frequency

Since 1 cm^3 of iron weighs 7·9 g, 1 kg of iron occupies $\frac{1000}{7{\cdot}9}$ cm^3 = 126·5 cm^3.

Therefore, iron loss in watts per cubic centimetre

$$= 0{\cdot}00000179B^{1{\cdot}8}f^{1{\cdot}6} \quad . \quad . \quad . \quad (19.57)$$

where B is in kilolines per square centimetre

The permissible watts per square centimetre of stator barrel surface at the gap below 12m/sec

= 0·033 × peripheral speed in metres per second + 0·08

above 12 m/sec

= 0·0408 × peripheral speed in metres per second

If z_1 = conductors in series per slot

I = current per conductor in amperes

h_s = height of slot in centimetres

w_s = width of slot in centimetres

A = area of conductor in square centimetres

$\triangle$ = current density in amperes per square centimetre

Consider unit length of slot, i.e. 1 cm length axially.

The copper loss per centimetre

$$= \frac{I^2 z_1 \rho}{A} \quad . \quad . \quad . \quad . \quad (19.58)$$

where ρ = specific resistance of copper at the temperature considered. Copper loss per centimetre

$$= \frac{I}{A} \times I z_1 \rho$$

$$= \triangle \times I z_1 \rho \quad . \quad . \quad . \quad . \quad (19.59)$$

The specific electric loading

$$q = \frac{I z_1 \times \text{number of slots}}{\pi D} \quad . \quad . \quad (19.60)$$

$$q = \frac{I z_1}{\text{slot pitch}} = \frac{I z_1}{K \omega_s} \quad . \quad . \quad . \quad (19.61)$$

where K = some constant

$$\therefore \qquad \frac{Iz_1}{\omega_s} = q \times K \qquad . \quad . \quad . \ (19.62)$$

The copper loss per square centimetre at the gap surface

$$= \frac{\triangle \rho I z_1}{\omega_s} \qquad . \quad . \quad . \quad . \ (19.63)$$

$$= \triangle \rho q \times K \qquad . \quad . \quad . \ (19.64)$$

The iron loss in the teeth at $B = 16 \times 10^3$ and $f = 50$ c/s

$$= 0{\cdot}00000179 \times 16^{1{\cdot}8} \times 50^{1{\cdot}6}$$

$$= 0{\cdot}135 \text{ W/cm}^3 \qquad . \quad . \quad . \quad . \ (19.65)$$

Therefore, the iron loss per square centimetre at the gap, assuming all the tooth loss to be dissipated at the gap surface

$$0{\cdot}135 \times h_s \text{ watts} \qquad . \quad . \quad . \ (19.66)$$

Therefore, up to a peripheral speed of 12 m/sec, we have

$$\triangle \rho q K + 0{\cdot}135 h_s = 0{\cdot}033 v_s + 0{\cdot}08 \ . \qquad . \ (19.67)$$

where v_s = peripheral speed of rotor in metres per second

and above 12 m/sec

$$\triangle \rho q K + 0{\cdot}135 h_s = 0{\cdot}0408 v_s \qquad . \quad . \ (19.68)$$

From these two equations, we can draw a series of curves, for several values of h_s, giving q as a function of v_s. It will be seen that the product of current density and specific electric loading is sensibly constant for a given peripheral speed and depth of slot. If we take a value for $\triangle$ of 350 A/cm² (a very usual value) and calculate q for several values of h_s, we get the series of lines given in Fig. 19.3.

On 25 c/s machines, the iron loss is considerably reduced; it follows that, on 25 c/s machines, we can increase the electric loading to a certain extent.

The iron loss, at 25 c/s, per cubic centimetre, at $B = 16{,}000 = 0{\cdot}049$.

$$\therefore \qquad \triangle \rho q K + 0{\cdot}049 h_s = 0{\cdot}0408 v_s$$

$v_s > 12$ m/sec

and $\qquad \triangle \rho q K + 0{\cdot}049 h_s = 0{\cdot}033 v_s + 0{\cdot}08$

$v_s < 12$ m/sec

The curves are based on a value of $K = 2$.

This derivation is imperfect, for it assumes that the whole of the copper and iron loss is dissipated at the gap surface. It is well known that a certain amount of heat is dissipated from the ducts and end-plates, and it would appear that the values of q from curve 3 may be increased by about 10 per cent. Such curves are, by nature, approximate only, but they do give a suitable working basis for arriving at a suitable value for q. Incidentally, the effect of slot depth on the value of q is shown. We have now determined the specific electric and magnetic loadings, and it now remains to determine the power factors and efficiencies, which may be reasonably expected from a well-designed line of machine.

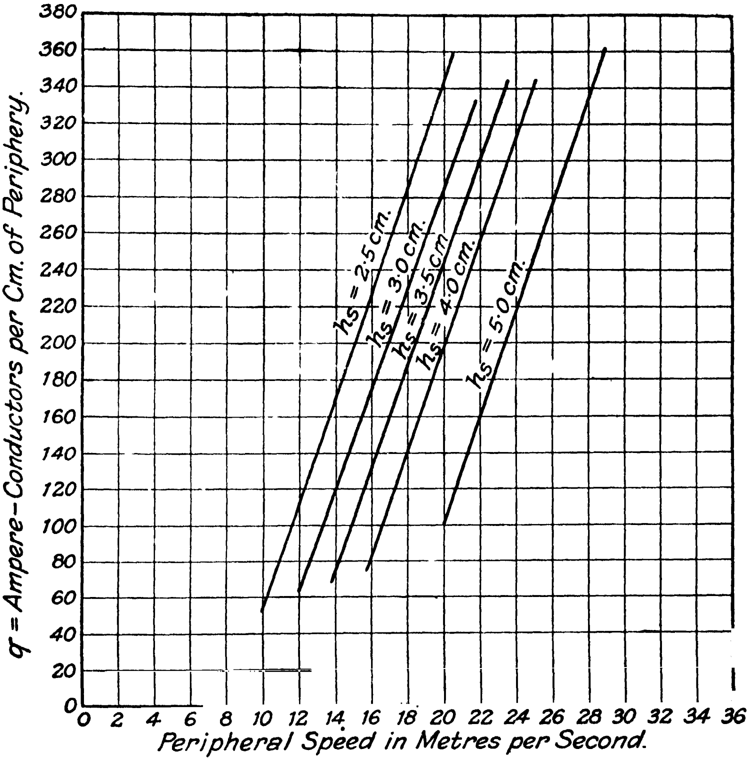


FIG. 19.3. RELATION BETWEEN q AND PERIPHERAL SPEED FOR DIFFERENT SLOT DEPTHS

THE RELATION BETWEEN *D* AND *L* FOR BEST POWER FACTOR

We shall investigate first the relation which exists between D and L to give the best power factor. As is well known, the ratio

$$\frac{\text{magnetizing current per phase}}{\text{the ideal short-circuit current per phase}} = \sigma$$

which is called the dispersion coefficient, is fundamental in determining the power factor. The smaller this ratio, the higher the power factor. In calculating the magnetizing current per phase, we shall use the method due to Dr. Max Kloss. He has shown that the ratio of the flux density, in the air-gap, at a point one-third of the pole pitch from the zero value, to the average density is nearly constant, and, in calculating the magnetizing current, we shall take all densities at one-third of the pole pitch from the zero value.

For a three-phase machine

$$\frac{B_{60}}{B_{av}} = 1{\cdot}28 \quad . \quad . \quad . \quad . \quad (19.69)$$

For a two-phase machine

$$\frac{B_{60}}{B_{av}} = 1{\cdot}18 \quad . \quad . \quad . \quad . \quad (19.70)$$

For a single-phase machine

$$\frac{B_{60}}{B_{av}} = 1{\cdot}57 \quad . \quad . \quad . \quad . \quad (19.71)$$

The resultant m.m.f. for the machine, at 60° from zero value,

$= 2{\cdot}12\bar{I}q_1z_1$ *per pole* pair for a three-phase machine
$= 1{\cdot}25\bar{I}q_1z_1$ per pole pair for a two-phase machine
$= 0{\cdot}74\bar{I}q_1z_1$ per pole pair for a single-phase machine

where q_1 = slots per pole phase in the stator
z_1 = conductors in series per slot
$\bar{I}$ = current per conductor

The resultant m.m.f. at 60°, for a complete magnetic circuit,

$$= \frac{1}{0{\cdot}4\pi} \times B_{60} \times 2\delta$$

$$= 0{\cdot}796 \times B_{60} \times 2\delta \quad . \quad . \quad . \quad (19.72)$$

where B_{60} = flux density in the air-gap, 60 electrical degrees from the zero value

δ = equivalent air-gap length in centimetres. This includes increase due to slot openings, and also increase due to saturation

$$\delta = \delta_1 + \delta_t + \delta_c \quad . \quad . \quad . \quad (19.73)$$

where δ_1 = actual gap length, increased by Carter's factor
δ_t = equivalent increase in gap length, due to tooth saturation in stator and rotor
δ_c = increase in gap length, due to saturation of stator and rotor cores

The magnetizing current per phase in amperes for three-phase machine

$$= \bar{I}_\mu = \frac{B_{av} \times 1{\cdot}28 \times 2\delta}{0{\cdot}4\pi \times 2{\cdot}12 \times q_1 z_1} \quad . \quad . \quad (19{\cdot}74)$$

Also the ideal short-circuit current per phase

$$= \bar{I}_{sci} = \frac{\hat{\phi}}{0{\cdot}4\pi \times 2\sqrt{2} \times \lambda \times q_1 \times z_1}$$

$$= \frac{\pi D \times L \times B_{av}}{p \times 0{\cdot}4\pi \times 2\sqrt{2} \times \lambda \times q_1 \times z_1} \quad . \quad (19{\cdot}75)$$

therefore, for three-phase machines

$$\frac{\bar{I}_\mu}{\bar{I}_{sci}} = \sigma = \frac{B_{av} \times 1{\cdot}28 \times 2\delta \times p \times 0{\cdot}4\pi \times 2\sqrt{2} \times \lambda \times q_1 z_1}{0{\cdot}4\pi \times \pi D \times L \times B_{av} \times 2{\cdot}12 \times q_1 \times z_1}$$

$$. \quad . \quad (19{\cdot}76)$$

$$\therefore \qquad \sigma = \frac{1{\cdot}09 \times p \times \lambda \times \delta}{D \times L} \quad . \quad . \quad (19{\cdot}77)$$

Now $$\lambda = \lambda_S + \lambda_T + \lambda_Z \quad . \quad . \quad (19{\cdot}78)$$

$$\lambda_S = \left\{\frac{\lambda_1}{q_1} + \frac{\lambda_2}{q_2}\right\} \times L \quad . \quad . \quad . \quad (19{\cdot}79)$$

and $$\lambda_1 = \frac{h_s}{3w_s} \text{ and } \lambda_2 = \frac{h_r}{3w_r} \text{ approximately} \quad . \quad (19{\cdot}80)$$

where h_s = height of stator slot
w_s = width of stator slot
h_r = height of rotor slot
w_r = width of rotor slot

$$\lambda_T = 0{\cdot}73(0{\cdot}36\tau + A) \quad . \quad . \quad . \quad (19{\cdot}81)$$

where τ = pole pitch (*see* section on calculation of short-circuit current)
A = length of overhang—pole pitch

$$\lambda_Z = \frac{Lt\,T}{48 \times \delta \times q_1 q_2}$$

$$\therefore \ \lambda = \left[\frac{h_s}{3w_s q_1} + \frac{h_r}{3w_s q_2}\right] \times L + 0{\cdot}73(0{\cdot}36\tau + A) + \frac{Lt\,T}{48 \times \delta \times q_1 q_2}$$

$$. \quad . \quad (19{\cdot}82)$$

Our object is to obtain the best ratio of D to L, and not so much to estimate λ with great accuracy. We may, therefore, without much error, make the following assumptions—

(a) h_s and h_r will be assumed equal, and equals h,
(b) width of tooth equals width of slot,
(c) the overhang permeance will be replaced by $m\tau$,

$$3q_1w_s \text{ and } 3w_rq_2 = \text{slot space per pole} = \frac{\pi D}{2p}$$

Also if τ_1 = stator slot pitch
τ_2 = rotor slot pitch

$$q_1 = \frac{\pi D}{3\tau_1 \times p} \text{ and } q_2 = \frac{\pi D}{3\tau_2 \times p} \quad . \quad . \quad (19.83)$$

$$\therefore \quad \lambda = \frac{4hpL}{\pi D} + \frac{m\pi D}{p} + \frac{q\varepsilon\tau_1\tau_2 \times p \times L}{48\pi D\delta} \quad . \quad (19.84)$$

$$\therefore \; \sigma = \frac{1\cdot09p \times \delta}{D^2L}\left[\left(\frac{4hp}{\pi} + \frac{q\varepsilon\tau_1\tau_2 \times p}{48 \times \pi \times \delta}\right)L + \frac{m\pi D^2}{p}\right] \quad . \quad (19.85)$$

Now D^2L is a constant for any given output and speed $= \beta$, say, and the expression in brackets

$$= aL + bD^2 \quad . \quad . \quad . \quad . \quad (19.86)$$

$$= y, \text{ suppose}$$

$$y = \frac{a\beta}{D^2} + bD^2 \quad . \quad . \quad . \quad (19.87)$$

$$\frac{1}{y} = \frac{D^2}{a\beta + bD^4} \quad . \quad . \quad . \quad (19.88)$$

$$\frac{d\left(\frac{1}{y}\right)}{d(D)} = \frac{2D(a\beta + bD^4) - 4bD^5}{(a\beta + bD^4)^2} \quad . \quad . \quad (19.89)$$

for a maximum

$$a\beta + bD^4 = 2bD^4 \quad . \quad . \quad . \quad (19.90)$$

i.e. $$a\beta = bD^4$$

i.e. $$\frac{a\beta}{D^2} = bD^2 \quad . \quad . \quad . \quad . \quad (19.91)$$

i.e. $$aL = bD^2 \quad . \quad . \quad . \quad . \quad (19.92)$$

i.e. y is a minimum, when $\frac{1}{y}$ is a maximum.

We arrive at the conclusion that minimum y of σ occurs, when the *slot leakage flux equals the end connection leakage flux*, for the two expressions represent the slot permeance and overhang permeance respectively. This then gives us the best ratio of D to L for best power factor.

$$\therefore \quad \frac{m\pi D^2}{p} = \left[\frac{4hp}{\pi} + \frac{q\varepsilon\tau_1\tau_2 \times p}{48 \times \pi \times \delta}\right] \times L \quad . \quad . \quad (19.93)$$

$$\therefore \qquad \frac{D^2}{L} = \frac{p^2}{m\pi^2}\left[4h + \frac{q\varepsilon\tau_1\tau_2}{48\delta}\right] \qquad . \qquad . \qquad . \quad (19{\cdot}94)$$

$$\frac{1}{L} = \frac{p^2}{m\pi^2 D^2}\left[4h + \frac{q\varepsilon\tau_1\tau_2}{48\delta}\right] . \qquad . \qquad . \quad (19{\cdot}95)$$

$$= \frac{1}{m\tau^2}\left[4h + \frac{q\varepsilon\tau_1\tau_2}{48\delta}\right] \qquad . \qquad . \qquad . \quad (19{\cdot}96)$$

$$\frac{\tau}{L} = \frac{1}{\tau}\left[\frac{4h}{m} + \frac{q\varepsilon\tau_1\tau_2}{48 \times \delta \times m}\right] \qquad . \qquad . \quad (19{\cdot}97)$$

Taking average values for the various quantities, viz. $h = 2{\cdot}5$ cm, $m = 1{\cdot}5$, $\tau_1 = \tau_2 = 1{\cdot}5$ cm, and $\delta = 0{\cdot}035$ cm (minimum value).

$$\frac{\tau}{L} = \frac{1}{\tau}\left[\frac{10}{1{\cdot}5} + \frac{9 \times 1{\cdot}6 \times 1{\cdot}5 \times 1{\cdot}5}{48 \times 0{\cdot}035 \times 1{\cdot}5}\right]$$

$$= \frac{19{\cdot}47}{\tau} \qquad . \qquad . \qquad . \qquad . \qquad . \qquad . \qquad . \quad (19{\cdot}98)$$

This relation will hold for small machines.

For large machines, $h_s = 4{\cdot}5$ and $\delta = 0{\cdot}09$ cm.

$$\frac{\tau}{L} = \frac{1}{\tau}\left[\frac{18}{1{\cdot}5} + \frac{9 \times 1{\cdot}6 \times 1{\cdot}5 \times 1{\cdot}5}{48 \times 0{\cdot}09 \times 1{\cdot}5}\right]$$

$$= \frac{17}{\tau} \qquad . \qquad . \qquad . \qquad . \qquad . \qquad . \qquad . \quad (19{\cdot}99)$$

We shall not be far from the truth, if we take an average value of $\frac{\tau}{L} = \frac{18}{\tau}$ for three-phase machines. This relation is plotted in Fig. 19.4.

It will be observed that the relation between $\frac{\text{pole pitch}}{\text{core length}}$ and pole pitch is a rectangular hyperbola. It shows clearly that the larger the pole pitch, the smaller the ratio of D to L becomes, and the smaller the pole pitch, the larger the ratio of D to L.

Since for three-phase machines, $\frac{\tau}{L} = \frac{18}{\tau}$

$$\tau^2 = 18L \ . \qquad . \qquad . \qquad . \quad (19.100)$$

i.e.

$$\frac{\pi^2 D^2}{p^2 L} = 18 \ . \qquad . \qquad . \qquad . \quad (19.101)$$

or

$$D^2 = \frac{18p^2 L}{\pi^2} \qquad . \qquad . \qquad . \quad (19.102)$$

$$\therefore \qquad D = 1{\cdot}35p\sqrt{L} \qquad . \qquad . \qquad . \quad (19.103)$$

This determination of the ratio of $\frac{D}{L}$ for best power factor was first given by the author, in the first edition of this book, in 1924, and

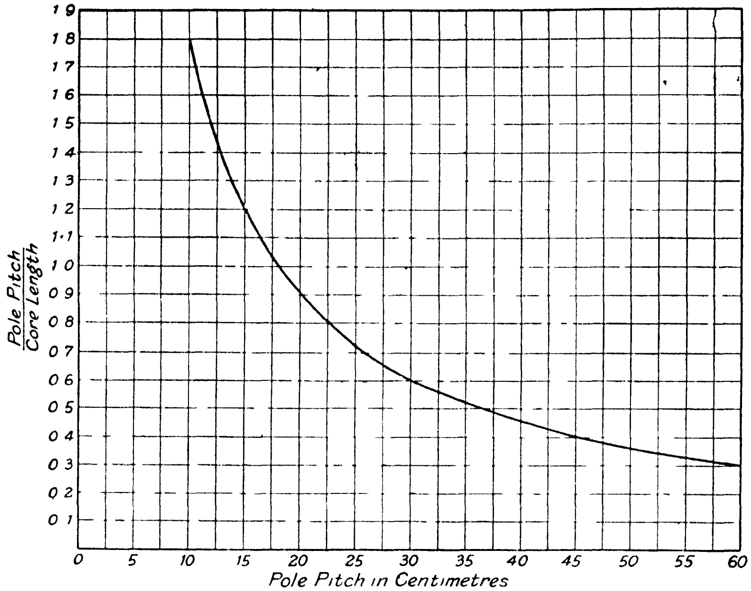


Fig. 19.4 Relation of (Pole Pitch)/(Core Length) and Pole Pitch for Best Power Factor $\tau/L = 18/\tau$

it was, to the best knowledge of the author, the first attempt to determine this ratio. It does not follow, of course, that one must adhere to this ratio, for there are other factors which may decide, such as

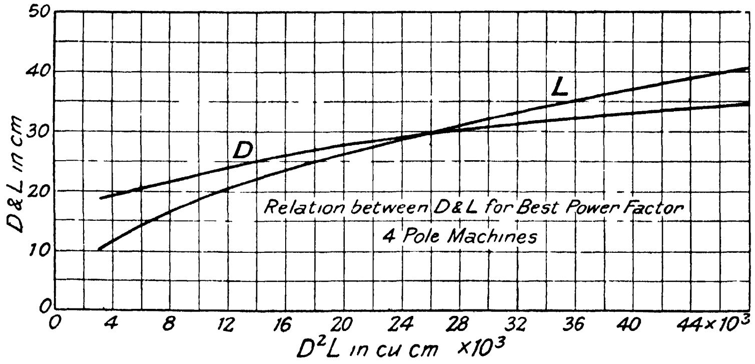


Fig. 19.5

the existence of frame sizes, with their necessary tools, or even considerations of ventilation, for, on high-speed machines, this rule may lead to lengths of core too long for cooling purposes, and it is possible to alter this ratio on high-speed machines quite a lot, without suffering any great loss of power factor, but on slow-speed machines, the use of the rule above, should save much time. This will be appreciated, when one is recommended—even by standard works—to work out several designs and choose the one with best performance.

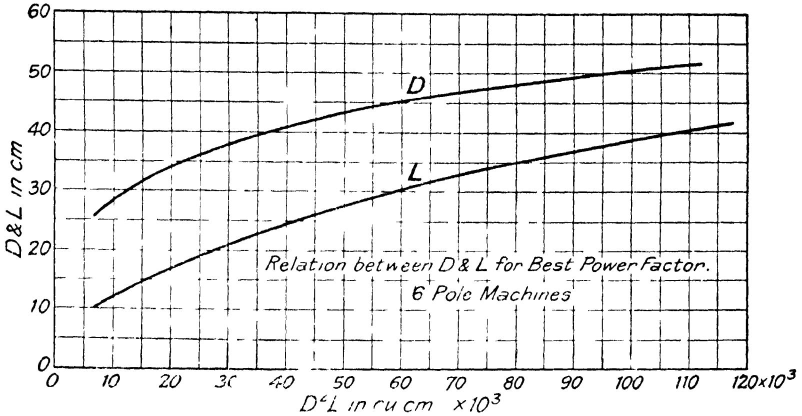


FIG. 19.6

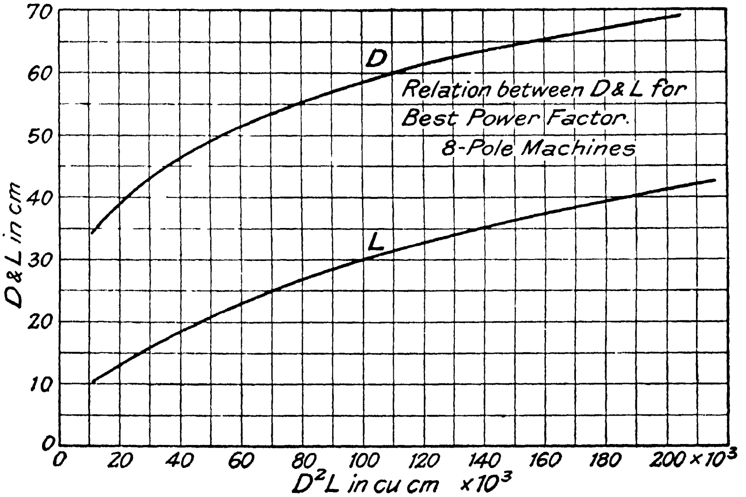


FIG. 19.7

The curves, given in Figs. 19.5 to 19.9, inclusive, show this relation for various numbers of poles.

It is now our purpose to determine the values of the dispersion coefficient σ for different values of D and L, and for different numbers of poles, and, from this value, we shall deduce the power factor which can be expected at full load. In order to do this with the greatest possible accuracy, we require to determine the relation between the diameter and length of core and length of air-gap. As is well known, the shorter the air-gap the better. This is determined by the safe mechanical clearance.

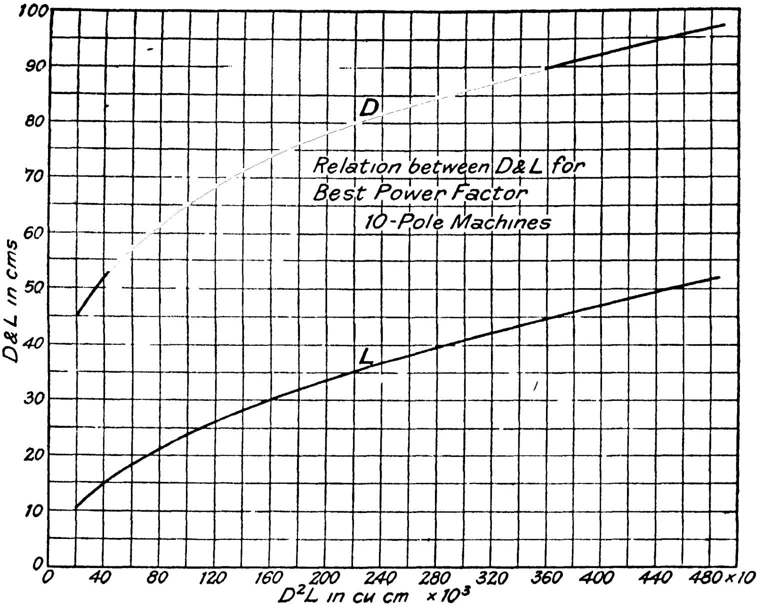


FIG. 19.8

Various values have been given for the air-gap length, among which are the following—

(*a*) (Kapp)
$$\underset{\text{cm}}{\delta} = 0{\cdot}02 + \frac{D_2}{1000}\ \text{cm} \quad . \quad . \quad . \quad (19.104)$$

(*b*)
$$\underset{\text{cm}}{\delta} = \frac{240D_2 + 1{\cdot}5D_2^2}{\left(6 - \dfrac{l}{\tau}\right)\dfrac{p}{2}} \times 10^{-4}\ \text{cm} \quad . \quad . \quad (19.105)$$

where D_2 = rotor diameter in centimetres
l = core length in centimetres
τ = pole pitch
p = number of poles

(*c*) (Fischer-Hinnen)
$$\underset{\text{cm}}{\delta} = \frac{D_2}{1000}\left(\frac{10}{p/2} + 0{\cdot}5\right)\ \text{cm} \quad . \quad . \quad (19.106)$$

(*d*) $$\underset{\text{in.}}{\delta} = 0{\cdot}005 + 0{\cdot}00035D + 0{\cdot}001L_g + 0{\cdot}003V \quad (19.107)$$

where δ = gap length in (*d*) in inches
D = diameter of rotor in inches
L_g = core length in inches
V = peripheral velocity of rotor in thousands of feet per minute

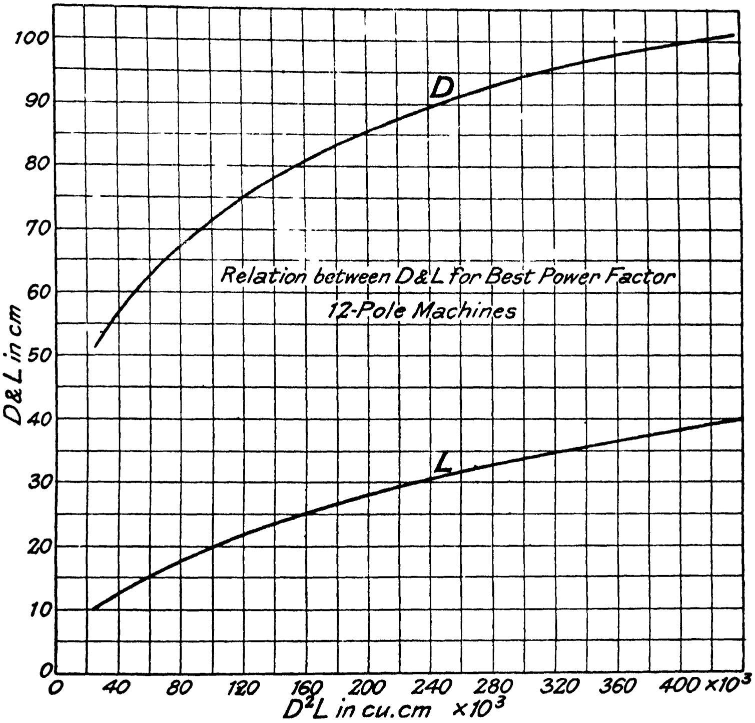


FIG. 19.9

In (*a*), (*b*), and (*c*) all dimensions are in centimetres. In (*d*) the dimensions are in inches.

The following are the mechanical clearances for four-pole machines—

D cm	δ cm	D cm	δ cm
15	0·035	45	0·13
21	0·050	55	0·18
25	0·060	65	0·25
30	0·070	80	0·40

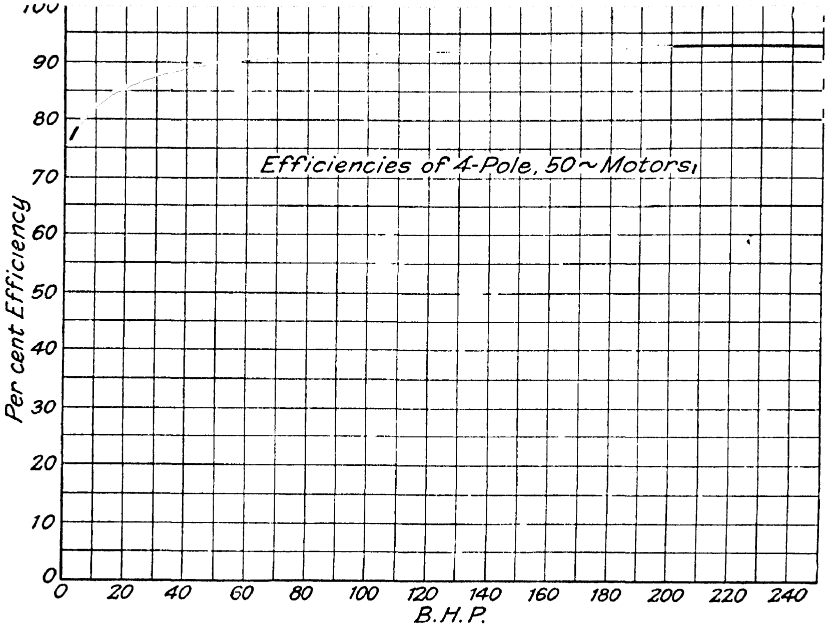


FIG. 19.10

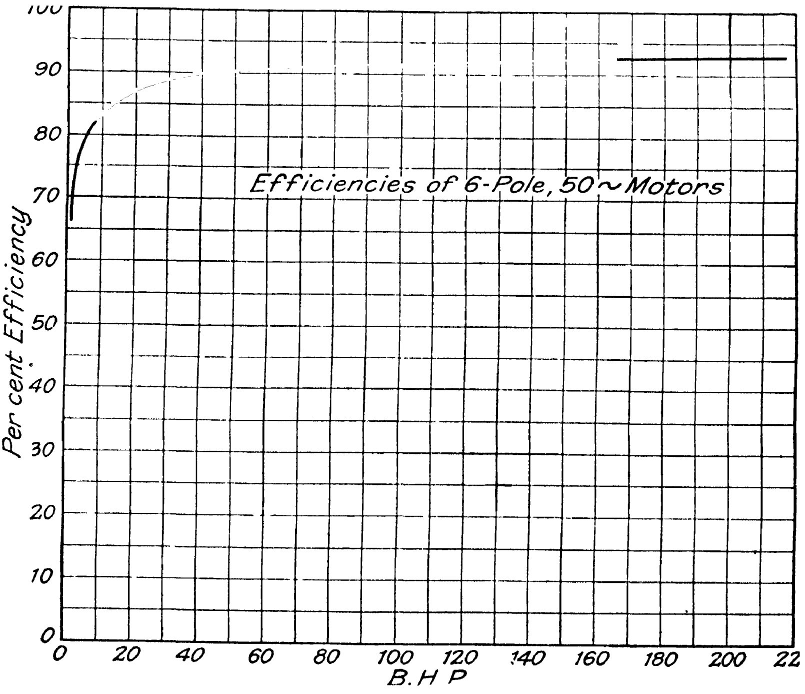


FIG. 19.11

The values of σ are calculated from the following equation—

$$\sigma = \frac{1{\cdot}09 \times p \times \delta}{D^2L}\left[\left\{\frac{4hp}{\pi} + \frac{q\varepsilon\tau_1\tau_2 p}{\delta \times 48 \times \pi}\right\}L + \frac{m\pi D^2}{p}\right] \quad (19.108)$$

For minimum value of σ

$$aL = bD^2 \quad . \quad . \quad . \quad (19.109)$$

$$\sigma = \frac{1{\cdot}09 \times p \times \delta}{D^2L} \times \frac{2m\pi D^2}{p} \quad . \quad . \quad . \quad (19.110)$$

$$\therefore \quad \sigma = \frac{1{\cdot}09 \times \delta \times 2m\pi}{L} \quad . \quad . \quad . \quad (19.111)$$

It would appear, therefore, whatever the number of poles, if the ratio of D to L were fixed to give minimum values of σ, that value would vary between the limits of

$$\sigma = \frac{1{\cdot}09 \times 0{\cdot}035 \times 2 \times 1{\cdot}5 \times \pi}{L} = \frac{1{\cdot}09 \times 0{\cdot}035 \times 3 \times \pi}{10} = 0{\cdot}036$$

taking 10 cm as minimum core length and

$$\sigma = \frac{1{\cdot}09 \times 0{\cdot}4 \times 3 \times \pi}{80} = 0{\cdot}041$$

taking extreme limits for δ of 0·035 cm and 0·4 cm and limits for L of 10 cm to 80 cm.

If such conditions could be realized in a standard line of machines, we should have power factors at full load of over 90 per cent, and the much-debated question of power factor correction would be dismissed forever. Unfortunately, such conditions cannot be realized. One has to build, on one frame, all numbers of poles up to the limit of that frame. It is true that, even with one frame, the internal diameter of the stator bore is increased as the number of poles is increased, but the increase can only be slight. If one chooses, for a given frame, the most suitable ratio of D to L for six poles or eight poles, one gets much poorer power factor on the other numbers of poles. Now a line of machines is invariably standardized for numbers of poles up to twelve, and for outputs up to about 200 h.p. Beyond this range, one is fairly free to choose a ratio of D to L which is best from power factor considerations, always paying regard to the question of cost.

The efficiencies, which may be expected on a range of induction motors, are given in Figs. 19.10 to 19.14 inclusive.

We shall now develop an expression for the power factor of an induction motor at any load.

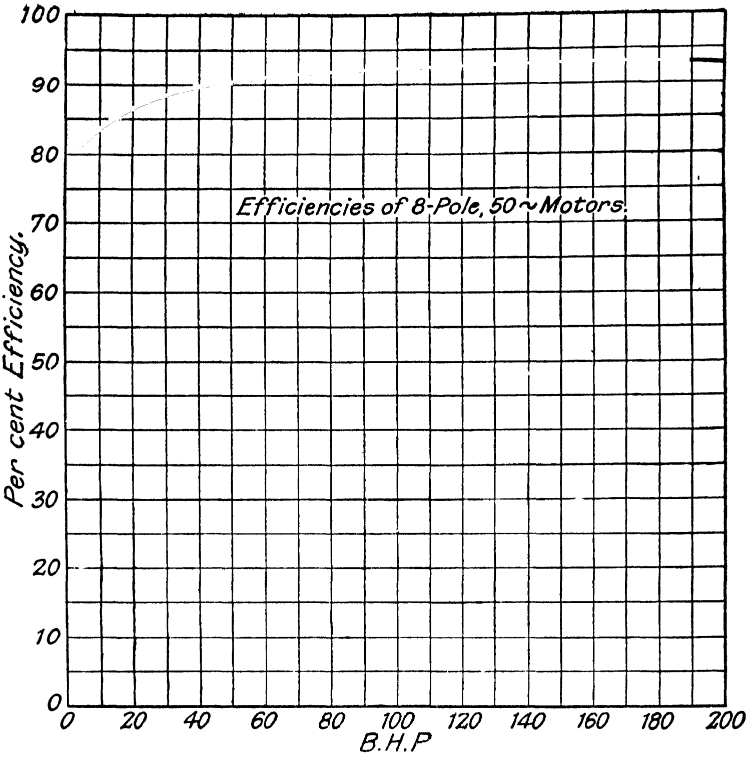


FIG. 19.12

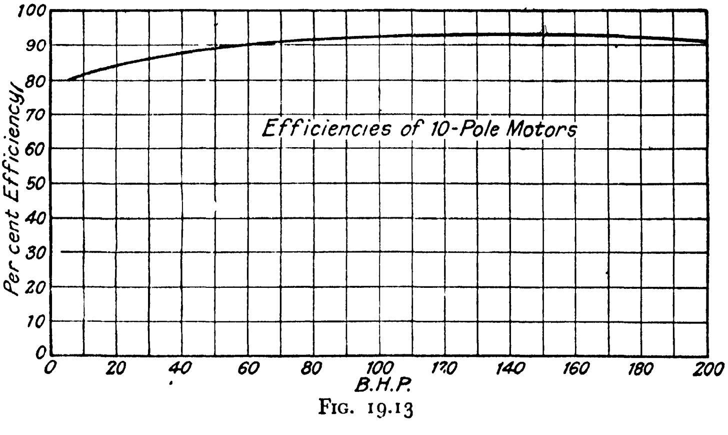


FIG. 19.13

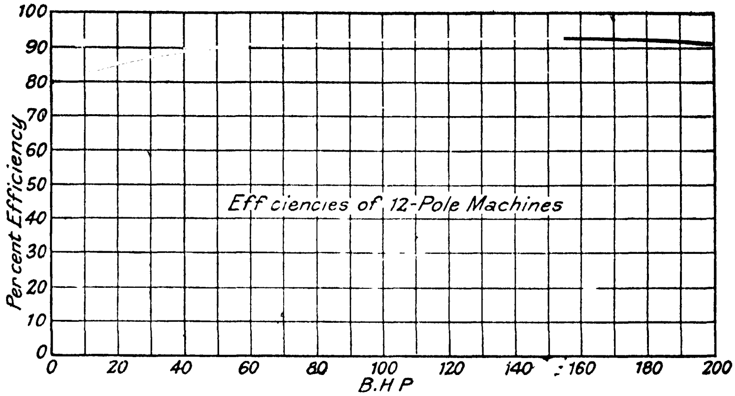


FIG. 19.14

The line KC represents the output line.

The maximum h.p.

$$= \frac{m_1 V \times EF}{746} \qquad (19.112)$$

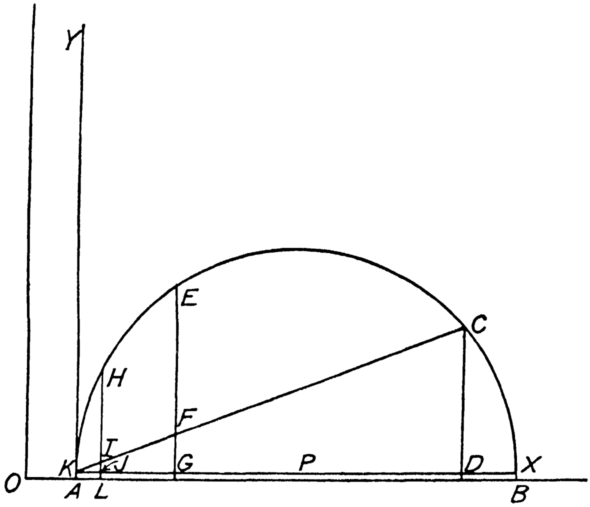


FIG. 19.15

where m_1 = number of stator phases

V = applied volts per phase

$$EF = \frac{KC}{2(1 + \cos C\hat{K}Y)} = \frac{KC}{2(1 + \sin C\hat{K}X)} \qquad (19.113)$$

Take the origin at K and the axis of x along OX.

Our purpose is to determine the power factor at any given load, which is a definite fraction of the maximum load.

The primary current is OH, and the power factor

$$= \frac{HL}{OH} = \frac{HL}{\sqrt{HL^2 + OL^2}}$$

The x co-ordinate of G is determined thus

$$(x - r)^2 + y^2 = r^2 \quad . \quad . \quad . \quad (19.114)$$

i.e.
$$y^2 = 2rx - x^2 \quad . \quad . \quad . \quad (19.115)$$

i.e.
$$y = \pm \sqrt{2rx - x^2} \quad . \quad . \quad . \quad (19.116)$$

where r = radius of the circle

$$FG = y_1 = m_1'x$$

where m_1 = tangent $C\hat{K}D$

$$\therefore \quad EF = y - y_1 = \sqrt{2rx - x^2} - m_1'x \quad . \quad (19.117)$$

This is a maximum when $\frac{d}{dx}(y - y_1) = 0$.

$$\frac{d}{dx}(y - y_1) = \frac{r - x}{\sqrt{2rx - x^2}} - m_1' \quad . \quad . \quad (19.118)$$

Therefore, for a maximum

$$r - x = m_1'\sqrt{2rx - x^2} \quad . \quad . \quad . \quad (19.119)$$

$$\therefore \quad r^2 - 2rx + x^2 = (m_1')^2 2rx - (m_1')^2x^2 \quad . \quad (19.120)$$

i.e.
$$x^2(1 + (m_1')^2) - x(2r(m_1')^2 + 2r) + r^2 = 0 \quad . \quad (19.121)$$

$$\therefore \quad x = \frac{2r(m_1')^2 + 2r \pm \sqrt{(2r(m_1')^2 + 2r)^2 - 4(1 + (m_1')^2)r^2}}{2(1 + (m_1')^2)r^2)}$$

$$= r\left[1 \pm \frac{m_1'}{\sqrt{1 + (m_1')^2}}\right] \quad . \quad . \quad . \quad . \quad . \quad (19.122)$$

Also, since the angle in a semicircle is a right angle, we have

$$EG^2 = KG \times GX \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.123)$$

$$= x(2r - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.124)$$

$$= r\left(1 - \frac{m_1'}{\sqrt{1 + (m_1')^2}}\right)\left(2r - r + \frac{rm_1'}{\sqrt{1 + (m_1')^2}}\right) \quad (19.125)$$

$$EG^2 = \frac{r^2}{1 + (m_1')^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.126)$$

It will simplify our calculations, if we put in the values of m_1', with which we have to deal.

For small motors, $m_1 \triangleq \tan 30^\circ = 0{\cdot}5774$. For large motors, $m_1 \triangleq \tan 14^\circ\ 30' = 0{\cdot}2586$.

Therefore, for small motors

$$EG = \frac{r_1}{\sqrt{1 + (m_1')^2}} = \frac{r}{\sqrt{1 + 0{\cdot}334}} = \frac{r}{1{\cdot}152}$$

$$= 0{\cdot}866r \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.127)$$

and for large machines

$$EG = 0{\cdot}967r$$

Also

$$EF = \frac{KC}{2(1 + \sin C\hat{K}D)} = \frac{2r \cos C\hat{K}D}{2(1 + \sin C\hat{K}D)} \quad . \quad (19{\cdot}128)$$

$$\cos C\hat{K}D = \frac{1}{\sqrt{1 + (m_1')^2}} \text{ and } \sin C\hat{K}D = \frac{m_1'}{\sqrt{1 + (m_1')^2}} \quad . \quad (19{\cdot}129)$$

$$\therefore \qquad EF = \frac{r}{m_1' + \sqrt{1 + (m_1')^2}} \quad . \quad . \quad . \quad (19{\cdot}130)$$

For small machines

$$EF = 0{\cdot}58r_1 \quad . \quad . \quad . \quad (19{\cdot}131)$$

and for large machines

$$EF = 0{\cdot}775r \quad . \quad . \quad . \quad (19{\cdot}132)$$

For small machines

$$FG = EG - EF = 0{\cdot}866r - 0{\cdot}58r \quad . \quad . \quad (19{\cdot}133)$$
$$\simeq 0{\cdot}286r$$

and for large machines

$$FG \simeq 0{\cdot}192r \quad . \quad . \quad . \quad (19{\cdot}134)$$

Let $HI = K \times EF$, and let us require the power factor, at a load which is K times the maximum load. In practice we shall require the values of $\cos \phi$ chiefly at full load, three-quarters, half, and quarter of full load.

If $\dfrac{\text{full load}}{\text{maximum load}} = \frac{1}{2}$

then at full load $K = 0{\cdot}500$
at three-quarters full load $K = 0{\cdot}375$
at half full load $K = 0{\cdot}250$
at quarter full load $K = 0{\cdot}125$

Now

$$HJ = HI + IJ$$
$$= KEF + FG \times \frac{KJ}{KG} \quad . \quad . \quad (19{\cdot}135)$$
$$= KEF + FG \frac{x_1}{x} \quad . \quad . \quad . \quad (19{\cdot}136)$$

Also

$$HJ^2 = KJ \times JX \quad . \quad . \quad . \quad (19{\cdot}137)$$
$$= x_1(2r - x_1) \quad . \quad . \quad . \quad (19{\cdot}138)$$

For small machines

$$HJ = K \times 0{\cdot}58r + 0{\cdot}286r \times \frac{x_1}{r\left(1 - \dfrac{m_1'}{\sqrt{1 + (m_1')^2}}\right)} \quad (19{\cdot}139)$$

$$= 0{\cdot}58Kr + \frac{0{\cdot}286x_1}{1 - \dfrac{0{\cdot}5774}{1{\cdot}152}} \quad . \quad . \quad . \quad . \quad . \quad (19{\cdot}140)$$

$$= 0{\cdot}58Kr + 0{\cdot}572x_1 \quad . \quad . \quad . \quad . \quad . \quad (19{\cdot}141)$$

$$\therefore \quad 0{\cdot}336K^2r^2 + 0{\cdot}663Krx_1 + 0{\cdot}327x_1^2 = 2rx_1 - x_1^2 \quad . \quad (19{\cdot}142)$$

$$\therefore \quad x_1 = r\left[2 - 0{\cdot}663K \pm \frac{\sqrt{4 - 2{\cdot}65K - 1{\cdot}35K^2}}{2{\cdot}654}\right] \quad . \quad (19{\cdot}143)$$

If we substitute in this expression the various values of K, viz. 0·5, 0·375, 0·25, 0·125, we have

K	x_1	$HJ = 0{\cdot}58Kr + 0{\cdot}572x_1$
0·500	$0{\cdot}054r$	$= 0{\cdot}3209r$
0·375	$0{\cdot}0272r$	$= 0{\cdot}233r$
0·250	$0{\cdot}0111r$	$= 0{\cdot}151r$
0·125	$0{\cdot}0035r$	$= 0{\cdot}074r$

The value of $\cos\phi$ at any load

$$= \frac{HJ + JL}{\sqrt{(HJ + JL)^2 + (OA + x_1)^2}} \quad . \quad . \quad (19{\cdot}144)$$

Let $HJ = \alpha r$, $x_1 = \beta r$, and $JL = \frac{1}{20}OA$.

$$\alpha = 0{\cdot}58K + 0{\cdot}572\left[\frac{2 - 0{\cdot}663K - \sqrt{4 - 2{\cdot}65K - 1{\cdot}35K^2}}{2{\cdot}654}\right] \quad (19{\cdot}145)$$

$$\beta = \left[\frac{2 - 0{\cdot}663K - \sqrt{4 - 2{\cdot}65K - 1{\cdot}35K^2}}{2{\cdot}654}\right] \quad . \quad . \quad . \quad (19{\cdot}146)$$

$\cos\phi$ at any load

$$= \frac{\alpha r + \frac{1}{20}OA}{\sqrt{(\alpha r + \frac{1}{20}OA)^2 + (OA + \beta r)^2}} \quad . \quad . \quad . \quad . \quad (19{\cdot}147)$$

$$= \frac{\alpha r + \frac{1}{20}OA}{\sqrt{r^2(\alpha^2 + \beta^2) + rOA\left(\frac{2\alpha}{20} + 2\beta\right) + OA^2\left(\frac{1}{400} + 1\right)}} \quad . \quad (19{\cdot}148)$$

but

$$\sigma = \frac{OA}{OA + 2r} = \frac{1}{1 + \frac{2r}{OA}} \quad . \quad . \quad (19{\cdot}149)$$

$$1 + \frac{2r}{OA} = \frac{1}{\sigma} \quad . \quad . \quad . \quad (19{\cdot}150)$$

$$\therefore \quad \frac{r}{OA} = \frac{1}{2}\left(\frac{1}{\sigma} - 1\right) \quad . \quad . \quad . \quad (19{\cdot}151)$$

$\cos\phi$ at any load

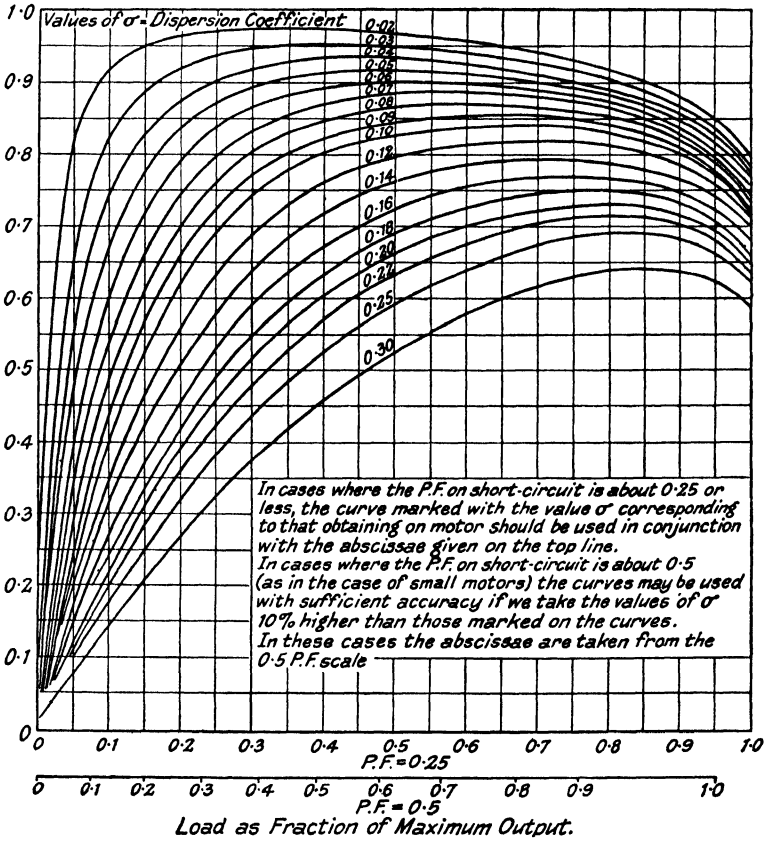
$$= \frac{\alpha\left\{\frac{1}{2}\left(\frac{1}{\sigma} - 1\right)\right\} + \frac{1}{20}}{\sqrt{(\alpha^2 + \beta^2)\left\{\frac{1}{2}\left(\frac{1}{\sigma} - 1\right)\right\}^2 + \frac{1}{2}\left(\frac{1}{\sigma} - 1\right)\left\{\frac{2\alpha}{20} + 2\beta\right\} + \frac{1}{400} + 1}}$$

$$. \quad . \quad (19{\cdot}152)$$

We will now tabulate the various values of the power factor at full, three-quarters, half, and quarter full load for various values of σ. This set applies to small motors, where the power factor at short-circuit = 0·5.

σ	cos φ		
	Full Load (%)	¾ Full Load (%)	½ Full Load (%)
0·03	94·00	93·0	90·0
0·04	93·00	91·5	86·5
0·06	89·00	86·0	78·0
0·07	86·50	83·0	73·5
0·08	84·25	79·5	69·0
0·09	81·75	76·5	64·5
0·10	80·00	73·5	60·5
0·12	75·50	69·0	54·0
0·14	71·00	63·0	49·5
0·16	66·00	58·0	43·5

The curves given in Fig. 19.16 are extremely useful in enabling one to predict, with fair accuracy, the power factor, at any given load, for various values of σ.



In cases where the P.F. on short-circuit is about 0·25 or less, the curve marked with the value σ corresponding to that obtaining on motor should be used in conjunction with the abscissae given on the top line.
In cases where the P.F. on short-circuit is about 0·5 (as in the case of small motors) the curves may be used with sufficient accuracy if we take the values of σ 10% higher than those marked on the curves.
In these cases the abscissae are taken from the 0·5 P.F. scale

FIG. 19.16

These curves show the rapid falling off of power factor at light load. They are plotted from equation (19.152). This falling off becomes serious for low-speed motors, i.e. machines with large values of σ, and they emphasize the importance of not choosing too large a machine for any particular job. Lightly loaded induction motors have much influence in lowering the power factor of the whole system.

THE MAGNETIZING CURRENT

The flux per pole can be found from the e.m.f. equation

$$\bar{E} = 2{\cdot}22 \times K_1 \times K_3 \times \hat{\phi} \times Z \times f \times 10^{-8}$$

$$\therefore \qquad \hat{\phi} = \frac{\bar{E} \times 10^8}{2{\cdot}22 \times K_1 \times K_3 \times Z \times f} \quad . \quad . \quad (19{\cdot}153)$$

$2{\cdot}22 \times K_1 = 2{\cdot}12$ for a three-phase machine

$2{\cdot}22 \times K_1 = 2{\cdot}02$ for a two-phase machine

$2{\cdot}22 \times K_1 = 1{\cdot}85$ for a single-phase machine with two-thirds of slots wound per pole

$$K_1 = \text{breadth factor} = \frac{\sin q\,\lambda/2}{q \sin \lambda/2}$$

and $$K_3 = \cos\frac{\varepsilon}{2}$$

for the fundamental.

The upper diagram of Fig. 19.17 shows the diagram of ampere-conductors over a double-pole pitch. Curve (*a*) shows the m.m.f. distribution when the current in phase I is at its maximum, and curve (*b*) shows the m.m.f. distribution, when the current in phase II is zero. The m.m.f. distribution changes from curve (*a*) to curve (*b*); six times during a cycle of current. Actually the m.m.f. curves are stepped curves, the m.m.f. rising, on passing a slot by the amount corresponding to the ampere-conductors per slot. Owing to saturation of the teeth and core, the flux curve becomes flattened, even if the m.m.f. curve is sinusoidally distributed. This is shown in Fig. 19.18.

The curves, in Fig. 19.18, are taken from a paper by Dr. S. P. Smith. They show the magnetization curve and also, below, the m.m.f. sinusoidal curves for different currents in the stator conductors. From these two curves, the curves of flux distribution are determined, as shown at the right of the magnetization curve.

The equations of the pointed curve in Fig. 19.17 is

$$y = \frac{18}{\pi^2} y'(\sin\theta + \tfrac{1}{25}\sin 5\theta - \tfrac{1}{49}\sin 7\theta \ldots)$$

where $y' = \frac{q_1}{\sqrt{2}} z_1 \times \bar{I}_{eff}$ in *ampere-turns* . . . . (19.154)

q_1 = slots per pole per phase in stator
z_1 = conductors in series per slot
$\bar{I}$ = effective or r.m.s. value of the current

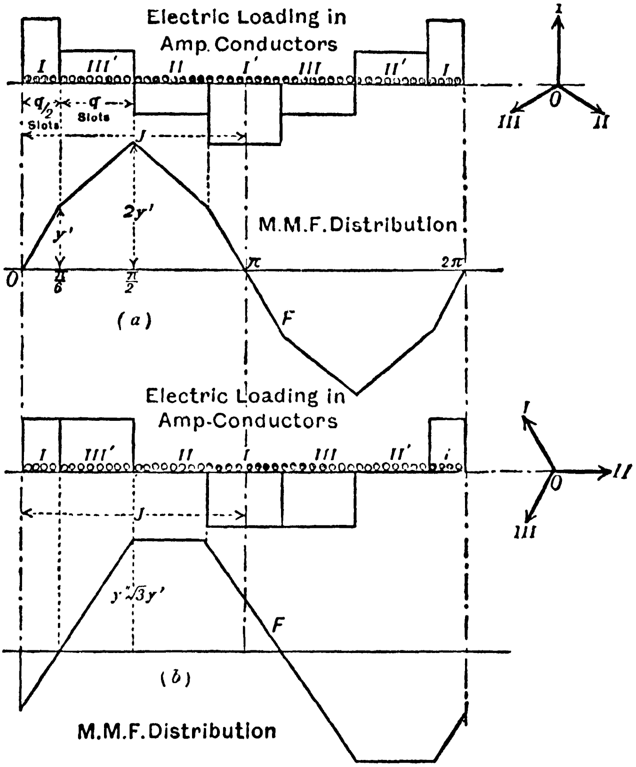


FIG. 19.17

The equation to the flat-topped curve, (*b*) in Fig. 19.17, is

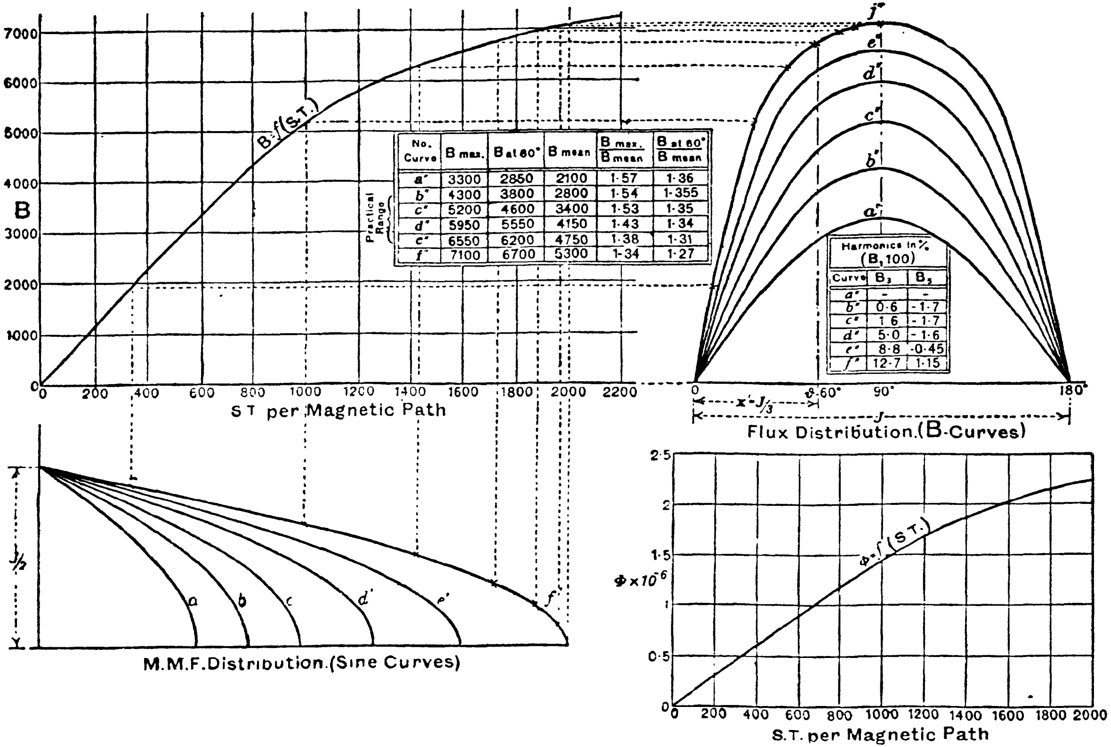
$$y = \frac{18}{\pi^2} y'(\sin\theta - \tfrac{1}{25}\sin 5\theta + \tfrac{1}{49}\sin 7\theta) \quad . \quad (19.155)$$

The fundamental has an amplitude per pole (in ampere-turns)

$$= \frac{18}{\pi^2} y' \text{ in each case}$$

$$= \frac{18}{\pi^2} \times \frac{1}{\sqrt{2}} q_1 z_1 \bar{I} \quad . \quad . \quad . \quad (19.156)$$

$$= 1{\cdot}29 q_1 z_1 \bar{I} \quad . \quad . \quad . \quad . \quad (19.157)$$



No. Curve	B max.	B at 60°	B mean	$\frac{B\ max.}{B\ mean}$	$\frac{B\ at\ 60°}{B\ mean}$
a″	3300	2850	2100	1·57	1·36
b″	4300	3800	2800	1·54	1·355
c″	5200	4600	3400	1·53	1·35
d″	5950	5550	4150	1·43	1·34
e″	6550	6200	4750	1·38	1·31
f″	7100	6700	5300	1·34	1·27

Harmonics in % (B, 100)

Curve	B_3	B_5
a″	-	-
b″	0·6	-1·7
c″	1·6	-1·7
d″	5·0	-1·6
e″	8·8	-0·45
f″	12·7	1·15

FIG. 19.18

All harmonics of order 3, and multiples of 3, vanish, and the amplitude of the largest is only 4 per cent of the fundamental. One may take the fundamental as representing the curve of mean m.m.f. distribution.

For a complete magnetic path the amplitude of the fundamental of the ampere-turns diagram $= 2{\cdot}58 q_1 z_1 I_{eff}$. For different values of the current, the sinusoidal ampere-turn diagrams are plotted over half a pole pitch in Fig. 19.18.

Due to the flattening of the flux curve, as saturation sets in, one can make a considerable error in the calculation of the magnetizing current, by assuming a sinusoidal distribution of flux. It has been customary to take a flux tube as the pole centre and calculate the ampere-turns required to drive the flux through the magnetic path. This method is allowable if no saturation is present, but it leads to very considerable error if saturation is present. Dr. Kloss, proceeding by graphical methods, plotted the distribution of m.m.f. over the pole pitch, and in order to be independent of scales he took, not the actual m.m.f. at any point on the periphery of the stator, but the value of the ordinate, expressed as a proportion of the mean ordinate of the curve. This ordinate he called y, and

$$y_1 = \frac{A}{A_{\text{mean}}}$$

Again he plotted the flux distribution over the pole pitch, and again the ordinate represented not the actual flux density, but the actual B expressed as a fraction of the mean density B_m,

i.e.
$$y_2 = \frac{B}{B_{\text{mean}}}$$

In the diagram, the mean value of the m.m.f. diagram and the mean value of the flux diagram coincide, then for each of their mean values, the ordinate is $x = y = 1$. Both curves then enclose the same area, but have different shapes. The law of the y_2 curve is not known. At the centre of the poles, the value of y_2 is less than the value of y_1, while in the two legs of the curve, the value of y_2 is greater than y_1, and the percentage deviation is greatest at the point of maximum value.

Therefore, the possible error in the calculation of the ampere-turns, by assuming a false value of y_2, is greatest at the pole centre. The most favourable flux tube is the one where the y_1 and y_2 curves intersect. Here y_1 and y_2 are both equal to y; y_1 is a known function of this position, and y_2 is given for this point. Dr. Kloss came to the conclusion, after setting out graphically the m.m.f. and flux density curves, that the flux tube which spans two-thirds of the pole pitch is the one in which y_1 and y_2 intersect. In other words, the ratio of $\frac{B_{60}}{B_{m\;n}}$ is more nearly constant than the ratio of the density at another point of the pole pitch to the mean density. So Dr. Kloss calculates the ampere-turns for a flux tube, which starts one-third of the pole pitch from the zero value.

The equation to the flattened flux curve is

$$y_2 = B_1 \sin \theta + B_3 \sin 3\theta + B_5 \sin 5\theta + \ldots \text{ etc.} \qquad (19.158)$$

$$B_{\text{mean}} = \frac{2}{\pi}\left(B_1 + \tfrac{1}{3}B_3 + \tfrac{1}{5}B_5 + \ldots\right) \qquad (19.159)$$

$$\frac{B_{60}}{B_{\text{mean}}} = \frac{\frac{\sqrt{3}}{2}\left(B_1 + 0 - B_5 + B_7 + \ldots\right)}{\frac{2}{\pi}\left(B_1 + \tfrac{1}{3}B_3 + \tfrac{1}{5}B_5 + \ldots\right)} \qquad (19.160)$$

For a pure sine wave

$$\frac{B_{60}}{B_{\text{mean}}} = \frac{B_{\max} \sin 60}{\frac{2}{\pi} \times B_{\max}} = 1{\cdot}36 = k_{60}\,. \qquad (19.161)$$

This ratio is more nearly constant at 60°, for the third harmonic vanishes there, i.e. $B_3 \sin 3\theta = B_3 \sin 180 = 0$.

From the large number of cases analysed by Dr. Kloss, the following values of $\frac{B_{60}}{B_{\text{mean}}}$ are derived—

For three-phase machines

$$\frac{B_{60}}{B_{\text{mean}}} = 1{\cdot}28 \qquad (19.162)$$

for two-phase machines

$$\frac{B_{60}}{B_{\text{mean}}} = 1{\cdot}18 \qquad (19.163)$$

for single-phase machines

$$\frac{B_{60}}{B_{\text{mean}}} = 1{\cdot}57 \qquad (19.164)$$

This method gives excellent results. The method is as follows—

(1) First find the flux per pole.

(2) Find the areas and lengths of the magnetic circuit, and take the length of path in the cores = two-thirds of pole pitch.

(3) Find the flux density in each part at 60 electrical degrees from zero point.

(4) Find the ampere-turns for each part at 60°.

(5) From the ampere-turns at 60°, calculate the magnetizing current.

The procedure is shown in tabular form below, for a three-phase machine—

Part	Area (cm²)	Length of Path (cm)	ϕ	B	$\bar{A}$ (per cm)	Total $\bar{A}$
Stator core	A_{sc}	$\frac{2}{3}l_{cs}$	$\frac{\phi}{2}$	$\frac{\phi}{2A_{sc}}$		
Stator teeth	$A_{t\,\mathrm{max}}$	$2l_t$	ϕ	$1{\cdot}28\frac{\phi}{A_{t\,\mathrm{max}}}$		
	$A_{t\,\mathrm{min}}$			$1{\cdot}28\frac{\phi}{A_{t\,\mathrm{min}}}$		
Air-gap	A_g	$2l_g$	ϕ	$1{\cdot}28\frac{\phi}{A_g}$		
Rotor teeth	$A_{r\,\mathrm{max}}$	l_{tr}	ϕ	$1{\cdot}28\frac{\phi}{A_{tr\,\mathrm{max}}}$		
	$A_{r\,\mathrm{min}}$			$1{\cdot}28\frac{\phi}{A_{tr\,\mathrm{min}}}$		
Rotor core	A_{rc}	$\frac{2}{3}l_{cr}$	$\frac{\phi}{2}$	$\frac{\phi}{2A_r}$		

The magnetizing current per phase

$$= \bar{I}_\mu = \frac{\text{ampere-turns at } 60^\circ}{2{\cdot}12 q_1 z_1} \tag{19.165}$$

where the ampere-turns at 60° are estimated for a complete magnetic circuit.

$\bar{I}_\mu$ for two-phase machines

$$= \frac{\text{ampere-turns at } 60^\circ \text{ per pole pair}}{1{\cdot}25 q_1 z_1} \tag{19.166}$$

$\bar{I}_\mu$ for single-phase machines

$$= \frac{\text{ampere-turns at } 60^\circ \text{ per pole pair}}{0{\cdot}74 q_1 z_1} \tag{19.167}$$

No-load Losses

In order to determine the no-load current, it is necessary to know the value of the energy component of current to overcome the no-load losses.

These losses are—

(1) Bearing friction loss.
(2) Windage loss.

(3) Iron loss.
(4) Brush friction loss with slip-ring motors.

(1) *The bearing loss* in watts

$$= (\text{speed of journal in metres per second})^{1\ 5} \times d_{j\,\text{cm}} \times l_{j\,\text{cm}} \times 0{\cdot}19 \quad . \quad . \quad (19.168)$$

where d_j = diameter of journal
l_j = length of journal

(2) *The windage loss* in watts

$$= 0{\cdot}17 \times 10^{-3} \times \text{rotor barrel surface in cm}^2 \times (\text{peripheral speed in metres per second})^2 \quad . \quad (19.169)$$

(4) *The brush friction loss*

$= 9{\cdot}81 \times$ rubbing velocity in metres per second $\times$ total brush area in cm^2 $\times$ brush pressure in kg/cm^2 $\times\ \rho$ watts

ρ = coefficient of friction = 0·15 to 0·2 depending on grade of brush

Brush pressure = 0·25 kg/cm^2, approximately.

In calculating (3), the iron losses, Dr. Kloss shows that the maximum flux density in the teeth = 1·362 times the average density. In finding the iron loss, therefore, one calculates the maximum density in the teeth, viz. $1{\cdot}362 \times B_{\text{mean}}$, and uses the iron-loss curves already given.

In addition to the iron losses in teeth and core, due to variation of flux at supply frequency, we have flux pulsation losses in stator and rotor teeth, which are of much higher frequency. The frequency of the pulsations in the stator teeth = number of rotor slots × revolutions per second. The frequency of the pulsations in the rotor teeth = number of stator slots × revolutions per second.

It is necessary to estimate the amplitude of flux pulsation in the teeth of stator and rotor, and for this purpose it is necessary to determine the mean, maximum, and minimum reluctance under the teeth for all positions of the rotor with respect to the stator.

There are eddy-current losses also in the surface of the stator and rotor near the gap surface. These losses are included in the experimental iron-loss curves (Fig. 19.2), and also average pulsation losses are included in the same curves for *machines with semi-enclosed slots*. It is very necessary to make an estimate of the pulsation losses for *open* slots in the stator, for these are *not* included in the curves (Fig. 19.2). These losses are very greatly increased with *open* stator slots, and it is absolutely necessary to include them, where such are used. In the author's judgment, it is better to avoid the use of open stator slots, for the disadvantages connected with their use more than outweigh their advantages.

THE SHORT-CIRCUIT CURRENT

The ideal short-circuit current per phase

$$= \frac{\text{applied volts per phase}}{\text{equivalent reactance per phase}}$$

$$= \frac{E}{X_1 + X_2 \times \left(\frac{q_1 z_1}{q_2 z_2}\right)^2 \times \frac{m_1}{m_2}} \quad . \quad . \quad . \quad (19.170)$$

where X_1 = stator *leakage* reactance per phase in ohms
X_2 = rotor *leakage* reactance per phase in ohms
q_1 = number of stator slots per pole per phase
q_2 = number of rotor slots per pole per phase
z_1 = number of conductors in series per slot in the stator
z_2 = number of conductors in series per slot in the rotor
m_1 = number of the stator phases
m_2 = number of the rotor phases
E = applied voltage per phase
f_1 = winding factor of stator
f_2 = winding factor of rotor

It is necessary to calculate X_1 and X_2, which are the *Leakage Reactances* of the primary and secondary.

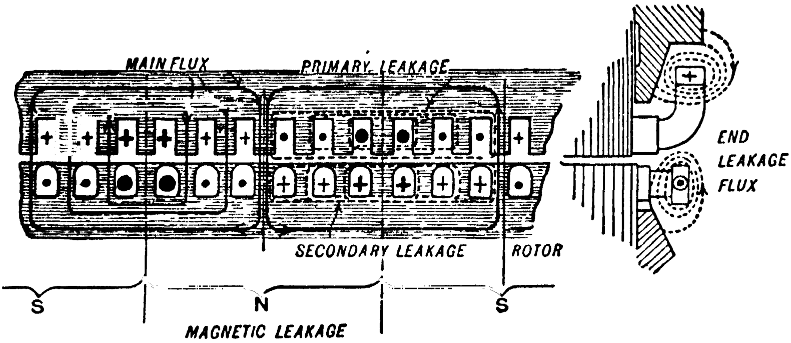


FIG. 19.19

The leakage fluxes for the primary and secondary are—

(1) Slot leakage flux.
(2) Tooth-tip leakage flux.
(3) End-connection leakage flux.
(4) Zigzag leakage flux.
(5) Belt leakage flux.

We will first calculate the inductance of a coil of T_c turns or z conductors per slot.

The inductance of the coil

$$= \frac{\Sigma \phi_x z}{i}$$

$$= \frac{0\cdot 4\pi z^2 \Sigma (l_x \lambda_x)}{10^8} \text{ henrys}$$

where λ_x = permeance per unit length of leakage path

l_x = length of part considered

The inductance of a coil in henrys

$$L = [0\cdot 4\pi z^2 2 l_c \lambda_S + 0\cdot 4\pi z^2 2 l_c \lambda_Z + 0\cdot 4\pi z^2 2 l_e \lambda_T] \times 10^{-8}$$

where λ_S = specific permeance per centimetre length of the slot

λ_Z = specific permeance of the zigzag path per centimetre of the slot

λ_T = specific permeance of the overhang per centimetre

With q_1 slots per pole per phase, there are $q_1 z_1$ conductors producing leakage flux associated with a pole-pair.

The leakage flux has now to cross q_1 slots, and, therefore, the effective permeance

$$= \frac{\lambda_S}{q_1} \text{ and } \frac{\lambda_Z}{q_1}$$

Hence the inductance of a group of coils of one phase

$$\begin{aligned} &= 0\cdot 4\pi \times (q_1 z_1)^2 2 l_c \times \frac{\lambda_S}{q_1} \times f_1^2 \times \frac{1}{10^8} \\ &+ 0\cdot 4\pi (q_1 z_1)^2 \times 2 l_c \times \frac{\lambda_Z}{q_1} \times f_1^2 \times \frac{1}{10^8} \\ &+ 0\cdot 4\pi (q_1 z_1)^2 \times 2 l_e \times \lambda_T \times f_1^2 \times \frac{1}{10^8} \quad . \quad (19\cdot 171) \end{aligned}$$

where l_c = core length

l_e = length of one end connection . . . (19·172)

Since there are $\frac{p}{2}$ groups of such coils per phase, and these are assumed to be connected in series, the inductance per phase of the stator with a *hemi-tropic* winding

$$L_s = 0\cdot 4\pi (q_1 z_1)^2 \times \frac{p}{2} \times f_1^2 \times 10^{-8} \left[2 l_c \frac{\lambda_S}{q_1} + 2 l_c \frac{\lambda_Z}{q_1} + 2 l_e \lambda_T \right] \quad . \quad (19\cdot 173)$$

$$= 0\cdot 4\pi q_1^2 z_1^2 p \times f_1^2 \times 10^{-8} \left[l_c \frac{\lambda_S}{q_1} + l_c \frac{\lambda_Z}{q_1} + l_e \lambda_T \right] \quad . \quad (19\cdot 174)$$

$$\therefore\ L_s = 0{\cdot}4\pi q_1^2 z_1^2 \times p \times f_1^2 \times 10^{-8}[\lambda_L' + \lambda_Z' + \lambda_T'] \quad . \quad (19.175)$$

$$\lambda_L' = l_c \frac{\lambda_S}{q_1};\ \lambda_Z' = l_c \frac{\lambda_Z}{q_1};\ \lambda_T' = l_e \lambda_T \quad . \quad (19.176)$$

In the same way the rotor inductance per phase in henrys

$$= L_r = 0{\cdot}4\pi q_2^2 z_2^2 p \times f_2^2 \times 10^{-8}[\lambda_L'' + \lambda_T''] \quad . \quad (19.177)$$

where $\lambda_L'' = l_c \dfrac{\lambda_r}{q_2}$

and $\lambda_T'' = l_e \lambda_T'$

(this assumes $f_1 = f_2$ = winding factor of the rotor)

and referring the rotor *leakage* inductance to the stator, we have

$$\underset{\text{(referred)}}{L_r} = 0{\cdot}4\pi q_1^2 z_1^2 p \times f_1^2 \times 10^{-8}[\lambda_L'' + \lambda_T''] \quad . \quad (19.178)$$

The ideal short-circuit current

$$= \frac{\hat{E}}{2\pi f(L_s + \underset{\text{(referred)}}{L_r})} \quad . \quad . \quad . \quad (19.179)$$

$$= \frac{E}{2\pi f L_e} \quad . \quad . \quad . \quad . \quad (19.180)$$

where L_e = equivalent inductance in henrys
$= L_s + L_r$ (referred to stator)

$$\therefore\ \underset{\text{(ideal)}}{I_{sc}} = \frac{\hat{E}}{2\pi f \times 0{\cdot}4\pi q_1^2 z_1^2 \times p \times f_1^2 \times 10^{-8}[\lambda_1 + \lambda_2 + \lambda_3]} \quad . \quad (19.181)$$

where

$$\lambda_1 = \lambda_L' + \lambda_L'' = \left(\frac{\lambda_S}{q_1} + \frac{\lambda_r}{q_2}\right) l_c \quad . \quad . \quad (19.182)$$

$$\lambda_2 = \lambda_Z' = \left(\frac{\lambda_Z}{q_1}\right) l_c \quad . \quad . \quad . \quad (19.183)$$

$$\lambda_3 = \lambda_T' + \lambda_T'' = l_e \lambda_T + l_e \lambda_T' \quad . \quad . \quad (19.184)$$

$$\hat{E} = \sqrt{2}\,\pi\hat{\phi} \times q_1 z_1 \frac{p}{2} \times \frac{1}{10^8} \times f \times f_1 \quad . \quad (19.185)$$

$$\therefore\ \underset{\text{(ideal)}}{I_{sc}} = \frac{\sqrt{2}\,\pi\hat{\phi} \times q_1 z_1 \times p \times f \times f_1 \times 10^8}{2\pi f \times 0{\cdot}4\pi q_1^2 z_1^2 \times 2 \times f_1^2 \times 10^8 \times p \times \lambda} \quad . \quad (19.186)$$

$$= \frac{\hat{\phi}}{0{\cdot}4\pi \times 2\sqrt{2}\, q_1 z_1 \times f_1 \times \lambda} \quad . \quad . \quad (19.187)$$

$$= \frac{\hat{\phi}}{3{\cdot}55 q_1 z_1 \times f_1 \times \lambda} \quad . \quad . \quad . \quad (19.188)$$

where

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 \quad . \quad . \quad . \quad (19.189)$$

That is, the ideal short-circuit current per phase

$$= \frac{\text{working flux per pole}}{\text{leakage flux per pole per ampere}} \quad . \quad (19.190)$$

It would be well for the student to look now at the equivalent circuit, and to observe that, on short-circuit, i.e. at standstill, virtually all the flux in the machine exists as leakage flux. We see, therefore, why the dispersion coefficient becomes larger as the number of poles increases, for the working flux per pole becomes smaller and the leakage flux per pole per ampere increases. We have a smaller polar area as the number of poles increases, and, therefore, a smaller part available for the conductors per phase, and, therefore, a greater concentration of conductors in a smaller space per pole. The magnetizing current increases, the short-circuit current decreases and the power factor falls rapidly.

We have assumed a single-layer hemi-tropic winding in our calculations. Should a divided coil winding be used, the end connection reactance will be halved, for we have only half the number of conductors, per coil group, producing leakage flux. The end connection reactance of each coil group, with the divided coil winding, is one-quarter of the end connection reactance of a coil group of the hemi-tropic winding, but there are twice as many coil groups. With a divided coil winding, therefore, we must divide the end-connection reactance of the hemi-tropic winding by two.

Our calculations, in the first instance, will relate to the full-pitch winding. We will deal later with fractional-pitch windings.

SPECIFIC PERMEANCE OF VARIOUS TYPES OF STATOR AND ROTOR SLOTS

Consider first a *single-layer* winding. A tube of force of height δx, distant x cm from the top of the stator slot has a permeance per centimetre length of slot

$$= \frac{dx \times 1}{\omega_1} \quad . \quad . \quad . \quad . \quad (19.191)$$

The m.m.f. acting on the strip

$$= 0{\cdot}4\pi \times \frac{x}{h_1} \times z_1 \times i \text{ ampere-turns} \quad . \quad (19.192)$$

where h_1 = height of slot

z_1 = conductors per slot

The flux in strip = m.m.f. × permeance of strip

$$= 0{\cdot}4\pi \times \frac{xz_1}{h_1} \times i \times \frac{dx}{\omega_1} \quad . \quad . \quad . \quad (19.193)$$

The line linkages per ampere per centimetre length of slot

$$= 0{\cdot}4\pi \frac{z_1^2}{\omega_1} \int_0^{h_1} \frac{x^2}{h_1^2} dx \quad . \quad . \quad . \quad (19{\cdot}194)$$

$$= 0{\cdot}4\pi \times z_1^2 \frac{h_1}{3\omega_1} \quad . \quad . \quad . \quad (19{\cdot}195)$$

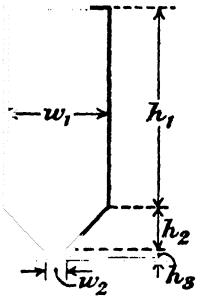


FIG 19 20

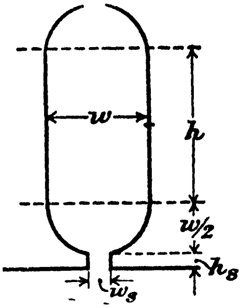


FIG. 19.21

The equivalent permeance for the straight part of slot

$$= \frac{h_1}{3\omega_1}$$

To this we must add the permeances

$$\frac{2h_2}{\omega_1 + \omega_2} \text{ and } \frac{h_3}{\omega_2}$$

For these portions of the slot, the full ampere-conductors of the slot act on them, so for a single-layer winding we have

$$\lambda_S = \frac{h_1}{3\omega_1} + \frac{2h_2}{\omega_1 + \omega_2} + \frac{h_3}{\omega_2} \quad . \quad . \quad (19{\cdot}196)$$

For a slot with semicircular ends, with the winding included in the portion h, the equivalent permeance for the straight part $= \frac{h}{3\omega}$ (*see* Fig. 19.21).

Permeance of Semicircular Bottom Portion

Permeance of the elementary strip per centimetre length (*see* Fig. 19.22)

$$= \frac{\delta x}{2R \cos \theta} \quad . \quad . \quad (19{\cdot}197)$$

where R = radius

$$x = R \sin \theta \quad . \quad . \quad (19{\cdot}198)$$

$$dx = R \cos \theta d\theta \quad . \quad . \quad (19{\cdot}199)$$

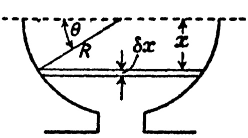


FIG. 19.22

permeance of strip $= \frac{R \cos \theta d\theta}{2R \cos \theta} = \frac{d\theta}{2}$ . . . (19.200)

Elementary flux in strip, per centimetre, per ampere

$$= 0{\cdot}4\pi z_1 \frac{d\theta}{2} \quad . \quad . \quad . \quad (19.201)$$

Line linkages per ampere, per centimetre of core, due to flux in strip

$$= 0{\cdot}4\pi z_1^2 \frac{d\theta}{2} \quad . \quad . \quad . \quad (19.202)$$

Total line linkages per ampere, per centimetre of core length, for circular part

$$= 0{\cdot}4\pi \frac{z_1^2}{2} \int_0^{\pi/2} d\theta = 0{\cdot}4\pi z_1^2 \times \frac{\pi}{4} \quad . \quad . \quad (19.203)$$

So that the equivalent permeance per centimetre

$$= \frac{\pi}{4} \text{ (lower circular part)}$$

$$\lambda_s = \frac{h}{3\omega} + \frac{\pi}{4} + \frac{h_s}{\omega_s} \quad . \quad . \quad . \quad (19.204)$$

Equivalent Permeance for Double-layer Windings. Full Pitch Windings

The slot permeance for a coil side in the upper part is different from that for the lower coil side. (*See* Fig. 19.23.)

The permeance for the top portion

$$= \frac{h_1'}{3\omega} + \frac{h_5 + h_1'}{\omega} + \frac{2h_2}{\omega + \omega_1} + \frac{h_3}{\omega_1} \quad . \quad (19.205)$$

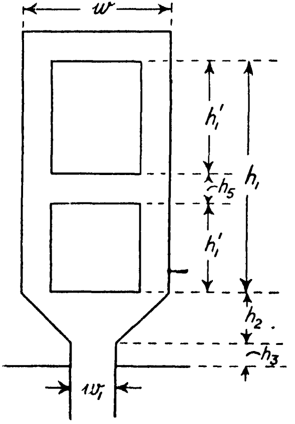


FIG. 19.23

The permeance for the lower layer

$$= \frac{h_1'}{3\omega} + \frac{2h_2}{\omega + \omega_1} + \frac{h_3}{\omega_1} \quad . \quad (19.206)$$

The presence of the two coil sides results in mutual inductance between them. The coefficient of mutual inductance is the number of line linkages with the top conductors, due to unit current in the bottom conductors. If there are t_c turns in the upper conductor, the m.m.f.

$$= 0{\cdot}4\pi i t_c$$

and the permeance of the flux strip, of

width δx and length 1 cm, in the part of the lower conductor, through which it passes

$$= \frac{1 \times \delta x}{\omega} \quad . \quad . \quad . \quad . \quad (19.207)$$

Leakage flux in strip

$$= \frac{0{\cdot}4\pi i t_c \times \delta x}{\omega} \quad . \quad . \quad . \quad (19.208)$$

Line linkages per ampere, due to mutual inductance, per centimetre

$$= \frac{0{\cdot}4\pi t_c^2}{\omega} \int_0^{h_1'} \frac{x}{h_1'}\,dx = \frac{0{\cdot}4\pi l_c^2 h_1'}{2\omega} \quad . \quad . \quad (19.209)$$

Therefore, equivalent permeance, due to mutual inductance, per centimetre

$$= \frac{h_1'}{2\omega} + \frac{2h_2}{\omega_1 + \omega} + \frac{h_3}{\omega_1} \quad . \quad . \quad (19.210)$$

Summing up, we have the equivalent permeances per centimetre length for the *double-layer* winding.

(*a*) Top layer

$$\lambda_{\text{top}} = \frac{h_1'}{3w} + \frac{h_5 + h_1'}{w} + \frac{2h_2}{w + w_1} + \frac{h_3}{w_1} \quad . \quad (19.211)$$

(*b*) Bottom layer

$$\lambda_{\text{bottom}} = \frac{h_1'}{3\omega} + \frac{2h_2}{\omega + \omega_1} + \frac{h_3}{\omega_1} \quad . \quad . \quad . \quad (19.212)$$

(*c*) Mutual inductive permeance

$$\lambda_{tb} = \frac{h_1'}{2\omega} + \frac{2h_2}{\omega + \omega_1} + \frac{h_3}{\omega_1} \quad . \quad . \quad . \quad (19.213)$$

For the *single*-layer winding the inductance per phase in henrys due to slot leakage only

$$= 0{\cdot}4\pi q_1^2 z_1^2 p \times l_c \times \lambda \times f_1^2 \times 10^{-8} \quad . \quad (19.214)$$

If $\quad N$ = number of turns per phase $= \dfrac{pq_1z_1}{2}$. (19.215)

$$L = 1{\cdot}6\pi \frac{N^2}{pq_1} \times l_c \times \lambda \times f_1^2 \times 10^{-8} \quad . \quad (19.216)$$

where p = poles

l_c = length of core in centimetres

q_1 = slots per pole per phase

$\lambda = \lambda_S + \lambda_Z$ . . . . (19.217)

For a full pitch, *double-layer winding*, we have for the coefficient of self-inductance for the top layer coil sides

$$L_{\text{top layer}} = 1{\cdot}6\pi \frac{\mathcal{N}^2}{4pq_1} l_c(\lambda_{\text{top}} + \lambda_{tb})f_1^2 \times 10^{-8} \quad . \quad (19.218)$$

$$L_{\text{bottom layer}} = 1{\cdot}6\pi \frac{\mathcal{N}^2}{4pq_1} l_c(\lambda_{\text{bottom}} + \lambda_{tb})f_1^2 \times 10^{-8} \quad (19.219)$$

Total coefficient of self inductance per phase, due to *slot leakage* only

$$= 1{\cdot}6\pi \frac{\mathcal{N}^2}{4pq_1} l_c(\lambda_{\text{top}} + \lambda_{\text{bottom}} + 2\lambda_{tb})f_1^2 \times 10^{-8} \text{henrys} \quad (19.220)$$

In Fig. 19.24 $\quad \omega = \omega_1 - (\omega_1 - \omega_2)\dfrac{x}{h_1}$ . . . (19.221)

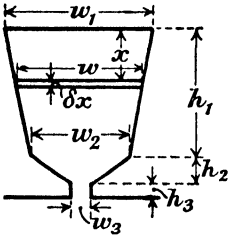


FIG. 19.24

The permeance of a strip of width ω and thickness δx and unit axial length $= \dfrac{\delta x}{\omega}$

If z_1 = conductors per slot. Conductors, which produce flux in the strip,

$$= \frac{\dfrac{\omega_1 + \omega}{2} \times x \times z_1}{\dfrac{\omega_1 + \omega_2}{2} \times h_1} \quad . \quad (19.222)$$

Flux in strip per centimetre per ampere

$$= 0{\cdot}4\pi \left[\frac{\omega_1 + \omega}{\omega_1 + \omega_2} \times \frac{x}{h_1} \times z_1\right] \times \frac{\delta x}{\omega} \quad . \quad (19.223)$$

Line-linkages per ampere per centimetre

$$= 0{\cdot}4\pi \left[\frac{\omega_1 + \omega}{\omega_1 + \omega_2} \times \frac{x}{h_1} \times z_1\right]^2 \times \frac{\delta x}{\omega} \quad . \quad (19.224)$$

Total line linkages per ampere per centimetre

$$= 0{\cdot}4\pi z_1^2 \int_0^{h_1} \left[\frac{\omega_1 + \omega}{\omega_1 + \omega_2} \times \frac{x}{h_1}\right]^2 \times \frac{\delta x}{\omega} \quad . \quad . \quad (19.225)$$

$$= \frac{0{\cdot}4\pi\ z_1^2}{(\omega_1 + \omega_2)^2 h_1^2} \int_0^{h_1} \left(\frac{\omega_1^2 x^2}{\omega} + 2\omega_1 x^2 + \omega x^2\right) dx \quad . \quad (19.226)$$

$$= \frac{0{\cdot}4\pi \times z_1^2}{(\omega_1 + \omega_2)^2 h_1^2} \int_0^{h_1} \left\{\frac{\omega_1^2 x^2}{\omega_1 - (\omega_1 - \omega_2)\dfrac{x}{h_1}} + 2\omega_1 x^2 + \omega_1 x^2 - (\omega_1 - \omega_2)\frac{x^2}{h_1}\right\} dx \quad . \quad (19.227)$$

Let
$$\frac{\omega_1 - \omega_2}{h_1} = a \quad . \quad . \quad . \quad (19.228)$$

Then equation (19.227) becomes

$$\frac{0{\cdot}4\pi z_1^{\,2}}{(\omega_1 + \omega_2)^2 h_1^{\,2}} \int_0^{h_1} \left(\frac{\omega_1^{\,2} x^2}{\omega_1 - ax} + 3\omega_1 x^2 - ax^2 \right) dx \ . \quad (19.229)$$

The equivalent permeance per centimetre length of slot for part h_1

$$= \frac{h_1}{(\omega_1 - \omega_2)^2} \left[\frac{-\omega_1^{\,2}}{2(\omega_1 - \omega_2)} - \frac{\omega_1^{\,3}}{(\omega_1 - \omega_2)^2} + \frac{\omega_1^{\,4}}{(\omega_1 - \omega_2)^3} \log \frac{\omega_1}{\omega_2} + \omega_1 + \frac{\omega_1 - \omega_2}{4} \right] \ . \quad (19.230)$$

To this must be added

$$\frac{2h_2}{\omega_2 + \omega_3} + \frac{h_3}{\omega_3}$$

Rotor Circular Slot for Squirrel-cage Machines

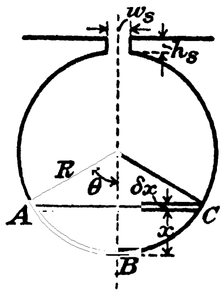


FIG. 19.25

Area $ABC = R^2\theta - R^2 \sin \theta \cos \theta \quad . \quad (19.231)$

Permeance of elementary flux strip per centimetre length of slot

$$= \frac{\delta x}{2R \sin \theta} \quad . \quad . \quad (19.232)$$

Now
$$x = R(1 - \cos \theta) \quad . \quad (19.233)$$
$$dx = R \sin \theta d\theta \ . \quad . \quad (19.234)$$

Permeance of strip, per centimetre

$$= \frac{d\theta}{2} \quad . \quad . \quad . \quad . \quad (19.235)$$

Portion of conductor producing flux in strip

$$= \frac{R^2\theta - \frac{1}{2}R^2 \sin 2\theta}{\pi R^2} \quad . \quad . \quad . \quad (19.236)$$

$$= \frac{\theta - \frac{1}{2} \sin 2\theta}{\pi} \quad . \quad . \quad . \quad . \quad (19.237)$$

Flux in strip per centimetre

$$= 0{\cdot}4\pi \times \left(\frac{\theta - \frac{1}{2} \sin 2\theta}{\pi} \right) \frac{d\theta}{2} \quad . \quad . \quad (19.238)$$

Therefore, total line linkages per centimetre length of core

$$= \frac{0{\cdot}4\pi}{2\pi^2}\int_0^\pi (\theta^2 - \theta \sin 2\theta + \tfrac{1}{4}\sin^2 2\theta)d\theta \quad . \quad (19{\cdot}239)$$

$$= \frac{0{\cdot}4\pi}{2\pi^2}\left[\frac{\pi^3}{3} + \frac{\pi}{2} + \frac{\pi}{8}\right] \quad . \quad . \quad . \quad . \quad (19{\cdot}240)$$

$$= 0{\cdot}4\pi \times 0{\cdot}62 \quad . \quad . \quad . \quad . \quad . \quad (19{\cdot}241)$$

Total permeance per centimetre

$$= 0{\cdot}62 + \frac{h_s}{\omega_s} \quad . \quad . \quad . \quad (19{\cdot}242)$$

Zigzag Permeance

This refers to the flux which zigzags from stator to rotor teeth (*see* Fig. 19.26).

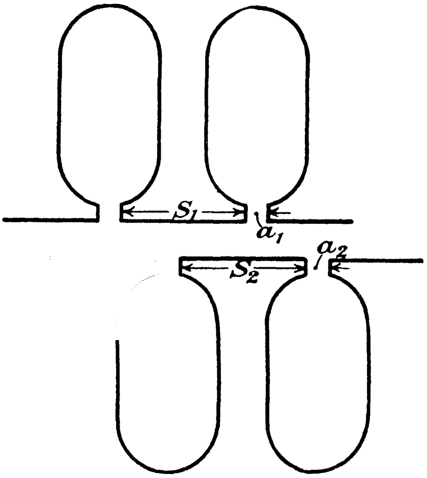


Fig. 19.26

Let δ = gap length in centimetres.

The permeance of the stator zigzag path per centimetre

$$P_s = \frac{s_2 - a_1}{2} \times \frac{1}{2\delta} \times \frac{1}{0{\cdot}96 q_1} \times \frac{1}{2} \quad . \quad . \quad (19{\cdot}243)$$

The permeance of the rotor

$$P_r = \frac{s_1 - a_2}{2} \times \frac{1}{2\delta} \times \frac{1}{q_2} \times \frac{1}{2} \quad . \quad . \quad (19{\cdot}244)$$

Average permeance per centimetre

$$= \frac{1}{16\delta}\left\{\frac{s_2 - a_1}{0{\cdot}96 q_1} + \frac{s_1 - a_2}{q_2}\right\} \quad . \quad . \quad (19{\cdot}245)$$

Average permeance per centimetre

$$= \frac{\tau}{m_1 \times 16 q_1 q_2 \delta}\left[\frac{s_2 - a_1}{0{\cdot}96 q_1} \times \frac{m_1 q_1 q_2}{\tau} + \frac{s_1 - a_2}{q_2} \times \frac{m_1 q_1 q_2}{\tau}\right] \quad . \ (19.246)$$

$$= \frac{\tau}{m_1 \times 16 q_1 q_2 \delta} \times \left[\frac{s_2 - a_1}{0{\cdot}96 \tau_2} + \frac{s_1 - a_2}{\tau_1}\right] \quad . \ (19.247)$$

where τ = pole pitch
q_1 = stator slots per pole per phase
q_2 = rotor slots per pole per phase
τ_2 = rotor slot pitch in centimetres at the gap
τ_1 = stator slot pitch in centimetres at the gap

Let
$$\varepsilon = \frac{s_2 - a_1}{0{\cdot}96 \tau_2} + \frac{s_1 - a_2}{\tau_1} \quad . \quad . \quad . \ (19.248)$$

then
$$\lambda_{Z \text{ per cm}} = \frac{\varepsilon \tau}{m \times 16 \times \delta \times q_1 \times q_2} \quad . \quad . \quad . \ (19.249)$$

Total zigzag permeance

$$\lambda_Z = \frac{l_c \varepsilon \tau}{m_1 \times 16 \times q_1 \times q_2 \times \delta} \quad . \quad . \ (19.250)$$

where l_c = core length in centimetres

For three-phase machines

$$\lambda_Z = \frac{l_c \varepsilon \tau}{48 \times q_1 \times q_2 \times \delta} \quad . \quad . \ (19.251)$$

for two-phase machines

$$\lambda_Z = \frac{l_c \varepsilon \tau}{32 \times q_1 \times q_2 \times \delta} \quad . \quad . \ (19.252)$$

for single-phase machines

$$\lambda_Z = \frac{l_c \varepsilon \tau}{16 q_1 \times q_2 \times \delta} \quad . \quad . \quad . \ (19.253)$$

More Accurate Calculation of Zigzag Permeance

A more accurate calculation of zigzag permeance is as follows (*see* Fig. 19.27)—

Let t_1 = width of primary tooth plus fringe
t_2 = width of secondary tooth plus fringe
τ_1 = tooth pitch, assumed equal in primary and secondary
x = displacement of the centre of the secondary tooth from the centre of the primary slot

Consider the series permeance from t_1 to t_2 and back to the next primary tooth-tip, neglecting the iron part of the path. This permeance takes account of the total tooth-tip leakage for primary and secondary.

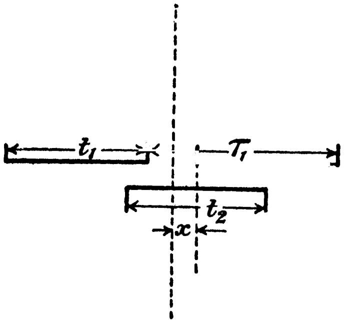


Fig. 19.27

Consider 1 cm of core length, parallel to the shaft. The left-hand overlap

$$= \frac{t_2}{2} - x - \frac{\tau_1 - t_1}{2}$$

$$= \tau_1 \left(\frac{t_1 + t_2}{2\tau_1} - \frac{1}{2} \right) - x \tag{19.254}$$

$$= m - x \tag{19.255}$$

Right-hand overlap

$$= \frac{t_2}{2} + x - \frac{\tau_1 - t_1}{2} \tag{19.256}$$

$$= \tau_1 \left(\frac{t_1 + t_2}{2\tau_1} - \frac{1}{2} \right) + x \tag{19.257}$$

$$= m + x \tag{19.258}$$

where

$$m = \tau_1 \left(\frac{t_1 + t_2}{2\tau_1} - \frac{1}{2} \right) \tag{19.259}$$

The series reluctance

$$= \frac{\delta}{m - x} + \frac{\delta}{m + x} \tag{19.260}$$

$$= \frac{2m\delta}{m^2 - x^2} \tag{19.261}$$

where δ = air-gap length

Series permeance

$$= \frac{m^2 - x^2}{2m\delta} \tag{19.262}$$

The average value of the series permeance

$$= \frac{2}{\tau_1} \times \frac{1}{2m\delta} \int_{x=0}^{x=m} (m^2 - x^2)\,dx \tag{19.263}$$

$$= \frac{1}{m\tau_1\delta} \left[m^2 x - \frac{x^3}{3} \right]_0^m \tag{19.264}$$

$$= \frac{2}{3} \frac{m^2}{\delta\tau_1} \tag{19.265}$$

$$= \frac{2}{3\tau_1\delta} \left[\tau_1 \left(\frac{t_1 + t_2}{2\tau_1} - \frac{1}{2} \right) \right]^2 \tag{19.266}$$

Squirrel-cage Rotor

The coefficient of self-inductance of one bar of a squirrel-cage rotor

$$\underset{\text{(per bar)}}{L_r} = 0{\cdot}4\pi l_c \times \lambda \times 10^{-8} \text{ henrys} \quad . \quad . \ (19.267)$$

where $\lambda = \lambda_s + \lambda_e$

where λ_s = slot permeance per centimetre

λ_e = end connection permeance

The self-inductance *per bar* of the end winding of a squirrel-cage rotor

$$= 0{\cdot}4\pi \times \frac{N_2}{m_1 p} \times \frac{2}{3}[(l_b - l_c) + K \times \tau'] \times 10^{-8} \text{ henrys}$$

where N_2 = number of rotor bars

l_b = length of a bar between rings (centimetres)

l_c = length of core (centimetres)

$K = 0{\cdot}36$ for $p = 2$ and $K = 0{\cdot}18$ for $p > 2$

$\tau' = \frac{\pi D'}{p}$ = pole pitch at the middle of the ring

$$L_r = \frac{0{\cdot}4\pi l_c \times \lambda_s}{10^8} + 0{\cdot}4\pi \frac{N_2}{m_1 p} \times \frac{2}{3}[(l_b - l_c) + K \times \tau'] \times 10^{-8} \quad . \quad . \ (19.268)$$

= inductance *per bar* in henrys

The above formula is given by Dr. Liwschitz-Garik. It would appear to be semi-empirical.

Usually the leakage flux of the end ring is neglected.

The equivalent reactance of the squirrel-cage rotor, *referred to the stator*

$$= 12 \times (T_1 K_1)^2 \times 2\pi f \times 10^{-8}\left(\frac{l_c \lambda_n}{N_2} + \frac{l_e \lambda_e}{2\pi^2 p^2}\right) \ . \ (19.269)$$

where T_1 = turns in series per phase for the three-phase stator winding

l_c = length of core (centimetres)

λ_n = slot, tooth, and zigzag permeance per centimetre

λ_e = end connection permeance per centimetre and is of the order of 0·5

p = number of poles

N_2 = number of rotor bars

K_1 = winding factor for the stator winding

This applies to three-phase machines.

OVERHANG LEAKAGE FLUX IN INDUCTION MOTORS

The following is an article on this subject by Dr. Max Kloss. It gives an excellent survey of this subject, and it is reproduced by permission of Dr. Kloss.

Translation

THE CALCULATION OF THE OVERHUNG LEAKAGE FLUX IN INDUCTION MOTORS

By Max Kloss

Preliminary Particulars

The circle diagram of an induction motor is, as is well known, fixed by two quantities—the magnetizing current, and the ideal short-circuit current. The magnetizing current produces in the stator winding such ampere-turns as are necessary to induce the normal flux in the main magnetic circuit through the stator, air-gap, and rotor. The short-circuit current produces such ampere-turns as are necessary to produce the normal flux (constant e.m.f. assumed) through the leakage field *path*. Now for the connection between flux and ampere-turns producing it we have the well-known fundamental expression

$$\phi = 0{\cdot}4\pi \text{ A.W.} \sqcap \quad . \quad . \quad . \quad (19.270)$$

where $\sqcap$ denotes the magnetic conductivity of the flux path encircled by the ampere-turns, A.W.

For the case of the short-circuit current with q slots per pole and phase, and z conductors in series per slot,

$$\phi = 0{\cdot}4\pi q z i_K \sqrt{2}\, 2(\Lambda_n + \Lambda_z + \Lambda_s) \quad . \quad (19.271)$$

There the various values of Λ denote the partial magnetic conductivities of the flux path encircling a coil side, so that, in order to obtain the total conductivity (magnetic) of the stray flux path, the factor 2 is introduced before the bracket.

Individually,

Λ_n denotes the conductivity of *slot* stray flux path
Λ_z the conductivity of so-called zigzag flux path
Λ_s the conductivity of the path of stray flux round the *overhang*

For the calculation of the short-circuit current, it is consequently necessary, in the next place, to calculate the magnetic conductivity of the stray flux path. The two first portions of this are easily calculated from known formulae.* The overhang leakage has formerly been simplified by the assumption that it is small as compared with the slot stray flux. The calculation is not so easily accessible as that, however.

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* For example, Arnold, *Wechselstromstechnik*, vol. 5, Part I, page 51

In the following, a simple formula for the overhang stray flux shall be deduced, which has proved itself of great value throughout a lengthy period of use and, therefore, is worthy of a further sphere of interest and value.

If one considers the overhang of a normal induction motor stator winding projecting beyond the iron, with its straight portions, bent portions, and with its repeated subdivisions, and if one attempts to form a picture of the flux paths, taking into account the combined effect of the three phases, the problem appears to be so entangled that it might almost be thought impossible to commence the calculation.

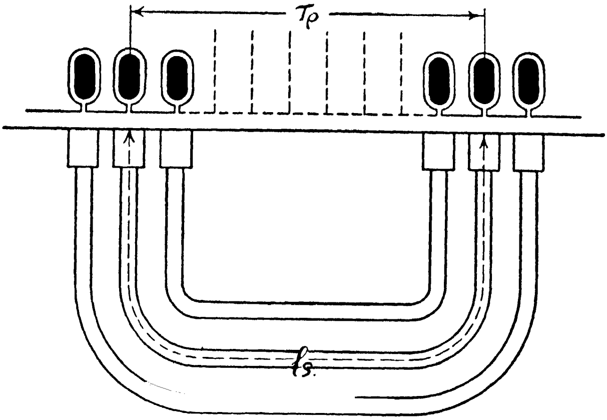


FIG. 19.28

In order to make the complicated portions of the calculation to a certain extent clearer, we must reduce them to a simpler form by the use of certain assumptions. At the same time, it must be sought to make clear in what direction these simplifications may influence the final result.

In order to be able to check the specified guarantees for the output factor of the overload capacity of an induction motor, we must, as far as possible, avoid the estimation of too large a short-circuit current, or that is to say, too small magnetic conductivity (for stray flux).

If we, therefore, by our estimation make such assumptions that the magnetic conductivity may be obtained larger than it actually is, then we shall be on the safe side.

Assumptions for Simplification of the Calculation

We wish, in the first place, to consider a straight coil overhang (Fig. 19.28) of one phase, without regard to the influence of the other phases. It consists of a group of q individual coils ($q = 3$ is assumed in Fig. 19.28), and each of these coils has a rectangular-shaped section corresponding to the type of slot. Further, the plan of the coil top shows a rectangle with considerably rounded corners. For the simplification of the calculation, we shall now make the following assumptions—

(1) The plan of the coil top will be considered as a rectangle of equal area to the actual plan and having sharp corners.

(2) The q separated coils will be considered as combined into one.

(3) The section of this equivalent coil will be taken as of circular form, and of area equal to the sum of the q individual coil sections (including the conductor insulation and the surrounding air spaces).

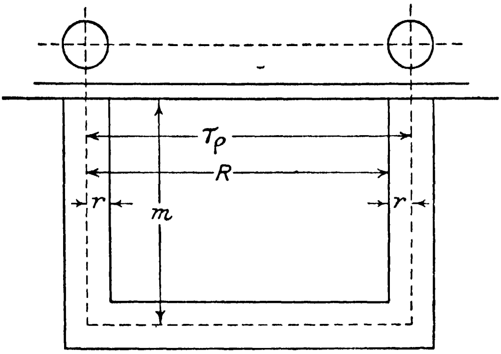


FIG. 19.29

With z turns per slot and with d as diameter of individual wires, we have—

$$\pi r^2 = qzd^2 \quad . \quad . \quad . \quad (19.272)$$

Also $$r = d\sqrt{\frac{qz}{\pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.273)$$

= radius of equivalent coil side bundles

Coil overhang = equivalent rectangle

This simplified equivalent coil end is illustrated in Fig. 19.29. The two sides of the rectangle are—

m = in direction of axis; and

T_p' = in direction of circumference, and measured in coil centre.

The length T_p' is somewhat greater than the pole pitch measured along the stator bore, and the difference between these two dimensions is greater, the less the diameter of the motor is.

A further assumption is that the path of the flux lines round the coil side (conductor bundle) is circular.

The Magnetic Conductivity for a "Conductor Bundle" of Circular Section

The internal and external magnetic conductivity must together be considered.

Since the flux lines in the interior of the conductor group interlink with only a portion of the total number of conductors, so the full effect (of total internal flux and total conductors) must not be taken as producing e.m.f. In consequence thereof the actual magnetic conductivity of the interior stray flux paths must not be calculated, but the equivalent conductivity for the case, where all lines link with all the turns.

Fig. 19.30 shows the cross-section of the circular conductor group with radius r. The internal flux tube with radius x and section $1 \text{ cm} \times \delta x$ interlinks with $(qz)\,\frac{x^2}{r^2}$ conductors.

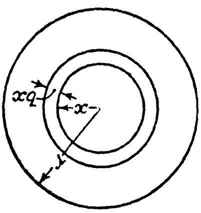


FIG. 19.30

The flux produced in this elementary flux tube has a maximum instantaneous value of

$$\Delta\phi = 0{\cdot}4\pi\left(qz\frac{x^2}{r^2}\right)i\sqrt{2}\,\frac{dx}{2\pi x} \quad . \quad . \quad (19.274)$$

where $\frac{dx}{2\pi x}$ is the magnetic conductivity of the elementary flux tube. This elementary flux produces in the conductors which it embraces an elementary voltage.

$$\Delta_e = 4{\cdot}44c\left(qz\frac{x^2}{r^2}\right)\Delta\phi 10^{-8}$$

$$= 4{\cdot}44c \,.\, 0{\cdot}4\pi i\sqrt{2}(qz)^2\,\frac{x^3}{2\pi r^4}\,dx 10^{-8} \quad . \quad (19.275)$$

where c represents the frequency.

By integrating over the whole group, i.e. between the limits $x = 0$. and $x = r$. we get

$$e = 4{\cdot}44c \,.\, 0{\cdot}4\pi i\sqrt{2}(qz)^2\,\frac{1}{2\pi r^4}\int_0^r x^3 dx 10^{-8}$$

$$= 4{\cdot}44cqz(0{\cdot}4\pi qzi\sqrt{2})\left(\frac{1}{2\pi}\cdot\frac{1}{4}\right)10^{-8} \,. \quad . \quad (19.276)$$

Now $$e = 4{\cdot}44cqz\phi 10^{-8} \quad . \quad . \quad . \quad (19.277)$$

and $$\phi = 0{\cdot}4\pi qzi\sqrt{2}\lambda_i \quad . \quad . \quad . \quad (19.278)$$

wherein λ_i is the magnetic conductivity per centimetre length. So we find, from equations (19.276), (19.277), and (19.278), that the magnetic conductivity for the interior of the conductor groups is, per centimetre length,

$$\lambda_i = \frac{1}{8\pi} = 0{\cdot}040\,\frac{\text{cm}}{\text{cm}} \quad . \quad . \quad . \quad (19.279)$$

and, therefore, independent of the radius r.

For a flux tube with radius x and breadth dx outside the conductor group, the magnetic conductivity per centimetre length of conductor $= \frac{dx}{2\pi x}$. (This is strictly true only for the case where r is small as compared with the length of conductor. We, therefore, calculate the conductivity greater than it is actually.)

Now all the conductors are embraced by the outside flux, and we have merely to integrate between the limits r and R, where R is the distance from the middle point of the conductor group to the outside point of the opposite groups. (*See* Fig. 19.31.)*

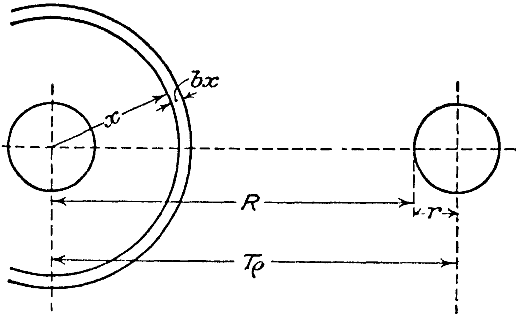


FIG. 19.31

From this we obtain for the magnetic conductivity of the external path, per centimetre conductor length,

$$\lambda_a = \frac{1}{2\pi}\int_r \frac{dx}{x} = \frac{1}{2\pi}\log_e\frac{R}{r} = \frac{2{\cdot}3}{2\pi}\log_{10}\frac{R}{r}$$

$$\therefore \qquad \lambda_a = 0{\cdot}367\log_{10}\frac{R}{r}\,\frac{\text{cm}}{\text{cm}} \qquad . \quad . \quad . \quad (19{\cdot}280)$$

For the two axial legs of the coil overhang, the length is now $= m$ (*see* Fig. 19.29); and for the tangential closing side of the rectangle of length $T_p{}'$ the upper limit is taken as $= m$ in place of the value R. The assumption is here made that the stray flux from this portion of the coil overhang only stretches in air up to the iron core, whilst the lines of force which enter the iron combine with the "slot" stray flux.

In consequence of this, we arrive at the following expression for the magnetic conductivity of the stray flux path for a coil overhang, the influence of other phases being neglected.

$$\Lambda = \left(0{\cdot}040 + 0{\cdot}367\log_{10}\frac{R}{r}\right)2m + \left(0{\cdot}040 + 0{\cdot}367\log_{10}\frac{m}{r}\right)T_p{}' \qquad . \quad . \quad (19{\cdot}281)$$

---

[*] Had the full pole pitch $T_p{}'$ been taken as upper limit instead of R, then the areas in Fig. 19.32, indicated by sloping hatched lines, would have been shown by full lines, whereas they in reality (*tatsachlich*) interlink with only a portion of the conductors. On the other hand, with R as upper limit, we obtain a somewhat small value. However, the error is so small that we may neglect it.

The Magnetic Conductivity of a Coil Overhang of a Normal Three-phase Winding

We shall yet essentially simplify the formula which has just been arrived at, but we wish yet to find the influence of the two other phases.

Fig. 19.32 illustrates the distribution of the induction in the coil planes for the instant of maximum current $= i\sqrt{2}$ in phase I, whilst the current in both other phases II and III is of half this value. From equation (19.274) it follows that in the interior of the conductor group the flux density $\frac{\Delta\phi}{dx}$ is directly proportional to the distance x from the centre point. The field distribution will, therefore, for the portion o to r be represented by a straight line. For the outside of the conductor group the flux density is, on the other hand, inversely proportional to the distance x, the distribution will, therefore, be represented by a rectangular (like sided) hyperbola. Similarly, the same holds good, also for the flux lines issuing from the second axial leg of the coil overhang, the distribution for this is shown by the dotted line in Fig. 19.32, so that the total flux produced by both legs will be given by the sum of these two. In the diagram this is represented by the chain-dotted line. Similarly the field distribution

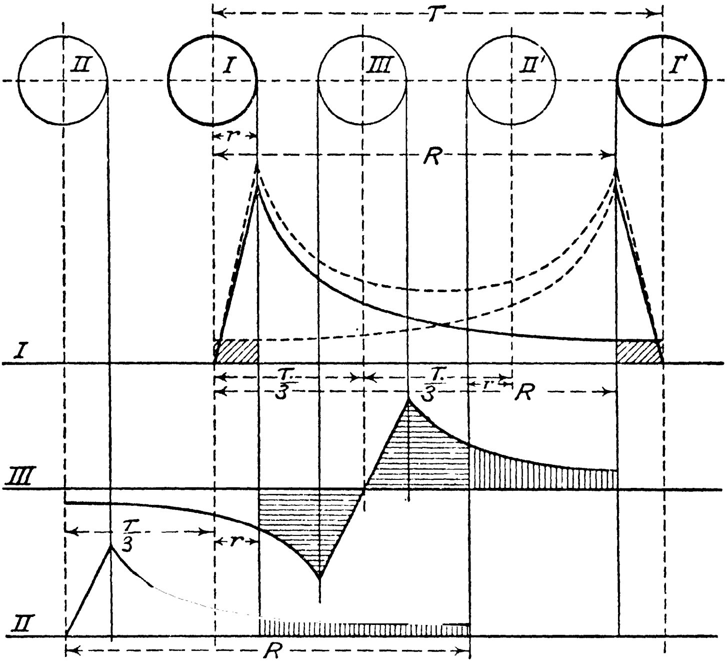


FIG. 19.32

diagram for the other two phases may also be plotted, only the amplitude, for the chosen instant, must be halved; and besides, the whole diagram laterally from left to right, or vice versa, are shifted by one-third of the pole pitch.

Since the magnetic conductivity for an axial coil leg is calculated from formula (19.279) and (19.280), so we need here also only allow for the influence of the coil sides II and III, which are next to the one under consideration; whereas, the coil sides II′ and III′ will be automatically taken account of if we multiply the magnetic conductivity per centimetre length in formula (19.281) by the length $2m$ of both coil sides.

The simplest method of allowing for the influence of coil sides II and III will be now to add the flux going through coil 1–1′ and produced by these (II and III) to the flux produced by I alone, and then consider this whole flux as produced by I alone; that is to say, we must simply increase the magnetic conductivity of coil side I in the ratio of the total flux to the portion produced by I alone.

On account of their special distribution, the flux produced by II and III must not, relatively to 1, be taken at their full values. The horizontally-hatched portions of the diagram, therefore, cancel each other, and only the vertically-hatched portions are taken account of. For the estimation of the true values we must, therefore, form the integral between limits chosen from the diagram in place of the values R and r used in formula (19.280).

The flux produced by 1 is (per centimetre length)

$$\phi_{I} \quad 0{\cdot}4\pi AW_1 0{\cdot}367 \log_{10}\frac{R}{r} \quad . \quad . \quad (19.282)$$

Further, the flux produced by II and effective on I (by half the ampere-turns) is

$$\phi_{II} = 0{\cdot}4\pi\frac{AW_1}{2}0{\cdot}367 \log_{10}\frac{R}{\dfrac{T'}{3}+r} \quad . \quad (19.283)$$

and by III, the effective flux on 1 is

$$\phi_{III} = 0{\cdot}4\pi\frac{AW_2}{2}0{\cdot}367 \log_{10}\frac{R-\dfrac{T'}{3}}{\dfrac{T'}{3}-r} \quad . \quad (19.284)$$

We have, therefore, arrived at, in place of the expression in equation (19.280), the total expression for the sum $\phi_{I}+\phi_{II}+\phi_{III}$.

We obtain, then,

$$\lambda_a = 0{\cdot}367\left[\log_{10}\frac{R}{r}+\frac{1}{2}\log_{10}\frac{R-\dfrac{T'}{3}}{\dfrac{T'}{3}-r}+\frac{1}{2}\log_{10}\frac{R}{\dfrac{T'}{3}+r}\right] \quad (19.285)$$

By replacing T' by the equivalent sum $R + r$, the following simplified expression is arrived at—

$$\lambda_a = 0{\cdot}367\left[\log_{10}\frac{R}{r} + \frac{1}{2}\log_{10}\frac{\dfrac{2R}{r} - 1}{\dfrac{R}{r} - 2} + \frac{1}{2}\log_{10}\frac{3\dfrac{R}{r}}{\dfrac{R}{r} + 4}\right] \quad (19.286)$$

where R and r have values as given in Fig. 19.33 or 19.32.

It is somewhat more difficult to calculate the influence of the two other phases on the tangential portion of the overhang of coil of phase I.

Let us consider, in the first place, the actually impossible case in which the three coil heads lie on a cylindrical surface; the axis of their tangential portions will then fall together, so that only in the middle third the current group of phase I would have its complete effect. (Fig. 19.33.) Whilst in the outer thirds the effect will be halved by reason of the current of half value flowing in the opposite direction in phases II and III. The true value of the excitation due to the tangential portion would, therefore, be $\frac{1 + \frac{1}{2} + \frac{1}{2}}{3} = \frac{2}{3}$ of that due to the AT_s of phase I.

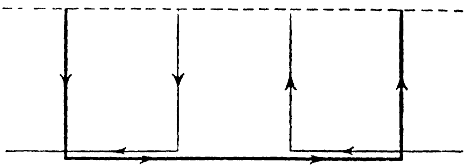


FIG. 19.33

In reality, however, the coil heads of phases II and III are not in the same plane as that of coil 1, so that on these grounds the demagnetizing value will be reduced. We may, with some degree of accuracy, assume that the demagnetizing values may be reduced to about a third of their total value, so that for the parts concerned the reduction on phase I is $\frac{1}{3} \times \frac{1}{2} \times (0{\cdot}4\pi qzi\sqrt{2})$; and the resulting A.W., therefore, are reduced to five-sixths of normal. But this diminution does not truly extend over the two outer thirds, as shown in Fig. 19.33; but, on account of the rounding of the coil, overhangs at their corners over really a shorter length which shall be taken as a sixth of the pitch.

In place of the current distribution diagram of Fig. 19.33, we obtain, therefore, the following distribution—

In the middle four-sixths the full excitation $0{\cdot}4\pi qzi\sqrt{2}$, whilst in both outer sixths only $\frac{5}{6} \times 0{\cdot}4\pi qzi\sqrt{2}$, so that the effective total value is

$$\frac{4 \times 1 + 2 \times \frac{5}{6}}{6}(0{\cdot}4\pi qzi\sqrt{2}) = \frac{17}{18}(0{\cdot}4\pi qzi\sqrt{2}) \quad . \quad (19.287)$$

Hence the influence of the two other phases is, in effect, to multiply the magnetic conductivity of the tangential part of the coil overhang by approximately 0·94.

Hence, in place of the values found in formula (19.281), we arrive at the following expression for the magnetic conductivity of a coil overhang.

$$\Lambda_s = \left[0{\cdot}040 + 0{\cdot}367 \left(\log_{10} \frac{R}{r} + \frac{1}{2} \log_{10} \frac{\frac{2R}{r} - 1}{\frac{R}{r} - 2} + \frac{1}{2} \log_{10} \frac{\frac{3R}{r}}{\frac{R}{r} + 4} \right) \right]$$

$$2m + 0{\cdot}94 \left[0{\cdot}040 + 0{\cdot}367 \log_{10} \frac{m}{r} \right] T_p' \quad . \qquad . \quad (19.288)$$

And now to simplify the formula for practical use, we may graph the values r and R/r.

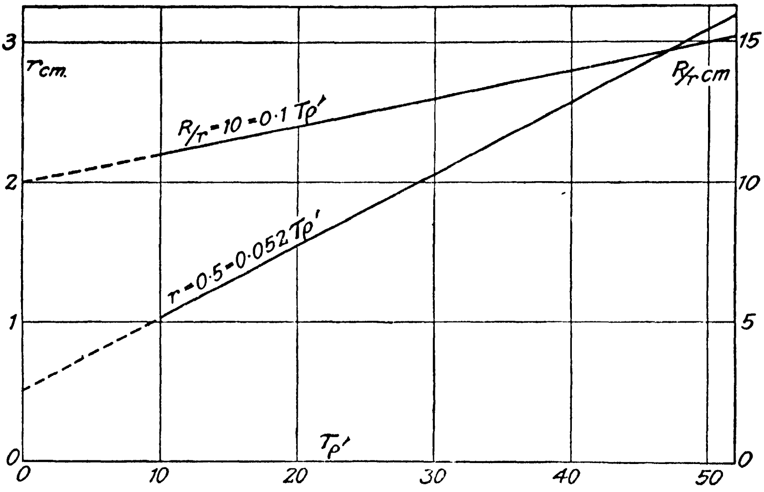


Fig. 19.34

These values should be ascertained for a line of motors. In Fig. 19.34 these are plotted as functions of T_p'.

It is evident that we can put, with sufficient accuracy for practical purposes,

$$r = 0{\cdot}50 \text{ cm} + 0{\cdot}052\,T_p' \quad . \qquad . \quad (19.289)$$

and

$$\frac{R}{r} = 10 + 0{\cdot}10\,T_p' \quad . \qquad . \qquad . \quad (19.290)$$

Putting these values in formula (19.288), we obtain

$$\sqcap_s = \left[0{\cdot}040 + 0{\cdot}367 \left\{ \log_{10} (10 + 0{\cdot}1\,T_p') + \frac{1}{2} \log_{10} \frac{19 + 0{\cdot}2\,T_p'}{8 \times 0{\cdot}1\,T_p'} \right.\right.$$

$$\left.\left. + \frac{1}{2} \log \frac{30 + 0{\cdot}3\,T_p'}{14 + 0{\cdot}1\,T_p'} \right\} \right] 2m$$

$$+ \left[0{\cdot}038 + 0{\cdot}345 \log_{10} \frac{m}{0{\cdot}5 + 0{\cdot}052\,T_p'} \right] T_p' \quad . \quad (19.291)$$

We have now reduced the formula to a function of two independent quantities T_p' and m. To a certain extent, m is, of course, dependent on T_p', since the combined section of the copper in the overhang will increase as T_p' is increased, whereby the overhang m will also be increased, however; clearance and such dimensions as are dependent on the voltage of the m/c are, of course, independent of T_p'.

The formula (19.291) just deduced is not suitable for practical use; it has the form

$$\Lambda_s = x2m + yT_p' \quad . \quad . \quad . \quad (19.292)$$

where, however, X and Y are not constants, but are functions of T_p' and of m and T_p'.

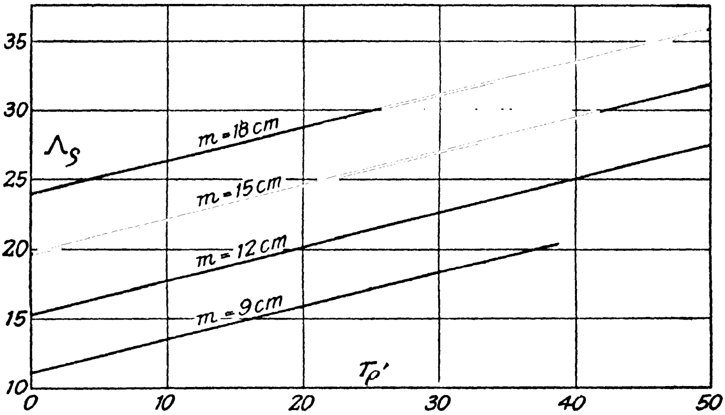


FIG. 19.35

It remains to be shown that for the usual practical range of T_p' and m, the effective values of x and y may be assumed as constants with a fair degree of accuracy. To arrive at these constants, we shall plot Λ_s as a function of T_p' with m as "parameter," so that, therefore, we obtain a series of curves, wherein each individual curve corresponds to a certain value of m. (*See* Fig. 19.35.)

In Fig. 19.35, calculated values are given in graphical form for $m = 9$, $m = 12$, $m = 15$, and $m = 18$ cm, and for $T_p' = 10$, 20, 30, 40, 50 cm, which correspond to the most frequently-occurring cases in practice.

If we wish to replace the somewhat intricate formula (19.291) by a simpler one, we must draw, with the greatest possible accuracy, straight lines through the points in Fig. 19.35; and, moreover, these straight lines must be parallel to one another if the factor y is to be constant throughout the range of T_p'. Similarly, the distance between the neighbouring lines must be equal if the factor x depending on m is to be a constant. It turns out that for practical cases these conditions are fairly well fulfilled. From Fig. 19.35 the equations of the series of straight lines may easily be derived. It is evident that the first term of this $\left\{\begin{matrix}\text{substitute}\\ \text{equivalent}\end{matrix}\right\}$ equation is not exactly proportional to m, but to $(m-1)$.

The equation, therefore, is of the simple form,

$$\Lambda_s = 1{\cdot}41(m-1) + 0{\cdot}24T_p' \quad . \quad . \quad (19.293)$$

The quantity in this equation is still inconvenient in so far as it must be estimated from the actual measurements of the coil overhang. For practical purposes, it is advantageous to arrive at this from a quantity which is readily obtainable from constructional particulars of the windings. It is now the general custom to express the length of mean turn of the winding in the form $lm = 2(L + T_p' + A)$, where L is the active iron length of core, T_p' the pole pitch measured at centre of slots, and A a quantity which is dependent on the voltage of the stator and in general practice fixed from standard tables. It may be shown that a simple relationship exists between A and m, which obtained empirically may be expressed as in the following equations,

$$m = 1{\cdot}0 + 0{\cdot}5A + 0{\cdot}04T_p' \quad . \quad . \quad (19.294)$$

By putting this in equation (19.293), we have

$$\begin{aligned}\Lambda_s &= 1{\cdot}41(0{\cdot}5A + 0{\cdot}04T_p') + 0{\cdot}24T_p' \\ &= 0{\cdot}705A + 0{\cdot}296T_p' = 0{\cdot}7A + 0{\cdot}3T_p' \text{ approx.} \quad . \quad (19.295)\end{aligned}$$

If one is accustomed to use the mean length l_s of the coil overhang (*see* Fig. 19.28), then equation (19.295) may be altered to suit, since $l_s = A + T_p'$, so we have

$$\Lambda_s = 0{\cdot}7l_s - 0{\cdot}4T_p' \quad . \quad . \quad . \quad (19.296)$$

This gives, therefore, the magnetic conductivity of stator coil overhang, taking into consideration the influence of the other two phases, but still, however, without consideration of the influence of the rotor.

The Influence of the Rotor Winding

The flux produced by the rotor ampere-turns takes effect on the magnetic conductivity of the stator winding overhang in different ways, according as the direction or the position of these ampere-turns is considered. Because of their direction, they assist in the magnetization of the air space between the stator and rotor windings, since the rotor currents are in the opposite direction to the stator currents, and so form a winding with an m.m.f. in the same direction. Their effect is, therefore, to increase the magnetic conductivity. On the other hand, because of their position, they limit the section available for the stray flux, in so far as they crowd the stray lines inter-linking with the stator current groups into a comparatively small section between the stator and rotor conductors, instead of allowing the free circular path for the stray flux, as before assumed. From this point of view they, therefore, act as a limiting factor on the magnetic conductivity for the stray flux.

It is, therefore, necessary to settle which of the two influences will be greater, that is to say, in what manner we have to correct our formula.

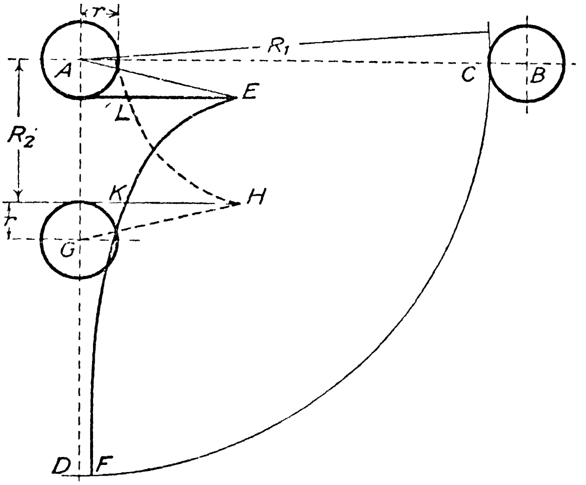


FIG. 19.36. INFLUENCE OF ROTOR WINDING

Fig. 19.36 illustrates further (as in Fig. 19.31) the section through the two axial "coil legs" *A* and *B*. In the derivation of our formula, we had assumed that the lines round leg *A* take a circular path. The outermost line with radius *R* is shown in the figure.

The assumption made by us of a circular path would be correct if the lines could spread themselves not merely between the bundles *A* and *B* lying on the stator periphery, but in a like manner also in the direction of the stator radius, that is to say, towards *D* in Fig. 19.36, whereby then the induction over the distance *AD* would be distributed accordingly to the curve *AEF*. In reality, however, there is in the direction *AD* a second current group *G* parallel to *A*, and with opposing current direction. By this the available section for the stray flux will be reduced from the limit *R* to the limit R_2. Simultaneously, however, the group *G* magnetizes the intervening space in the same sense as *A*, according to the diagram *HI*, which, when the radius of the second group is *r*, becomes, as it were, a reflected maze of the portion *EK* of the diagram of flux issuing from *A*. The resultant flux between the two groups now becomes the sum of the areas bounded by *EK* and *HL*.

In the case where the second group G is at the same distance as the group B, then $R_2 = R_1$, then the stray flux calculated from $0{\cdot}4\pi \times AW_1 \times 0{\cdot}367 \log_{10} \frac{R_1}{r}$ will be doubled. Should, however, the second group fall together with the first in one axis, then the stray flux issuing from A would be completely nullified. It is now clear that there must be between these two limiting values a position for G, for which the resulting flux will be exactly as great as that produced by A alone across a distance R_1. We shall take this as the "neutral" distance of G. For this condition the expression becomes (equal ampere-turns in both groups assumed),

$$2 \times 0{\cdot}4\pi AW_1 \times 0{\cdot}367 \log_{10} \frac{R_2}{r} = 0{\cdot}4\pi AW_1 \times 0{\cdot}367 \log_{10} \frac{R_1}{r} \quad . \quad . \quad (19{\cdot}297)$$

or

$$2 \log_{10} \frac{R_2}{r} = \log \frac{R_1}{r} \quad . \quad . \quad . \quad (19{\cdot}298)$$

so that

$$\frac{R_2}{r} = \sqrt{\frac{R_1}{r}} \quad . \quad . \quad . \quad (19{\cdot}299)$$

When the second group is brought nearer to the first, the resultant flux between the two, on account of the diminished value of the section, is smaller than that for the first group alone, under the assumption of circular propagation of lines up to the limit R_1. When the second group is removed farther from the first, so will the resultant flux between the two be greater than the value calculated for the first group alone to the limiting distance R_1.

Now applying this to the case of the induction motor, we find that neutral distance of the equivalent rotor conductor group is

$$R_2 = r\sqrt{\frac{R_1}{r}} \quad . \quad . \quad . \quad (19{\cdot}300)$$

From Fig. 19.34 the value of $\frac{R_1}{r}$ varies between 11 and 15, according to the pole pitch, and r between 1·0 and 3·0. We obtain, therefore, with these limiting values

$$R_2 = 1\sqrt{11} = 3{\cdot}3 \text{ cm and } 3 \times \sqrt{15} = 11{\cdot}6 \text{ cm}$$

In reality, however, the rotor winding is considerably nearer to the stator winding. We have, therefore, to deal with the case that the assumption of circular leakage lines, which does not truly hold, produces too great values for the magnetic conductivity of a stator winding overhang.

But this requires still another correction. We have, in the above investigation, assumed that the second group of conductors is parallel to the first. It happens, however, that this is not so in the usual case of basket-wound (drum-wound) slip-ring rotors. Here the rotor currents do not run parallel to the stator currents, but approximately at an angle of 45° to these. The effect of this will be that the influence of rotor winding will be further reduced; that is to say, the reduction in the value of the flux, just deduced, will not be so great as was to be expected. We shall, therefore, make no great error if, for slip-ring motors with drum winding in the rotor, we leave unaltered the formula already found, though it must be kept in mind that herein lies a source of deviation between calculation and test results.

If in two motors of similar type, the rotor winding of one is cylindrical, that is to say, lying close to the stator winding, and that of the other, on the other hand, is more radial and arched towards the centre, and, therefore, more cone shaped, then the first motor will show less stray flux than the second. Consequently, under certain circumstances, it may be necessary to modify the constants in our formula.

In the case of squirrel-cage rotor, one must differentiate between the case where the short-circuiting ring lies close to the iron core and the case where, in order to increase the ventilation in an axial direction, the bars project considerably from the core. When the rings lie close to the core, it may be assumed that the influence of the rotor on the formula for the magnetic conductivity of the stator winding overhang (calculated for circular path) may be neglected, so that the formula may be used without alteration.

On the other hand, for short-circuited rotors, with long bars and rings considerably removed from the core, the rotor currents flow sensibly parallel to those of the stator, consequently it must be expected that for this case the field weakening effect of the rotor must make itself apparent. For such motors, therefore, the constant of our stray formula must, therefore, be diminished.

All of these considerations will be confirmed by trial. For short-circuited rotors with long bars, the constants of formula (19.295) must be multiplied by a constant varying from 0·8 to 0·9.

Split-phase Windings—Three-plane Windings

Up to this period the inquiry into the stray flux has been for a winding according to that shown in Fig. 19.28, that is to say, all the end windings for one pole and phase are grouped together to form one coil group. For long pole pitches, the split phase winding is, however, often made use of, whereby the coil overhangs of the several phases lie in different winding planes perpendicular to the axis. This winding is diagrammatically illustrated in Fig. 19.37.

No alteration occurs in the portion of the coil which projects from the core in an axial direction. It remains, therefore, to leave the portion with $2m$—in formula (19.288) unaltered.

It is, however, a different matter in the case of the part of the formula dealing with the magnetic conductivity of the portion of the coil overhang lying along the periphery. This portion carries only one-half of the ampere-turns and produces, therefore, only one-half of the stray flux. To obtain all fluxes for cases of equal ampere-turns per pole and phase, we must, therefore, halve the magnetic conductivity obtained from this part of the formula. In this connection, it must still be noticed also that the equivalent radius r must be reduced approximately in the ratio of 1 to 2. Further, the shortened coil pitch $\frac{5}{6}T_p'$ (*see* Fig. 19.37) must be substituted in place of the pole pitch T_p'. Finally the influence of the two other phases may be arrived at by an investigation similar to that in connection with Fig. 19.33. It will be found that this influence is slightly greater and may be expressed by replacing the factor 0·94 of formula (19.288) by the factor 1·06.

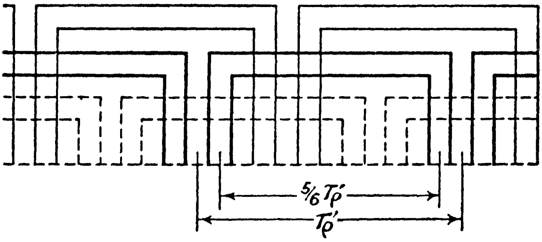


FIG. 19.37

We arrive, therefore, at the following expression for the portion of the overhang lying along the periphery—

$$\frac{1}{2} \times 1{\cdot}06\left[0{\cdot}040 + 0{\cdot}367 \log_{10}\frac{m}{r}\right]\frac{5}{6}\,T_p'$$

$$= \left[0{\cdot}0176 + 0{\cdot}162 \log_{10}\frac{m}{r}\right] T_p' \quad . \quad (19.301)$$

Further, the value of Λ_s for several values of m and T_p' may now be calculated and plotted as functions of T_p' with m as parameter in similar manner to Fig. 19.35. It will then be found that one may use the following expression with sufficient accuracy.

$$\Lambda_s = 0{\cdot}65A + 0{\cdot}20T_p' \quad . \quad . \quad (19.302)$$

where

$$A = l_s - (\tfrac{5}{6}T_p') \quad . \quad . \quad . \quad (19.303)$$

When using length of the coil overhang as l_s, the formula becomes

$$\Lambda_s = 0{\cdot}65l_s - 0{\cdot}34T_p' \quad . \quad . \quad . \quad (19.304)$$

where T_p' is the pole pitch measured at the middle of the slots (not the coil pitch). The length l_s has naturally a different value for each of the three phases.

Review of the Assumptions Made in the Derivation of Formulae

The assumptions of Figs. 19.28 and 19.29 that we may take the coil sides as equivalent to a circular conductor group of equal area, is somewhat arbitrary. If greater accuracy is required, it must be noted that the coils proceeding from the several slots do not lie quite close together. A sort of "winding factor" must, therefore, be introduced. If this be neglected, then presumably the magnetic conductivity is calculated too large and, therefore, on the safe side.

Further, no difference has been made between straight and bent coils, but the neglect of the point gives again a result on the safe side, since the area enclosed by the overhang of the bent coils is less than that of the straight coils.

A contrary effect is exercised by the vicinity of the iron of the end shields, whereby the effect of the first two approximations will be somewhat nullified.

An exact calculation must naturally include all these points. Now, in practice, it is not necessary to obtain a great accuracy; instead, it is much more important to arrive at a handy formula, by making simplifying assumptions, which will give an estimation of the stray flux in the majority of cases.

The formula agrees fully with this condition, especially since, for the calculations, they replace the inconvenient logarithmic terms by linear functions.

On the ground of experience in a given type of machine, the designer may slightly alter at his discretion the coefficients of the formulae.

Application to Generators

It need scarcely be mentioned that the formulae may, without much ado, be applied also to generators.

Summary

The short-circuit current of an induction motor is (equation (19.271))

$$i_k = \frac{\phi}{0{\cdot}4\pi 2\sqrt{2}qz(\Lambda_n + \Lambda_z + \Lambda_s)}$$

where Λ_n, Λ_z, Λ_s are the magnetic conductivities of a coil side for slot, zigzag, and overhang stray flux.

A simple formula is deduced, for the magnetic conductivity Λ_s of a coil overhang, under consideration of the influence of the other phases.

For the usual three-phase winding,

$$\Lambda_s = 0{\cdot}7A + 0{\cdot}3T_p' \quad . \quad . \quad . \quad (19{\cdot}305)$$

where $A = l_2 - T_p'$

or

$$\Lambda_s = 0{\cdot}7l_s - 0{\cdot}4T_p' \quad . \quad . \quad . \quad (19{\cdot}306)$$

and for split-phase winding (three-plane winding)

$$\Lambda_s = 0{\cdot}65A + 0{\cdot}2T_p' \quad . \quad . \quad . \quad (19.307)$$

where $A = l_s - \frac{5}{6}T_p'$

or

$$\Lambda_s = 0{\cdot}65l_s - 0{\cdot}34T_p' \quad . \quad . \quad . \quad (19.308)$$

These formulae are valid for motors with slip-ring drum windings, and for squirrel-cage motors with rings close up to core.

For squirrel-cage machines with rotor bars projecting far from the core, and for slip-ring machines with the rotor winding lying particularly close to the stator winding, the corresponding coefficients require to be somewhat reduced (about 10–20 per cent).

CHAPTER XX

Short-pitch Windings

SHORT-PITCH windings are sometimes used for induction motors, and by their use many advantages are secured. In the first place, a considerable reduction in the weight of copper required is effected; and, secondly, by correct chording of the winding, it is possible to eliminate harmonics from the flux wave which are so troublesome in producing crawling and other troubles. This is more fully dealt with in the section dealing with the nature of the rotating field.

The following investigation deals with the pitch of the coils which will give minimum copper weight for all values of the pole pitch and core lengths. It is taken from an article by the author published in *The Electrician* for 29th October, 1920.

In making the calculation, it is assumed that the current density is constant, and that there is room to deepen the slots to accommodate the increased numbers of conductors due to chording. In most cases of importance, this is possible.

SHORT-PITCH WINDINGS

Let E = e.m.f. generated per phase in volts

Z = number of conductors in series per phase

f = frequency of supply current

$\beta = \dfrac{\text{span of coil}}{\text{full pitch}}$

k = distribution factor × form factor

τ = pole pitch

a = area of conductor

$\hat{\phi}$ = flux per pole (max)

C.G.S. units are used throughout.

Then for a distributed winding,

$$E = 2 \times k \times Z \times \hat{\phi} \times f \times \sin\frac{\beta\pi}{2} \times 10^{-8} \text{ volts} \quad . \quad (20.1)$$

From this equation,

$$Z = \frac{p}{\sin \beta \frac{\pi}{2}} \quad . \quad . \quad . \quad . \quad (20.2)$$

where $p = \dfrac{E \times 10^8}{2 \times k \times \hat{\phi} \times f}$

p is constant for a given rating.

The mean length per turn with full pitch coils

$$= 2(l + \alpha\tau) \quad . \quad . \quad . \quad . \quad (20.3)$$

where l = length of core
and $\alpha\tau$ = length of overhang at one end of the coil

In any given case the value of α may be easily found, for the inclination of the coil end to the core with diamond-shaped coils is given by

$$\sin^{-1} \theta = \frac{\text{width of slot copper} + \text{clearance}}{\text{width of slot} + \text{width of tooth}}$$

The volume of copper with full-pitch coils, with m phases, is thus

$$= m \frac{Z'}{2} \times 2(l + \alpha\tau) \times a \quad . \quad . \quad . \quad (20.4)$$

$$= mZ' \times a \times (l + \alpha\tau) \quad . \quad . \quad . \quad (20.5)$$

where Z' = number of conductors in series per phase with full-pitch coils.

The volume of copper required with fractional pitch coils

$$= mZ \times a \times (l + \beta\tau\alpha) = \frac{mZ'}{\sin \beta \frac{\pi}{2}} \times a \times (l + \alpha\beta\tau) \quad . \quad (20.6)$$

Since $$Z \sin \beta \frac{\pi}{2} = Z' \quad . \quad . \quad . \quad . \quad (20.7)$$

Therefore, $\dfrac{\text{volume of copper with full-pitch coils}}{\text{volume of copper with fractional pitch coils}}$

$$= \sin \beta \frac{\pi}{2} \times \frac{l + \alpha\tau}{l + \alpha\beta\tau} \quad . \quad . \quad . \quad (20.8)$$

It is clear that this ratio will have a maximum value for minimum weight of copper; in other words, the most economical span of the coil will be that which gives the above ratio a maximum value. This span can be determined by differentiating the above expression with respect to β.

Calling this ratio A, we have

$$\frac{dA}{d\beta} = \frac{(l + \alpha\tau)\left(\frac{\pi}{2}\cos\beta\frac{\pi}{2}\right)(l + \alpha\beta\tau) - \alpha\tau\left(\sin\beta\frac{\pi}{2}\right)(l \times \alpha\tau)}{(l + \alpha\beta\tau)^2} \qquad (20.9)$$

for a minimum, $dA/d\beta = 0$

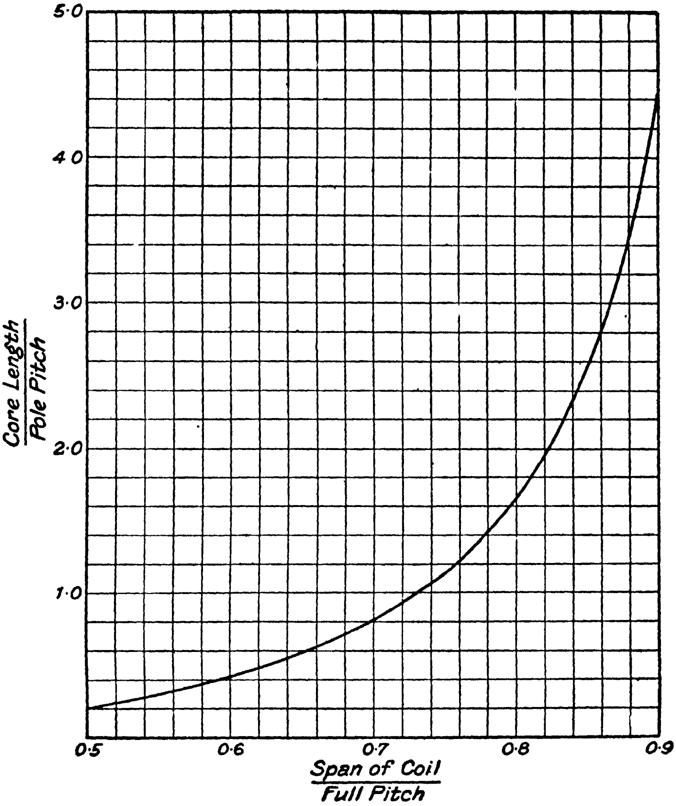


FIG. 20.1

$$\therefore \qquad \frac{\pi}{2}\cos\beta\frac{\pi}{2} \times (l + \alpha\beta\tau) = \alpha\tau\sin\beta\frac{\pi}{2}\,. \qquad (20.10)$$

Therefore
$$\tan\beta\frac{\pi}{2} = \frac{\pi}{2}\left(\frac{l + \alpha\beta\tau}{\alpha\tau}\right) \qquad (20.11)$$

$$= \beta(\pi/2) + (\pi l/2\alpha\tau) \qquad (20.12)$$

Let $\beta(\pi/2) = \theta$

then
$$\tan\theta = \theta + (\pi l/2\alpha\tau) \qquad (10.13)$$

Its solution is shown in the following curve, Fig. 20.1, which gives the most economical span of the coil for different ratios of core-length to pole-pitch.

In drawing this curve, a value of $\alpha = 1{\cdot}4$ has been taken, but in any given case its correct value can be substituted in the equation.

Inductance of Fractional Pitch Windings

Messrs. Adams, Cabot, and Irving have investigated this subject, and the following is an abstract of their work.

Slot Leakage

The reactance due to that part of the leakage flux which crosses a slot, linking with more or less of the conductors in the slot, may be expressed as follows

$$X_s = 2\pi f \times \phi_s q_1^2 \times l \times \frac{N_t}{m} \times 10^{-8} \quad . \quad . \quad (20.14)$$

where f = frequency

ϕ_s = flux per ampere conductor per unit length of slot in c.g.s. lines

q_1 = conductors per slot

l = core length

N_t = total number of slots

m = number of phases

Only ϕ_s in the above is affected by fractional pitch.

Let θ = pitch of the coil in electrical degrees

β = angle of pitch deficiency = $180 - \theta$

If the machine has a large number of phases, the two coil sides located in any given slot will, in general, carry currents differing in phase by β, and the component of one of these currents in phase with the other will be proportional to cos β.

If the product of the inductances of these two coil sides is equal to the square of the mutual inductance, i.e. if there is no relative leakage flux between them, the ratio of the average leakage flux linked with one to what it would be with a full pitch winding and no phase difference

$$= \frac{1 + \cos\beta}{2} = \cos^2\frac{\beta}{2} = \sin^2\frac{\theta}{2} = K_p \quad . \quad . \quad (20.15)$$

This may be called the slot pitch factor under the above-mentioned conditions.

Neither of these conditions exists in practice.

Consider, first, the relative leakage between the two coil sides in the same slot.

The flux linked with coil *b* at the bottom of the slot per centimetre length of slot for 1 A in *b* is

$$\phi_{bb} = 0{\cdot}4\pi\left(\frac{d_1}{6w} + \frac{d_1}{2w} + \frac{d_2}{\omega}\right) = \frac{0{\cdot}4\pi}{\omega}\left(\frac{d_1}{6} + \frac{d_1}{2} + d_2\right)\Big) \quad (20.16)$$

The flux linked with *b* per centimetre of slot for 1 A in *a* is

$$\phi_{ab} = \frac{0{\cdot}4\pi}{w}\left(\frac{d_1}{4} + d_2\right) \quad . \quad . \quad . \quad (20.17)$$

But the current in *a* differs in phase from that in *b* by an angle β, and the component of ϕ_{ab} in phase with ϕ_{bb} is $\phi_{ab}\cos\beta$.

Then the total in-phase flux linked with *b* per centimetre length of slot, and for 1 A distributed uniformly over the whole copper section of the slot, is

$$\phi_b = \frac{\phi_{bb} + \phi_{ab}\cos\beta}{2} \quad . \quad . \quad . \quad . \quad . \quad (20.18)$$

$$= \frac{0{\cdot}4\pi}{w}\left[\frac{d_1}{3} + \frac{d_2}{2} + \cos\beta\left(\frac{d_1}{8} + \frac{d_2}{2}\right)\right] \quad . \quad (20.19)$$

Similarly,

$$\phi_{aa} = \frac{0{\cdot}4\pi}{w}\left(\frac{d_1}{6} + d_2\right) \quad . \quad . \quad . \quad (20.20)$$

$$\phi_{ba} = \frac{0{\cdot}4\pi}{w}\left(\frac{d_1}{4} + d_2\right) \quad . \quad . \quad . \quad (20.21)$$

and

$$\phi_a = \frac{\phi_{aa} + \phi_{ba}\cos\beta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20.22)$$

$$= \frac{0{\cdot}4\pi}{w}\left[\left(\frac{d_1}{12} + \frac{d_2}{2}\right) + \cos\beta\left(\frac{d_1}{8} + \frac{d_2}{2}\right)\right] \quad . \quad (20.23)$$

Then since each coil has one side in the bottom and the other in the top of a slot, the average flux linkage per ampere per centimetre of slot will be

$$\phi_s = \frac{\phi_a + \phi_b}{2} = \frac{0{\cdot}4\pi}{w}\left[\left(\frac{5}{24}d_1 + \frac{d_2}{2}\right) + \cos\beta\left(\frac{d_1}{8} + \frac{d_2}{2}\right)\right] \quad (20.24)$$

If the winding were full pitch,

$$\phi_b = \frac{0{\cdot}4\pi}{w}\left[\frac{d_1}{3} + \frac{d_2}{2} + \frac{d_1}{8} + \frac{d_2}{2}\right] = \frac{0{\cdot}4\pi}{w}\left[\frac{11}{24}d_1 + d_2\right] \quad (20.25)$$

and

$$\phi_a = \frac{0{\cdot}4\pi}{w}\left[\frac{d_1}{12} + \frac{d_2}{2} + \frac{d_1}{8} + \frac{d_2}{2}\right] = \frac{0{\cdot}4\pi}{w}\left[\frac{5}{24}d_1 + d_2\right] \quad (20.26)$$

and

$$\phi_s = \frac{\phi_a + \phi_b}{2} = \frac{0{\cdot}4\pi}{w}\left[\frac{d_1}{3} + d_2\right] \quad . \quad . \quad (20.27)$$

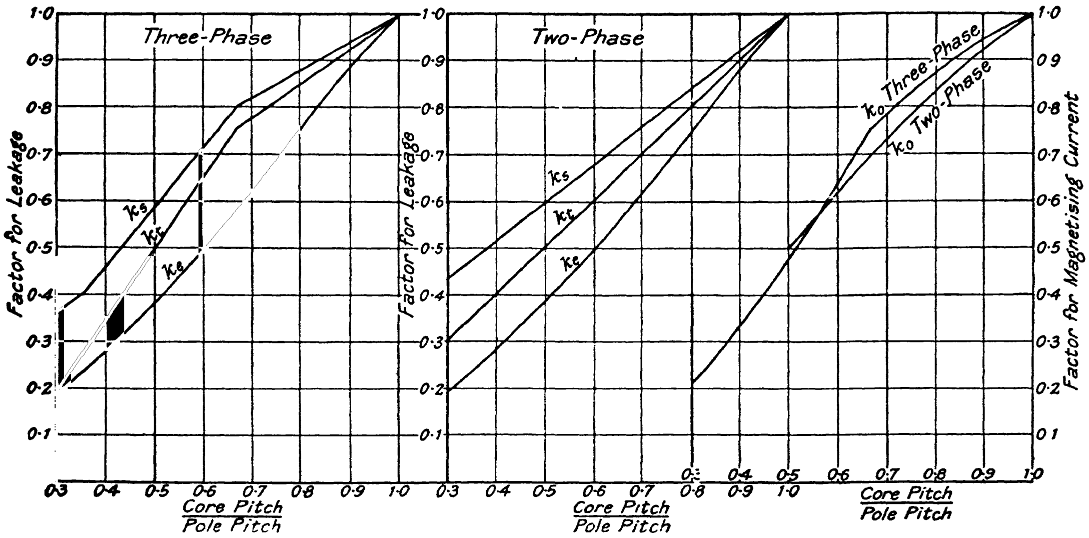


FIG. 20.2. CONSTANTS FOR FRACTIONAL PITCH WINDINGS (ADAMS)

k_0 = constant for magnetizing current
k_s = constant for slot leakage
k_t = constant for tooth-tip leakage
k_e = constant for end-winding leakage

The slot-pitch factor

$$K_{ps} = \frac{\frac{5}{24} d_1 + \frac{d_2}{2} + \cos \beta \left(\frac{d_1}{8} + \frac{d_2}{2}\right)}{\frac{d_1}{3} + d_2} \quad . \quad . \quad (20.28)$$

$$K_{ps} = \frac{\frac{5}{24} d_1 - \frac{1}{8} d_1 + \left(\frac{d_1}{4} + d_2\right)\left(\frac{1 + \cos \beta}{2}\right)}{\frac{d_1}{3} + d_2}$$

$$= \frac{\frac{d_1}{12} + \left(\frac{d_1}{4} + d_2\right)\left(\frac{1 + \cos \beta}{2}\right)}{\frac{d_1}{3} + d_2}$$

$$= \frac{\frac{d_1}{12} + \left(\frac{d_1}{4} + d_2\right)\left(\frac{1 + \cos \beta}{2}\right)}{\frac{d_1}{12} + \left(\frac{d_1}{4} + d_2\right)} {}^{*} \quad . \quad . \quad . \quad (20.29)$$

Tooth-tip or Zigzag Leakage

The tooth-tip leakage reactance

$$x_{tt} = 2\pi f \phi_{tt} q_1^2 \times l \times \frac{N_t}{m} \times 10^{-8} \quad . \quad . \quad (20.30)$$

ϕ_{tt} = tooth-tip flux per ampere conductor per centimetre length of slot for stator and rotor

As far as the fractional pitch effect is concerned, this element is exactly on a par with that part of the slot leakage which crosses the slots above the conductors, since it is wholly common to both coil sides.

Coil-end Leakage

The coil-end reactance may be expressed as follows—

$$x_f = 2\pi f \times \phi_f \times \frac{p}{2} \times \left(\frac{N}{p}\right)^2 \times l_e \times 10^{-8} \quad . \quad (20.31)$$

where ϕ_f = the flux per ampere-conductor per centimetre length of the whole phase bundle of coil ends

p = number of poles

N = conductors per phase

l_e = length of two ends of one coil

* d_1 = distance from the bottom of the slot to top of upper conductor.
d_2 = distance from top of upper conductor to top of slot.
w = width of slot; b = lower conductor.
a = upper conductor.

l_e varies with the coil pitch in a simple manner and ϕ_f in a manner not so obvious.

For a circular coil in air, ϕ_f the flux linked with 1 cm of the coil as a whole per ampere distributed uniformly throughout the section of the coil is proportional to the log of the ratio of coil diameter to the diagonal of its cross-section, provided this ratio is large.

For coils, such as are used on induction motors, this relation holds in a general way only, especially when the mutual inductive effect of neighbouring coils is taken into account and the pitch is fractional.

Within practical limits, ϕ_f would increase in approximate logarithmic relation to the coil pitch, were it not for mutually inductive effects of adjacent phases, and the curve representing this relation would tend to a zero which is *not* that of the pitch.

For pitches less than unity, the mutually inductive effect decreases in about the same ratio as the self-inductive effect, and this does not change much the general shape of the curve.

Thus, the curve showing the relation between ϕ_f and the coil pitch should be logarithmic in its general character, tending towards zero at some small (not zero) pitch and falling increasingly below the log curve for pitches greater than unity.

Belt Leakage

The belt-leakage reactance

$$X_b = 2\pi f \phi_b p \left(\frac{N}{h}\right)^2 . l \times 10^{-8} \quad . \quad . \quad (20.32)$$

ϕ_b = flux per ampere conductor per centimetre of belt, and is inversely proportional to the reluctance of the belt magnetic circuit, and proportional to $\sin^2$ of one-quarter of the angle of phase difference between the currents in the two opposing belts.

ϕ_b is proportional to the belt pitch and inversely proportional to the air-gap length.

The authors give, as the results of tests, a value for ϕ_f, the coil-end leakage flux, when all charged to the primary a value lying between 0·45 and 0·53 per centimetre length. The first value corresponds more nearly to a normal induction motor.

For squirrel-cage motors with bars extending well out from the core, the value of ϕ_f (all charged to the primary) is given as 0·395 per centimetre.

The curves in Fig. 20.2 (due to the authors mentioned) show the reduction factors due to chording.

CHAPTER XXI

Design for Medium-power Slip-ring Motor

THE method we have outlined for calculating the short-circuit current is fairly long, and it is desirable, in preliminary design work, to have a quick method of estimating this quantity, which will give us approximate values.

One method, used in design offices, that I have frequently employed and found satisfactory is to proceed as follows—

Ideal short-circuit current per phase in amperes

$$= \frac{1000}{z_1} \times \frac{\text{volts per conductor at 50 c/s}}{\text{short-circuit volts per conductor with 1000 ampere-conductors per slot}} \quad . \quad (21.1)$$

where z_1 = conductors in series per slot

The short-circuit volts per conductor, with 1000 ampere-conductors per slot, are divided into two parts—

(*a*) Short-circuit volts per conductor, for 1000 ampere-conductors in the slot

$$= 0{\cdot}04 \times \text{length of core in } \textit{inches} \quad . \quad . \quad (21.2)$$

where the number of rotor slots is *greater* than the number of stator slots.

The constant 0·04 is changed to 0·05 when the number of rotor slots is *less* than the number of stator slots.*

(*b*) Short-circuit volts per conductor (at 1000 ampere-conductors per slot) for the end-connections

$$= 0{\cdot}012 \times \frac{(\text{pole pitch in inches})^{1\cdot5}}{\text{slot pitch in inches}} \quad . \quad . \quad (21.3)$$

Total short-circuit volts per conductor (at 1000 ampere-conductors per slot)

$$= a + b \quad . \quad . \quad . \quad . \quad (21.4)$$

This is merely another form for the ideal short-circuit current, for we have seen that it is equal to the $\frac{\text{flux per pole}}{\text{leakage flux per pole per ampere}}$

---

* These values are given for 50 cycles

We have seen that the ideal short-circuit current per phase

$$= \frac{\phi}{0{\cdot}4\pi \times 2\sqrt{2} \times q_1 z_1 \times \lambda \times f_1}$$

$$= \frac{\text{flux per pole}}{\text{leakage flux per pole per ampere}}$$

Now $$\bar{E} = \frac{\pi}{\sqrt{2}} \times \hat{\phi} \times Z_1 \times f \times f_1 \times 10^{-8}$$

$\hat{\phi}$ = maximum flux per pole

Z_1 = conductors in series per phase

f = frequency

f_1 = winding factor = product of breadth factor and coil-span factor

$$\therefore \qquad \hat{\phi} = \frac{\bar{E} \times \sqrt{2} \times 10^8}{\pi \times Z_1 \times f \times f_1}$$

Leakage flux per pole per ampere

$$= 0{\cdot}4\pi \times 2\sqrt{2} q_1 z_1 \times \lambda \times f_1$$

q_1 = slots per pole per phase

z_1 = conductors per slot in series

p = poles

If $\bar{I}_{sci}$ = ideal short-circuit current

Then $$\bar{I}_{sc} \times 0{\cdot}4\pi 2\sqrt{2} q_1 z_1 \times f_1 \times \lambda = \hat{\phi}$$

Coefficient of self-inductance per phase

$$= L \text{ henrys} = 0{\cdot}4\pi \times \frac{(q_1 z_1)^2}{10^8} \lambda \times p \times f_1^2$$

Voltage generated on short-circuit per phase per ampere

$$= L\omega \times 1 = 0{\cdot}4\pi \times \frac{(q_1 z_1)^2}{10^8} \times \lambda \times p \times 2\pi f \times f_1^2$$

Short-circuit voltage per conductor for 1 A flowing

$$= \frac{L\omega \times 1}{f_1 q_1 z_1 \times p} = \frac{0{\cdot}4\pi q_1 z_1 \lambda \times 2\pi f \times f_1}{10^8}$$

Since $Z_1 = q_1 z_1 p f_1$

= number of effective conductors in series per phase

Volts per effective conductor

$$= \frac{E}{Z_1 f_1} = \frac{\pi \hat{\phi} \times f}{\sqrt{2} \times 10^8}$$

Short-circuit voltage per conductor for 1 A flowing

$$= 0{\cdot}4\pi q_1 z_1 \lambda \times 2\pi f \times f_1 \times \frac{1}{10^8}$$

Now the leakage pole per pole per ampere

$$= 0{\cdot}4\pi 2\sqrt{2}\, q_1 z_1 \lambda f_1 = \phi_l$$

Therefore, short-circuit voltage per conductor for 1 A flowing

$$= \frac{\phi_l}{2\sqrt{2}} \times \frac{2\pi f}{10^8}$$

$$\therefore \quad \frac{\text{volts per effective conductor}}{\text{short-circuit volts per conductor for 1 A flowing}}$$

$$= \frac{\pi \hat{\phi} \times f \times \sqrt{2}}{\sqrt{2} \times \phi_l \pi f} = \frac{\hat{\phi}}{\phi_l} = I_{sci}$$

Now our formula gives the short-circuit volts per conductor for 1000 ampere-conductors per slot. The current for 1000 ampere-conductors per slot

$$= \frac{1000}{z_1}$$

and it is obvious that the short-circuit volts per conductor *with* 1 *A flowing*

= short-circuit volts per conductor with 1000 ampere conductors per slot divided by $\frac{1000}{z_1}$

Therefore, ideal short-circuit ampere per phase

$$= \frac{\text{volts per conductor}}{\text{short-circuit volts per conductor at 1000 ampere-conductors per slot}} \times \frac{1000}{-}$$

It will be noted that the short-circuit volts per conductor for the slots and end connections *are given for* 50 *c/s*. These values obviously vary *with the frequency*.

It is obvious that this formula cannot give us accurate values, for the slot leakage and zigzag leakage are bunched together, and no account is taken of the widely varying shapes of slot. In starting a new design, it is invaluable, for one can obtain, by its use, the windings to give us the overload capacity desired at an early stage, and thus it saves a great deal of time. I have found it good enough for ordinary use. Another method, also empirical, but based on test results, is that due to Mr. G. Fletcher. It is as follows—

Ampere per phase, at $\cos\phi$

$$= 0{\cdot}5 \text{ at short-circuit}$$

$$= \frac{\text{diameter of rotor in inches} \times \text{mean gap density in lines per square centimetre}}{\text{conductors per phase}\left[K_1\left(\dfrac{1}{s_1}+\dfrac{1}{s_2}\right) + K_2\,\dfrac{\text{length of end connection}}{\text{length of embedded conductor}}\right]} \quad . \quad . \quad (21.5)$$

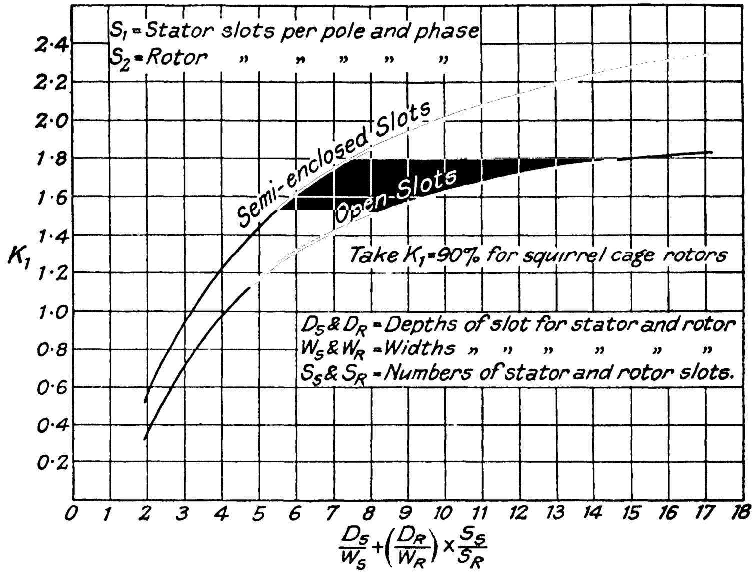


FIG. 21.1

The constant K_1 is given in Fig. 21.1.
The constant K_2 is given in tabular form below—

End Winding Constants

Type of Winding		K_2
Stator	Rotor	
Mush	Mush	0·240
Mush	Squirrel-cage	0·170
Mush	Diamond	0·220
Mush	Concentric	0·250
Concentric	Mush	0·250
Concentric	Squirrel-cage	0·175
Concentric	Concentric	0·260
Concentric	Diamond	0·240
Diamond	Mush	0·210
Diamond	Squirrel-cage	0·140
Diamond	Diamond	0·200
Diamond	Concentric	0·215

Another useful curve for the leakage permeance of the overhang of bar windings is given in Fig. 21.2.

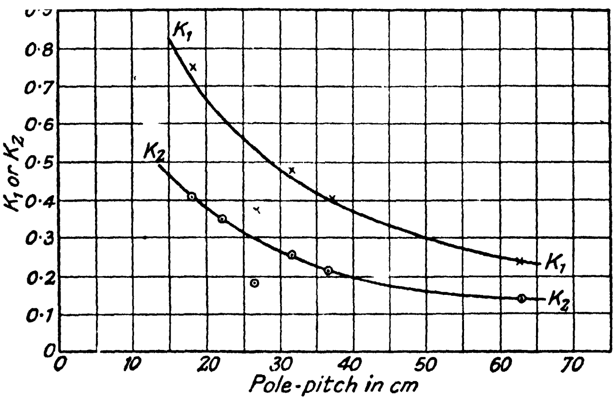


FIG. 21.2

To calculate the actual short-circuit current per phase, we need to know the stator and rotor resistances per phase (*see* Fig. 21.3).

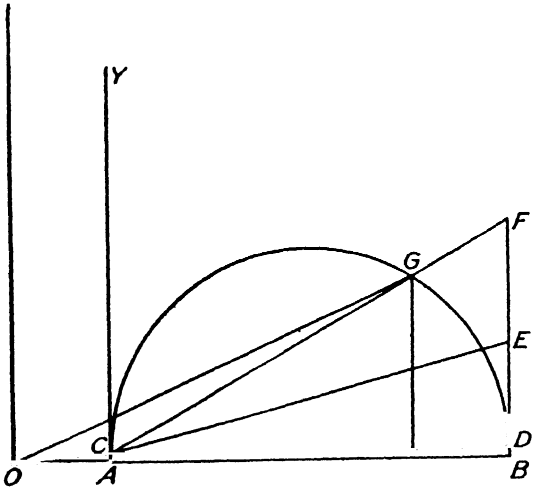


FIG. 21.3

OB = ideal short-circuit current per phase.
Set up, from B, a vertical line and make

$$DE = \frac{OB^2 \times R_s}{V} \qquad (21.6)$$

where R_s = stator resistance per phase (hot)
V = applied volts per phase

Make $$EF = \frac{\left(AB \times \frac{T_s}{T_r}\right)^2 \times R_r}{V} \quad . \quad . \quad . \quad (21.7)$$

where R_r = rotor resistance per phase (hot)
T_s = effective turns per phase in stator
T_r = effective turns per phase in the rotor

Join *OF*; then *OG* is the actual short-circuit current per phase in the stator.

CALCULATION OF LOSS IN SQUIRREL-CAGE ROTORS

Current per bar at short-circuit (ideal)

$$= AB \times \frac{\text{total conductors in series on stator}}{\text{total number of bars on rotor}} \quad . \quad (21.8)$$

This should, of course, be multiplied by the breadth factor and coil-span factor for the stator winding.

Current per bar (at ideal short-circuit)

$$= AB \times \frac{m_1 Z_1}{Q_2} \quad . \quad . \quad . \quad . \quad . \quad (21.9)$$

where m_1 = number of phases on the stator
Z_1 = conductors in series per phase
Q_2 = number of rotor bars

Loss in bars

$$= \left(AB \times \frac{m_1 Z_1}{Q_2}\right)^2 \times \text{resistance of rotor bars (hot)} \quad . \quad (21.10)$$

Resistance of rotor bars (hot) in ohms

$$= \frac{0{\cdot}017 \times Q_2 \times l_r}{a_r} \times 1{\cdot}2 \quad . \quad . \quad (21.11)$$

where l_r = length of bar in *metres*
a_r = area of rotor bar in *square millimetres*

Current and Loss in End Rings

Average current in each bar

$$= \frac{\text{effective current per bar}}{1{\cdot}11} \quad . \quad . \quad (21.12)$$

Maximum current in end rings

$$= \text{average current per bar} \times \frac{\text{bars per pole}}{2} \quad . \quad (21.13)$$

r.m.s. current *per ring*

$$= \frac{\text{effective current per bar}}{1{\cdot}11} \times \frac{\text{bars}}{2\ \text{poles}} \times \frac{1}{\sqrt{2}} \quad . \quad (21.14)$$

Resistance of copper ring when hot

$$= \frac{0{\cdot}017 \times \pi \times D_r}{a_{\text{ring}}} \times 1{\cdot}2 \quad . \quad . \quad . \quad (21.15)$$

where D_r = mean diameter of end ring *in metres*

a_{ring} = area of end ring in *square millimetres*

The constant 0·017 is for copper. If brass is used, the constant will be 0·0765.

TYPICAL DESIGNS

It is proposed here, and in the following chapters, to give typical designs of induction motors, to illustrate the application of the principles set out in previous chapters. In all cases examples are given of machines which have been built and are in service. The designs illustrated deal with both squirrel-cage and slip-ring types, and include both fractional h.p. machines as well as those of large output.

Design of Slip-ring Induction Motor

Specification: output = 250 b.h.p.; three-phase; 50 c/s; 440 V; 1500/1480 r.p.m.

Temperature rise not to exceed 40°C, by thermometer, after 6 hr full-load run.

Calculation—

$$\text{Number of poles} = \frac{120 \times \text{frequency}}{\text{r.p.m.}_{(\text{syn})}} = \frac{120 \times 50}{1500} = 4$$

$$D^2L = \frac{4{\cdot}06 \times 10^{11} \times \text{b.h.p.}}{B \times q \times \text{r.p.m.} \times \cos\theta \times \eta}$$

$$= \frac{4{\cdot}06 \times 10^{11} \times 250}{4500 \times 300 \times 1480 \times 0{\cdot}9 \times 0{\cdot}9} = 63\,000\ \text{cm}^3$$

B assumed = 4500; $q = 300$; $\cos\phi = 0{\cdot}9$; $\eta = 0{\cdot}9$.

For best power factor

$$\frac{\tau}{L} = \frac{18}{\tau}$$

$$\therefore \qquad \tau = \frac{\pi D}{p} \text{ and } \tau^2 = \frac{\pi^2 D^2}{p^2}$$

$$\therefore \qquad \frac{\pi^2 D^2}{p^2 L} = 18$$

$$\therefore \qquad D^2 = \frac{18 p^2 L}{\pi^2} = 1{\cdot}82 p^2 L$$

$$D = 1{\cdot}35 p\sqrt{L}$$

$$p = 4$$

$$D = 5{\cdot}4\sqrt{L}$$

$$29{\cdot}2 L^2 = 63\,000$$

$$L^2 = 2160$$

$$L = 46{\cdot}5 \text{ cm}$$

$$D = 36{\cdot}6 \text{ cm}$$

We see that a small-diameter machine, of relatively great core length is needed for best power factor. Such a machine could be built, but it would be better, from the standpoint of cooling qualities, to use a shorter core length and larger diameter. On four-pole machines, this can be done without losing much in the way of power factor. Actually this machine was built with a diameter of 45 cm and a core length of 34 cm gross, with four vent ducts, each 1 cm wide. A vent duct is usually provided for every 7 cm of core.

The apparent air-gap area per pole

= pole pitch × gross core length

The pole pitch $= \frac{\pi \times 45}{4} = 35{\cdot}4$ cm

The apparent air-gap area per pole

$= 35{\cdot}4 \times 34 = 1200 \text{ cm}^2$

Assuming an apparent air-gap density of 4500 lines/cm^2—which may need to be modified—we have the flux per pole

$= 4500 \times 1200$

$= 5{\cdot}4 \times 10^6$ lines

We have now to choose the number of stator slots. On a machine, with a pole pitch of 35·4 cm, we should expect to use 5 or 6 slots per pole per phase. If we choose 18 slots per pole, we get a slot-pitch of about 2 cm, and this is very suitable. This gives us 72 stator slots.

From our e.m.f. equation, we get the conductors in series per phase required, viz. Z,

$$E = 2{\cdot}12 \times \Phi \times Z \times f \times 10^{-8}$$

Assume delta connection.

Applied volts per phase = 440.

Allowing 2 per cent resistance drop

$$E = 430 \text{ approximately}$$

$$\therefore \qquad Z = \frac{430 \times 10^8}{2{\cdot}12 \times 5{\cdot}4 \times 10^6 \times 50} = 75$$

Slots per phase = 24.

We will try 3 conductors per slot, which will give us 72 conductors in series per phase.

With $Z = 72$, $\phi = 5{\cdot}62 \times 10^6$.

Before proceeding, we will check the overload capacity with this winding. Here the great advantage of the rough formula for the short-circuit current shows itself. We will save ourselves a great deal of work by its use.

Volts per conductor at 50 V

$$= \frac{440}{72} = 6{\cdot}12$$

Short-circuit volts per conductor, at 1000 ampere-conductors per slot,

$$(a) \qquad = 0{\cdot}04 \times 13{\cdot}4 = 0{\cdot}536$$

$$(b) \qquad = 0{\cdot}012 \times \frac{(13{\cdot}9)^{1{\cdot}5}}{0{\cdot}756} = 0{\cdot}795$$

Total short-circuit volts per conductor at 1000 ampere-conductors per slot

$$= a + b = 1{\cdot}331$$

Ideal short-circuit current *per phase*, at 1000 ampere-conductors per slot

$$= \frac{1000}{z_1} \times \frac{6{\cdot}12}{1{\cdot}331}$$

where z_1 = conductors per slot

Therefore, ideal short-circuit current *per line*

$$= \sqrt{3} \times \frac{1000}{3} \times \frac{6{\cdot}12}{1{\cdot}331} = 2650$$

Full-load line current

$$\frac{250 \times 746}{\sqrt{3} \times 440 \times 0{\cdot}92 \times 0{\cdot}92} = 290 \text{ A}$$

Magnetizing current = 73 approximately.

Therefore, actual short-circuit current (approximate line)

$$= (2650 - 73) \times 0{\cdot}866$$

$$= 2230 \text{ A (line)}$$

Maximum horse-power

$$= \frac{\sqrt{3} \times 440 \times 2230}{2\cdot8 \times 746} = 812 = 3\cdot25 \times \text{full load}$$

The power factor, at short-circuit, has been taken as 0·4. It would, probably, be lower, say, 0·25 to 0·3.

Now $$\sigma = \frac{73}{2650} = 0\cdot0276$$

A glance at the power-factor curves, already given, will show that this machine would have a good power factor, at full load, with this value of σ, but at light loads, it would fall off more rapidly than it ought.

It is clear that the overload capacity is too great. We will increase the number of turns per coil from 3 to 3·5, and this will reduce the maximum horse-power from 812 to $812 \times \left(\frac{3}{3\cdot5}\right)^2 = 595$. This will be quite suitable.

We will use 7 turns per coil, and connect each phase in two parallel circuits per phase, with the phases connected in delta. This will give us the equivalent of 3·5 turns per coil.

The value of ϕ is

$$\frac{430 \times 10^8}{2\cdot12 \times 24 \times 3\cdot5 \times 50} = 4\cdot84 \times 10^6$$

The full-load *line* current = 290 A approximately.

$$\text{Full-load amperes per phase} = \frac{290}{\sqrt{3}} = 167\cdot5 \text{ A}$$

Current per circuit = 83·75 A.

Assuming a current density in the conductors of 3·3 A/mm^2—a conservative value—we have the section of conductor of

$$\frac{83\cdot75}{3\cdot3} = 25\cdot4 \text{ mm}^2$$

If we use 8 wires in parallel, and each wire of 2 mm diameter bare, d.c.c. to 2·3 mm, the sectional area of the 8 wires, constituting one effective conductor, equals 25·12 mm^2. Since we are using a single-layer concentric winding there will be 56 wires per slot.

Space taken up by 56 wires

$$= 56 \times 2\cdot3 \times 2\cdot3 = 297 \text{ mm}^2$$

Allowing a space factor of 0·75, space required for the wires

$$= \frac{297}{0\cdot75} = 396 \text{ mm}^2$$

For 440 V, the slot insulation will consist of presspahn and leatheroid of total thickness *per side* of 0·7 mm. After trial, the slot shown is suitable.

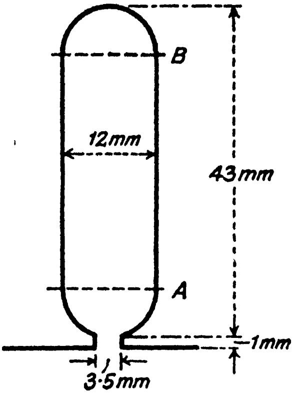


FIG. 21.4

The stator slot has semicircular ends.

Diameter at $A = 450 + 14$ mm $= 464$ mm

Pitch of slot at $A = \dfrac{\pi \times 464}{72} = 20{\cdot}3$

Width of tooth at $A = 20{\cdot}3 - 12$ mm $= 8{\cdot}3$ mm

Area of teeth per pole at A

$$= 0{\cdot}83 \times 18 \times 30 \times 0{\cdot}91 = 409 \text{ cm}^2$$

$$\beta_{60} \text{ at } A = \frac{4{\cdot}84 \times 10^6}{409} \times 1{\cdot}28 = 15\,100 \text{ lines/cm}^2$$

Area, at B, per pole $= 537$ cm^2

$$\beta_{60} \text{ at } B = 11\,500$$

These are suitable densities, and the slot will suit quite well. Assuming a flux density in the stator core of 7500 lines/cm^2, the area of core

$$= \frac{2{\cdot}42 \times 10^6}{7500} = 322 \text{ cm}^2$$

Let R = radial depth of core behind the stator teeth,

then $R \times 30 \times 0{\cdot}91 = 322$

$$R = 11{\cdot}75 \text{ cm}$$

Outside diameter of stator core

$$= 450 + 88 + 235 \text{ mm}$$
$$= 773 \text{ mm}$$

Rotor. A rotor winding, with 2 bars per slot, gives the soundest mechanical job. We might use a wave winding with a fractional number of slots per pole per phase, opened out in six places, and the opposite phases connected in series. In the present case, 7 slots per pole per phase was chosen.

The number of rotor slots = 84.

The rotor current per phase will be

$$\frac{250 \times 746}{3 \times 430 \times 0{\cdot}9} \times \frac{72 \times 3{\cdot}5}{84 \times 2} = 241 \text{ A}$$

and the volts between rings at start, approximately,

$$= \frac{250 \times 746}{\sqrt{3} \times 241 \times 0{\cdot}9} = 498 \text{ V}$$

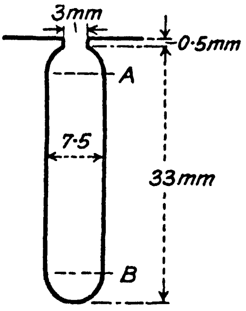


FIG. 21.5

Since the iron loss is small at full load, we may use a current density of 4 A/mm², and even higher.

Area of rotor bar $= \frac{241}{4} = 60{\cdot}25 \text{ mm}^2$

The rotor slot shown in Fig. 21.5 is suitable.

The slot has semicircular ends and the bar has one semicircular end as shown in Fig. 21.6.

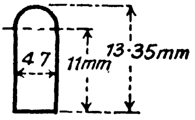


FIG. 21.6

Area of bar $= 60{\cdot}37 \text{ mm}^2$

There will be a slot lining of 0·15 mm thickness, and the insulation on the bars will be of paper 1 mm thick, ironed on to the coil.

Area of teeth at A per pole = 520 cm²

$$\beta_{60} \text{ at } A = 11\,900$$

$$\beta_{60} \text{ at } B = 15\,300$$

Area of rotor core

$$= \frac{2{\cdot}42 \times 10^6}{9000} = 268 \text{ cm}^2$$

Internal diameter of rotor

$$= 447{\cdot}4 - 67 - 196 = 184 \text{ mm}$$

Diameter of rotor shaft in inches

$$= 9\sqrt[3]{\frac{\text{h.p.}}{\text{r.p.m.}}} = 9\sqrt[3]{\frac{250}{1450}}$$

$$= 5 \text{ in.} = 127 \text{ mm}$$

Calculation of Magnetizing Current

We will first find the air-gap area per pole, corrected for slots and ventilating ducts. For this purpose, Carter's factor, for semi-enclosed slots, given in Fig. 21.7, is used.

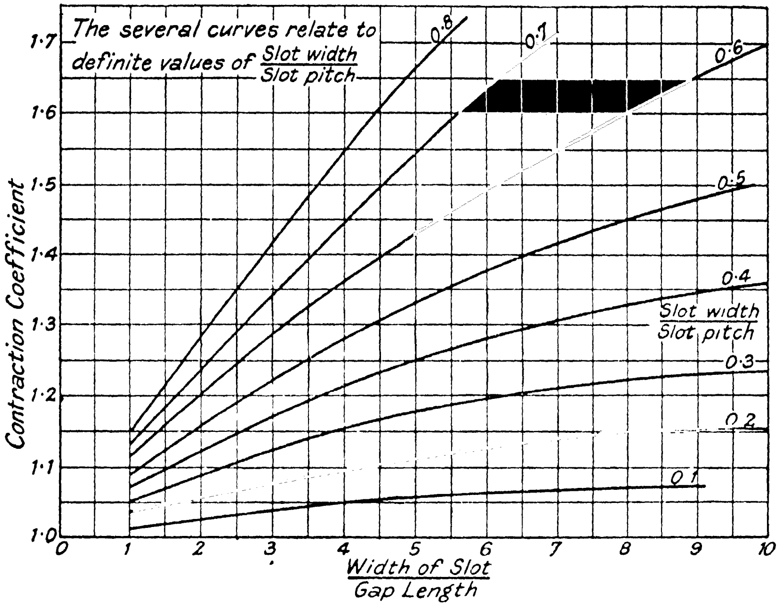


FIG. 21.7. FOR SEMI-ENCLOSED SLOTS

These curves, plotted by the late Prof. Miles Walker, from Carter's formula, are very useful.

The curves for "open" slots are given in Fig. 21.8.

We have,

$$\frac{\text{width of stator slot opening at the air-gap}}{\text{air-gap length}} = \frac{3{\cdot}5}{1{\cdot}3} = 2{\cdot}69$$

$$\frac{\text{width of stator slot opening at the gap}}{\text{pitch of stator slot}} = \frac{3{\cdot}5}{19{\cdot}6} = 0{\cdot}178$$

Carter's factor for stator slots $= 1{\cdot}085$

$$\frac{\text{width of rotor slot opening at the gap}}{\text{air-gap length}} = \frac{3{\cdot}0}{1{\cdot}3} = 2{\cdot}3$$

$$\frac{\text{width of rotor slot opening at the gap}}{\text{pitch of rotor slots at the gap}} = \frac{3{\cdot}0}{16{\cdot}75} = 0{\cdot}179$$

Carter's factor for rotor slots = 1·075

$$\frac{\text{width of vent duct}}{\text{gap length}} = \frac{10{\cdot}0}{1{\cdot}3} = 7{\cdot}7$$

$$\frac{\text{width of vent duct}}{\text{pitch of vent ducts}} = \frac{1{\cdot}0}{6{\cdot}8} = 0{\cdot}147$$

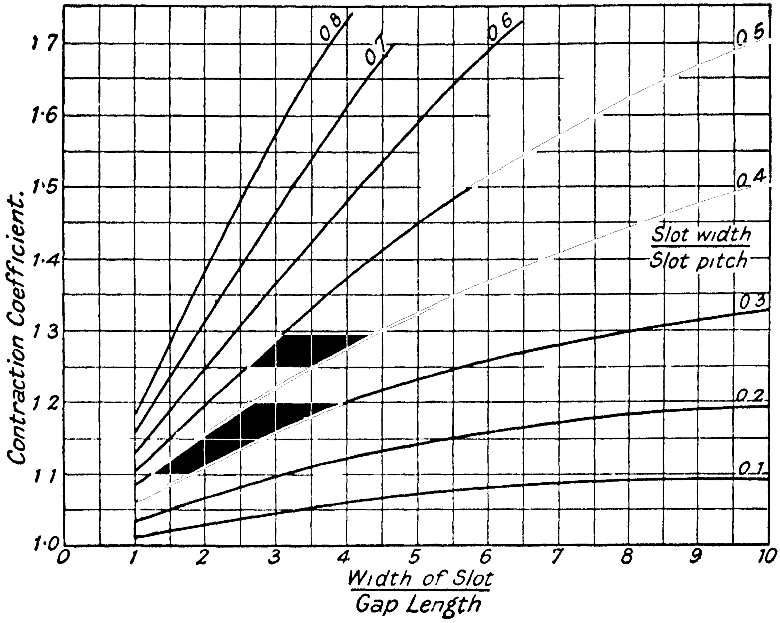


Fig 21.8. For Open Slots

Carter's factor for vents = 1·1

Actual air-gap area per pole

$$= \frac{\text{apparent gap area per pole}}{\text{product of Carter's factors}}$$

$$= \frac{1200}{1{\cdot}085 \times 1{\cdot}075 \times 1{\cdot}1} = 935 \text{ cm}^2$$

Magnetizing current. We require a magnetization curve, and this is given for tapered teeth in Fig. 21.9.

Total ampere-turns at 60° *per pole-pair* = 1810.

Magnetizing current *per line wire* $= \frac{\sqrt{3} \times 1810}{2{\cdot}12 \times 6 \times 3{\cdot}5} = 70{\cdot}5$ A.

Part	Area (cm²)	Length of Path (cm)	Flux per pole	B_{60}	$\bar{A}_{60}$ per pole pair
Stator core .	322	34·40 (⅔ of pole pitch in core)	$2{\cdot}42 \times 10^6$	7 500	113
Stator teeth	(A) 409 (B) 537	8·80	$4{\cdot}84 \times 10^6$	15 000 11 400	136
Air-gap . .	935	0·26	$4{\cdot}84 \times 10^6$	6 625	1370
Rotor teeth	(A) 520 (B) 405	6 70	$4{\cdot}84 \times 10^6$	11 900 15 300	113
Rotor core .	268	14·80	$2{\cdot}42 \times 10^6$	9 000	78
					1810

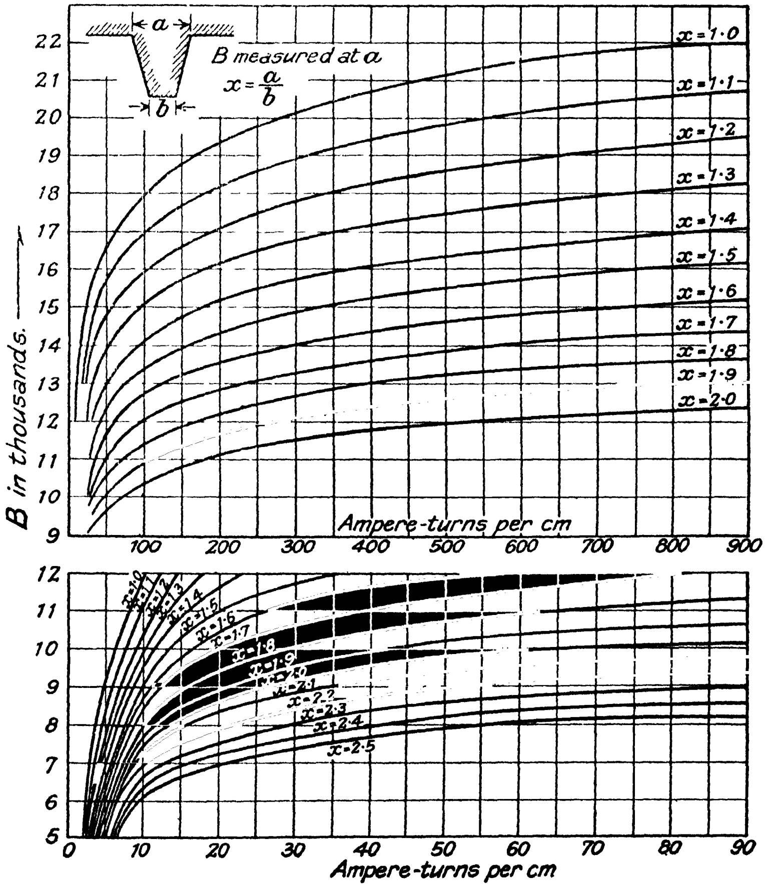


FIG. 21.9. MAGNETIZATION CURVES FOR TAPERED TEETH
Based on Leysaght's, Ltd., samples

The Ideal Short-circuit Current

Stator-slot permeance per centimetre length of core

$$= \frac{31}{36} + 0{\cdot}7854 + \frac{1{\cdot}0}{3{\cdot}5} = 0{\cdot}86 + 0{\cdot}7854 + 0{\cdot}286 = 1{\cdot}9314 = \lambda_s$$

Rotor-slot permeance per centimetre length of core

$$= \frac{25{\cdot}5}{22{\cdot}5} + 0{\cdot}7854 + \frac{0{\cdot}5}{2{\cdot}5} = 1{\cdot}13 + 0{\cdot}7854 + 0{\cdot}2 = 2{\cdot}115 = \lambda_r$$

$$\lambda_1 = \left[\frac{1{\cdot}93}{0{\cdot}96 \times 6} + \frac{2{\cdot}115}{7}\right] \times 30 = 19{\cdot}1$$

$$\lambda_2 = 0{\cdot}73(0{\cdot}36\tau + A) = 0{\cdot}73(0{\cdot}36 \times 35{\cdot}4 + 22) = 25{\cdot}3$$

$$\lambda_3 = \frac{L\varepsilon\tau}{48 \times \delta \times q_1 q_2} = \frac{34 \times 1{\cdot}715 \times 35{\cdot}4}{48 \times 0{\cdot}13 \times 6 \times 7} = 7{\cdot}9$$

$$\therefore \quad \lambda = \lambda_1 + \lambda_2 + \lambda_3 = 19{\cdot}1 + 25{\cdot}3 + 7{\cdot}9 = 52{\cdot}3$$

$$\underset{\text{(ideal)}}{I_{sc}} \text{ (line)} = \sqrt{3} \times \frac{4{\cdot}84 \times 10^6}{3{\cdot}55 \times 6 \times 3{\cdot}5 \times 52{\cdot}3} = 2145 \text{ A}$$

Our rough method gives—

Volts per conductor at 50 c/s $= \dfrac{440}{24 \times 3{\cdot}5} = 5{\cdot}24$

Short-circuit volts per conductor at 1000 ampere-conductors per slot = 1·331

Short-circuit amperes per line

$$= \sqrt{3} \times \frac{1000}{3{\cdot}5} \times \frac{5{\cdot}24}{1{\cdot}331} = 1950$$

The agreement is fairly good.

The dispersion coefficient $= \dfrac{70{\cdot}5}{2145} = 0{\cdot}0329$

Rotor Winding. Length of mean turn of rotor coil

$$= 2(34 + 50) = 168 \text{ cm}$$

Resistance per phase at 15°C

$$= \frac{0{\cdot}017 \times 28 \times 1{\cdot}68}{60{\cdot}37} = 0{\cdot}0132 \text{ ohm}$$

Resistance per phase (hot) = 0·0156 ohm

Weight of rotor copper

$$= 0{\cdot}0197 \times 84 \times 1{\cdot}68 \times 60{\cdot}37 = 168 \text{ lb}$$

Stator Winding. Length of mean turn

$$= 2(34 + 70{\cdot}5) = 209 \text{ cm}$$

Resistance per phase at 15°C

$$= \frac{0{\cdot}017 \times 12 \times 7 \times 2{\cdot}09}{25{\cdot}12 \times 4} = 0{\cdot}0296 \text{ ohm}$$

Resistance per phase (hot) $= 0{\cdot}035$ ohm

Weight of stator copper (total)

$$= 0{\cdot}0197 \times 36 \times 2{\cdot}09 \times 7 \times 25{\cdot}12 = 261 \text{ lb}$$

Power Factor at Short-circuit. Stator loss per phase with ideal short-circuit current

$$= \left(\frac{2145}{\sqrt{3}}\right)^2 \times 0{\cdot}035 = 53\,830 \text{ W}$$

Watt component of current corresponding to this

$$= \frac{53\,830}{440} = 122{\cdot}5 \text{ A}$$

Rotor ideal short-circuit current per phase

$$= 1195 \times \frac{72 \times 3{\cdot}5}{84 \times 2} = 1800 \text{ A}$$

Rotor loss, per phase, with ideal short-circuit current

$$= 1800^2 \times 0{\cdot}0156 = 50\,600 \text{ W}$$

Watt component of current, corresponding to this loss per phase,

$$= \frac{50\,600}{440} = 115 \text{ A} = FE$$

(*See* Fig. 21.3.) $FD = 122{\cdot}5 + 115 = 237{\cdot}5$

AB, per phase, $= \dfrac{2145 - 70{\cdot}5}{\sqrt{3}} = 1195$

and $\tan \phi' = \tan FCD = \dfrac{237{\cdot}5}{1195} = 0{\cdot}198$

$\therefore \qquad \phi' = 11° \, 35'$

$\cos \phi_{sc}$ at short-circuit $= 0{\cdot}2$

Maximum horse-power

$$= \frac{\sqrt{3} \times 440 \times 2010}{2{\cdot}4 \times 746} = 860 \text{ h.p.}$$

$$\frac{\text{full load}}{\text{maximum load}} = \frac{250}{860} = 0{\cdot}29$$

$\cos \phi$, at full load from the curves $= 93{\cdot}5$ per cent.

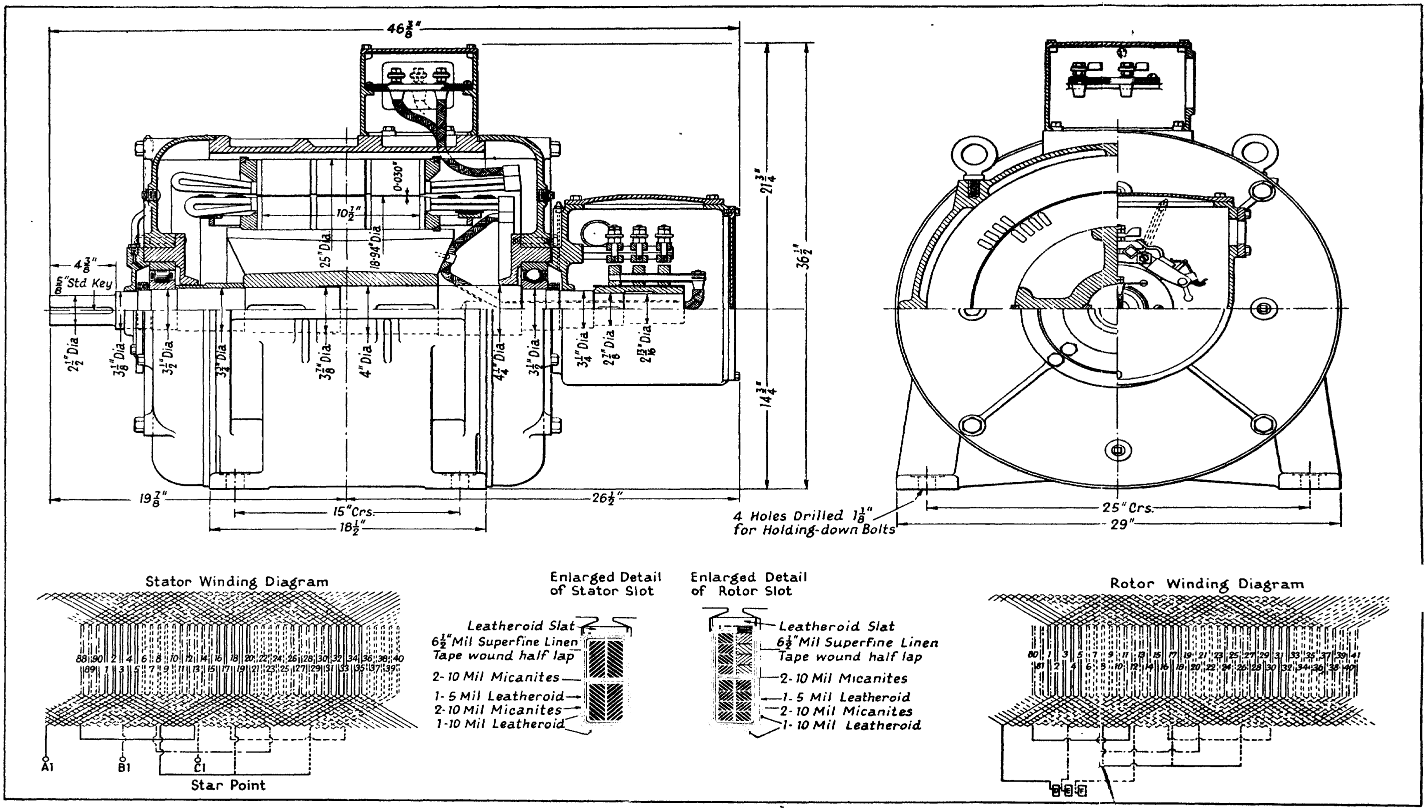


FIG. 21.11. A.C. SLIP-RING MOTOR, DETAILED DRAWING

(*Courtesy Clarke, Chapman & Co., Ltd.*)

The overload capacity is rather great, and the performance, at light loads, would be improved by increasing the number of turns per coil to 4. Four turns per coil would have given ample overload capacity. We have here an illustration of the fact that it is easy to get both overload capacity and also high power factor on a fairly large machine at high speed. We should find matters very different, were we to attempt to design a small machine at low speed.

Efficiency at Full Load. Stator copper loss

$$= 3 \times 164^2 \times 0\cdot0035 = 2830\ \text{W}$$

Rotor copper loss

$$= 3 \times 241^2 \times 0\cdot0156 = 2720\ \text{W}$$

Iron loss (teeth) = 1050 W
Iron loss (core) = 2130 W
Total iron loss = 3180 W

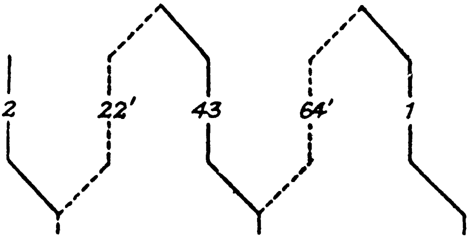


FIG. 21.10

Bearings. Journal 4 in. diameter and 10 in. long. Rubbing velocity

$$= 1515\ \text{ft/min}$$

Bearing loss (2 bearings)

$$= 0\cdot81 \times 4 \times 10 \times 15\cdot15^{1\,5} \times 2 = 3770\ \text{W}$$

Total losses = 12 500 W
Output = 186 500 W
Input = 199 000 W
Efficiency = 93·5 per cent

Slip, at full load

$$= \frac{2720}{192\,990} = 0\cdot0141 = 1\cdot41\ \text{per cent}$$

Full-load speed = 1480 r.p.m.

The stator will be connected in two parallel circuits per phase, and the phases will be connected in delta. The winding will be hemi-tropic, concentric type, with 1 coil group per *pair* of poles.

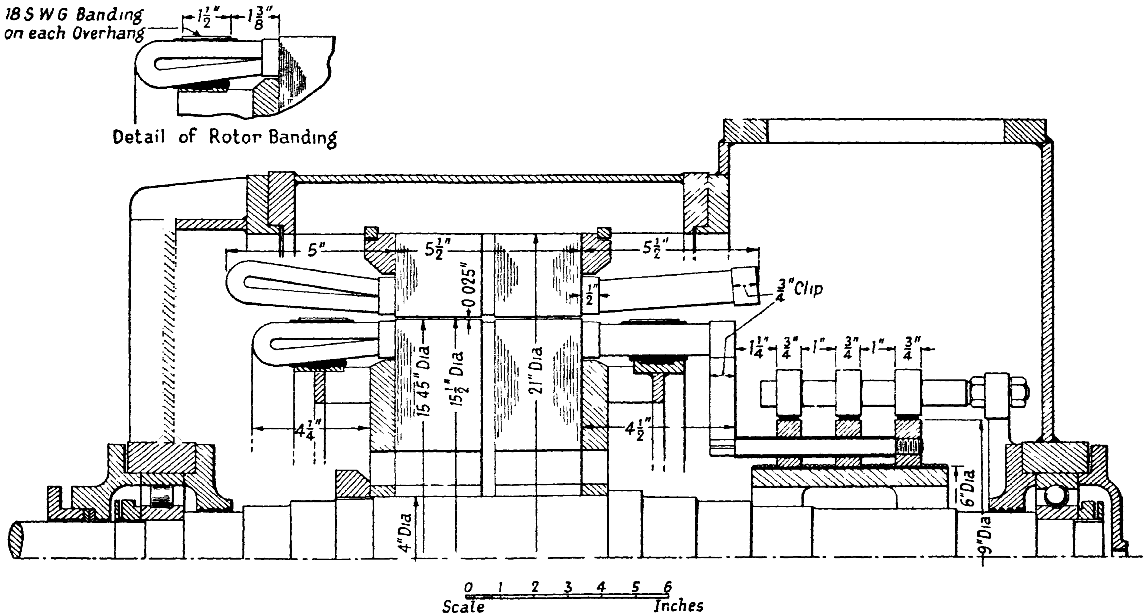


FIG. 21.12. DETAIL OF ROTOR BANDING IN MOTOR OF FIG. 21.11

(*Courtesy Clarke, Chapman & Co., Ltd.*)

Rotor Winding. Two bars per slot; 84 rotor slots; 168 bars; non-academic wave.

The bars are numbered, in the clockwise direction, at the slip-ring end.

Left-hand winding. Upper bars numbered 1, 2, 3, etc.; lower bars numbered 1′, 2′, 3′, etc.

Number of slots = 84.
Number of live bars = 168.
Slots per pole per phase = 7.
Step of winding—front = 21.
Step of winding—back = 21.
Abnormal step = 20.
Total number of lower bars, with abnormal step = 36.

Lower bars with abnormal step on front—

Phase I	*Phase II*	*Phase III*
22′–27′	78′–83′	50′–55′
43′–48′	15′–20′	71′–76′

Connections to slip-rings on upper bars—

Phase I	*Phase II*	*Phase III*
Bar 1	Bar 57	Bar 29

March of rotor winding is shown in Fig. 21.10.

Fig. 21.11 shows the design detail of an a.c. slip-ring motor, and Fig. 21.12 the detail of the rotor banding.

CHAPTER XXII

Design for Large Slip-ring Motor

THE design characteristics will be as follows: 3750 b.h.p.; 40 c/s, 2750 V; 400 r.p.m. synchronous, slip-ring type of induction motor.

$$\text{Poles} = \frac{120 \times \text{frequency}}{\text{r.p.m.}} = \frac{120 \times 40}{400} = 12$$

$$D^2L \text{ cm}^3 = \frac{4{\cdot}06 \times 10^{11} \times \text{b.h.p.}}{B \times q \times \text{r.p.m.} \times \cos\theta \times \eta}$$

$$\begin{aligned} B &= 4100 \text{ average} \\ q &= 310 \text{ ampere-conductors per centimetre} \\ \text{r.p.m.} &= 396 \text{ at full load} \\ \cos\theta &= 0{\cdot}86 \\ \eta &= 95{\cdot}5 \text{ per cent} \end{aligned}$$

$$\therefore \quad D^2L = \frac{4{\cdot}06 \times 10^{11} \times 3750}{4100 \times 310 \times 396 \times 0{\cdot}86 \times 0{\cdot}955} = 3{\cdot}68 \times 10^6 \text{ cm}^3$$

For best power factor $\frac{\tau}{L} = \frac{18}{\tau}$

$$\begin{aligned} \therefore \quad D &= 1{\cdot}35p\sqrt{L} = 16{\cdot}2\sqrt{L} \\ \therefore \quad 262L^2 &= 3{\cdot}68 \times 10^6 \\ L^2 &= 1{\cdot}42 \times 10^4 \\ \therefore \quad L &= 1{\cdot}19 \times 10^2 = 119 \text{ cm} \\ \text{and} \quad D &= 16{\cdot}2 \times 10{\cdot}9 = 177 \text{ cm} \end{aligned}$$

Actually this machine was built with $D = 205{\cdot}74$ cm and $L = 94$ cm, giving

$$D^2L = 205{\cdot}74^2 \times 94 = 3{\cdot}98 \times 10^6$$

No doubt from a cooling standpoint it is better to use the larger diameter and shorter core length, although the machine could well be built on the smaller diameter and would give better power factor.

We will chose $D = 205{\cdot}74$ cm and $L = 94$ cm.

There will be 14 ventilating ducts, each 1 cm wide.

Net iron length $= 80 \times 0{\cdot}91 = 72$ cm

Apparent air-gap area per pole

$$= \frac{\pi \times 205{\cdot}74}{12} \times 94 = 5050 \text{ cm}^2$$

Flux per pole $= 5050 \times 3760 = 19 \times 10^6$ lines

Pole pitch $= \dfrac{\pi \times 205{\cdot}74}{12} = 53{\cdot}8$ cm

We will use 12 slots per pole, giving a slot pitch of

$$\frac{53{\cdot}8}{12} = 4{\cdot}48 \text{ cm}$$

Number of stator slots $= 144$.

We will adopt a star-connected winding for this voltage.

Volts per phase $= \dfrac{2750}{\sqrt{3}} = 1585$

Now $E = 2{\cdot}12 \times Z \times \hat{\phi} \times f \times 10^{-8}$

$$\therefore \quad Z = \frac{1585 \times 10^8}{2{\cdot}12 \times 19 \times 10^6 \times 40} = 98$$

i.e. approximately 98 conductors in series per phase are required.

Now the number of slots per phase $= 48$.
Therefore, this suggests 2 conductors per slot.
With 2 conductors per slot, or 96 conductors in series per phase, the flux per pole

$$= \frac{1585 \times 10^8}{2{\cdot}12 \times 96 \times 40} = 19{\cdot}4 \times 10^6$$

Assuming a current density of $3{\cdot}1$ A/m², full-load stator current

$$= \frac{3750 \times 746}{\sqrt{3} \times 2750 \times 0{\cdot}86 \times 0{\cdot}955} = 710 \text{ A}$$

area of stator conductor

$$= \frac{710}{3{\cdot}1} = 229 \text{ mm}^2$$

Limiting the flux density in the teeth at minimum section to 10 500 lines/cm² average, we have—

width of tooth at minimum section

$$= \frac{19{\cdot}4 \times 10^6}{10\,500 \times 12 \times 72} = 2{\cdot}1 \text{ cm approximately}$$

pitch of teeth at minimum section

$$= \frac{\pi \times 213}{144} \; 4{\cdot}6 \text{ cm}$$

This gives a width of slot = 2·5 cm.

We will make the slot = 2·4 cm wide.

Now the slot insulation will consist of micanite troughs of 2 mm thickness for this voltage, so the width of copper will be

$$24 - 4 - 1 \text{ mm} = 19 \text{ mm}$$

and the depth of the copper (suitably laminated) will be 12 mm, giving an area of 228 mm^2.

The depth of slot will be, allowing for insulation and slack,

$$24 + 8 + 5 = 37 \text{ mm}$$

and our stator slot is shown in Fig. 22.1.

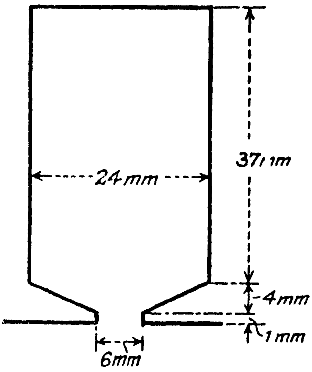


Fig. 22.1

Assuming a flux density in the core of 9200 lines/cm^2—

area of core

$$= \frac{\phi}{2 \times 9200} = \frac{19{\cdot}4 \times 10^6}{18\,400} = 1{\cdot}05 \times 10^3 \text{ cm}^2$$

radial depth of iron behind the teeth

$$= \frac{1050}{72} = 14{\cdot}6 \text{ cm}$$

external diameter of stator core

$$= 205{\cdot}74 + 8{\cdot}4 + 29{\cdot}2 = 243{\cdot}34 \text{ cm, say, } 244 \text{ cm}$$

L.M.T. of stator winding

$$= 2(122 + 78) \text{ cm} = 400 \text{ cm}$$

resistance per phase at 15°C

$$= \frac{0{\cdot}017 \times 48 \times 4{\cdot}0}{228} = 0{\cdot}0143$$

resistance (hot) $= 0{\cdot}0169\omega$

total weight $= 0{\cdot}0197 \times 144 \times 1 \times 4{\cdot}0 \times 228 = 2600$ lb

Rough Check of Overload Capacity

Volts per conductor at 50 c/s

$$= \frac{1585}{48 \times 2} \times \frac{50}{40} = 20{\cdot}6$$

Short-circuit volts per conductor—

(*a*) for slots $= 0{\cdot}040 \times 37{\cdot}0'' = 1{\cdot}480$

(*b*) end connection $= 0{\cdot}012 \times \dfrac{(21{\cdot}2)^{1{\cdot}5}}{1{\cdot}76} = \dfrac{0{\cdot}683}{\underline{\underline{2{\cdot}163}}}$

Ideal short-circuit current

$$= \frac{1000}{2} \times \frac{20{\cdot}6}{2{\cdot}163} = 4770 \text{ A}$$

Maximum horse-power (approximately)

$$= \frac{\sqrt{3} \times 2750 \times 4270}{2{\cdot}4 \times 746} = 11\,400 = 3 \text{ times full load}$$

This machine was intended for work requiring a large overload capacity, and so the above winding will suit. From the standpoint of sound mechanical construction and reliability, a two-bar per slot winding is best for the rotor.

We will use 7 slots per pole per phase for the rotor, which, with a three-phase winding, will give us 252 rotor slots.

The rotor current with all the conductors per phase in series

$$= \frac{3750 \times 746}{\sqrt{3} \times 2750 \times 0{\cdot}95} \times \frac{144 \times 2}{252 \times 2} = 373 \text{ A}$$

and the volts between rings at the start

$$= \frac{3750 \times 746}{\sqrt{3} \times 373 \times 0{\cdot}95} = 4550 \text{ approximately}$$

Obviously it would be better to use 4 parallel circuits per phase in the rotor, giving a rotor current of 1500 A and volts between rings at start = 1137.

This will be adopted. Since this machine has to be capable of dealing with heavy overloads, the current densities at normal full load will be kept lower than usual, and a density of 3·75 A/mm^2 will be used for the rotor conductors.

Current per conductor

$$= \frac{1500}{4} = 375 \text{ A}$$

Section of rotor conductor

$$= \frac{375}{3{\cdot}75} = 100 \text{ mm}^2$$

We will use a copper bar of 100 mm^2 section.

A bar 10 mm deep and divided into 5 strips, each 2 mm wide in parallel, will suit.

This bar will be insulated with micanite 1·5 mm thick. This leads to a rotor slot 14 mm wide (allowing for slack) and 28 mm in depth. The rotor slot is shown in Fig. 22.2.

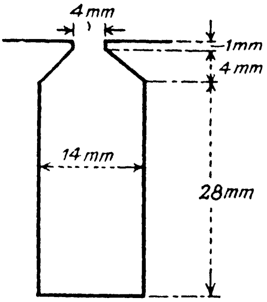


FIG. 22.2

L.M.T. of rotor coil = 3·76 m.

Resistance of rotor winding per phase at 15°C

$$= \frac{0{\cdot}017 \times 84 \times 1 \times 3{\cdot}76}{100 \times 16} = 0{\cdot}00335$$

Resistance per phase (hot)

$$= 0{\cdot}00396\omega$$

Weight of rotor copper

$$= 0{\cdot}0197 \times 252 \times 3{\cdot}76 \times 100 = 1870 \text{ lb}$$

MAGNETIZING CURRENT

Part	Area (cm^2)	Length of Path (cm)	$\phi \times 10^6$	B_{60}	A_{60}
Stator core . . .	1050	49·6	9·66	9 200	248·0
Stator teeth . .	1965	7·4	19·32	12 600	111·0
	1815	1·0	19·00	13 400	
	3360	—	—	7 150	6·5
Gap	4540	0·9	18·75	5 280	3810·0
Rotor teeth . .	3700	1·0	18·75	6 500	7·0
	1725	—	—	12 700	112·0
	1620	5·6	18·50	14 400	
Rotor core . . .	885	40·8	9·10	10 900	306·0

Total ampere-turns at 60° = 4600

$$\text{Magnetizing current} = \frac{4600}{2{\cdot}12 \times 4 \times 2} = 271 \text{ A}$$

The Ideal Short-circuit Current

$$\lambda_s = 1{\cdot}27 \qquad\qquad \lambda_r = 4{\cdot}29$$

$$\lambda_1 = \left[\frac{1{\cdot}27}{0{\cdot}96 \times 4} + \frac{4{\cdot}29}{7}\right] \times 87 = 82 \text{ cm}$$

$$\lambda_2 = 0{\cdot}6 \times 100 = 60$$

$$\lambda_3 = \frac{L\varepsilon\tau}{48 \times \delta \times q_1 \times q_2} = \frac{87 \times 1{\cdot}6 \times 53{\cdot}8}{48 \times 0{\cdot}45 \times 4 \times 7} = 12{\cdot}4$$

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 154{\cdot}4$$

Ideal short-circuit current

$$= \frac{19{\cdot}32 \times 10^6}{3{\cdot}55 \times 4 \times 2 \times 154{\cdot}4} = 4420$$

This also agrees very closely with our rough method.

Rotor Connection Diagram for 3750 h.p. Slip-ring Motor

Left-hand winding. Bars numbered in clockwise direction looking at slip-ring end.

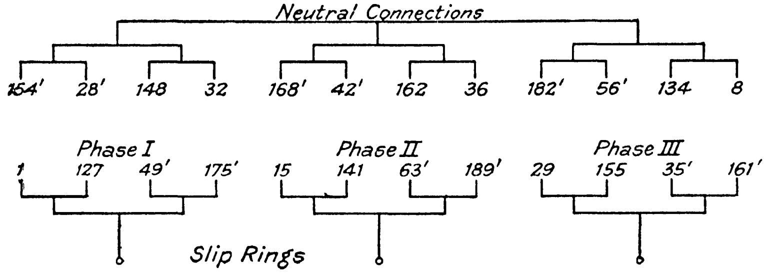


FIG. 22.3

Number of poles = 12.
Number of slots = 252.
Upper bars = 252 in number, numbered 1, 2, 3, etc.
Lower bars = 252 in number, numbered 1', 2', 3', etc.

Rotor Star-connected, with 4 Circuits in Parallel per Phase

Step of winding—back = 21.
Step of winding—front = 21.

Abnormal step of 20 on 84 lower bars.

$$\sigma = \frac{271}{4420} = 0{\cdot}061 \text{ and } \cos\phi = 86 \text{ per cent}$$

Losses: Full Load

Stator copper = 31·2 kW
Iron loss = 34·03 kW
Friction and windage loss = 25 kW
Total losses = 119·53 kW
Output kW = 2800 kW
Input kW = 2919·53 kW
Efficiency = 95·8 per cent
Slip = 1 per cent at full load

The stator conductor will be suitably laminated to reduce loss due to eddy currents.

CHAPTER XXIII

Design for Squirrel-cage Induction Motor

THIS time our motor will require to conform to the following ratings: 15 b.h.p.; three-phase; 50 c/s, 440 V; 1000 r.p.m. synchronous; $\cos\phi = 87{\cdot}5$ per cent; efficiency = 86 per cent.

Starting torque at full volts = full-load torque.

Temperature rise 40°C after 6-hr full-load run.

$$D^2L \text{ cm}^3 = \frac{4{\cdot}06 \times 10^{11} \times \text{b.h.p.}}{B \times q \times \text{r.p.m.} \times \cos\phi \times \eta}$$

$$= \frac{4{\cdot}06 \times 10^{11} \times 15}{4500 \times 215 \times 960 \times 0{\cdot}86 \times 0{\cdot}875}$$

$$= 8700 \text{ cm}^3$$

For best power factor

$$\frac{\tau}{L} = \frac{18}{\tau}$$

$$D = 1{\cdot}35p\sqrt{L}$$

$$p = 6$$

$$\therefore \quad D = 8{\cdot}1\sqrt{L}$$

$$\therefore \quad 65{\cdot}5L^2 = 8700$$

$$L^2 = \frac{8700}{65{\cdot}5} = 133$$

and

$$D = 27{\cdot}5 \text{ cm}$$

$$\therefore \quad L = 11{\cdot}5 \text{ cm}$$

This machine was built with $D = 25$ cm and $L = 14$ cm, which is quite close to the best proportions.

$$\text{Pole pitch} = \frac{\pi \times 25}{6} \text{ cm} = 13{\cdot}1 \text{ cm}$$

Apparent air-gap area per pole

$$= 13{\cdot}1 \times 14{\cdot}0 = 183 \text{ cm}^2$$

Flux per pole $= 183 \times 4500 = 0\cdot825 \times 10^6$

Conductors in series per phase

$$= \frac{430 \times 10^8}{2\cdot12 \times 0\cdot825 \times 10^6 \times 50} = 486$$

(allowing 10 V drop in stator winding at full load).

Using 3 slots per pole per phase in the stator, we have 54 stator slots, which gives us a suitable pitch of slot.

Conductors per slot $= \dfrac{486}{18} = 27$

The stator winding will be delta-connected

Full-load current (line) $= 19\cdot4$ A

Full-load current per phase $= 11\cdot2$ A

Actually 28 conductors per slot were used in this case.

Check on over-load capacity with 28 conductors per slot—

Volts per conductor at 50 c/s $= \dfrac{440}{18 \times 28} = 0\cdot87$

Short-circuit volts per conductor—

(*a*) for slot $= 0\cdot04 \times 5\cdot5$ (in.) $= 0\cdot22$

(*b*) end connection $= 0\cdot012 \times \dfrac{(5\cdot15)^{1\cdot5}}{0\cdot572} = 0\cdot258$

$$a + b = 0\cdot478$$

$$I_{sc} \text{ line} = \sqrt{3} \times \frac{1000}{28} \times \frac{0\cdot87}{0\cdot478} = 113 \text{ A}$$

Maximum horse-power

$$= \frac{\sqrt{3} \times 440 \times 93}{3\cdot0 \times 746} = 31\cdot8 = 2\cdot12 \text{ full-load h.p.}$$

Current per stator conductor at full load $= 11\cdot2$ A

Current density $= 3\cdot2$ A/mm^2

Area of conductor $= \dfrac{11\cdot2}{3\cdot2} = 3\cdot51$ mm^2

We will use 3 wires in parallel of 18 s.w.g. diameter bare

$= 1\cdot22$ mm d.c.c. to $1\cdot42$ mm

Space required in slot for conductors

$= 28 \times 3 \times 1\cdot42 \times 1\cdot42 = 170$ mm^2

Allowing a space factor of 0·75 for slot space required

$$= \frac{170}{0\cdot75} = 228 \text{ mm}^2$$

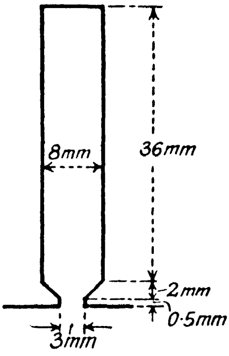


Fig. 23.1

The slot shown in Fig. 23.1 is suitable.

Slot insulation of presspahn and leatheroid troughs of 0·7 mm thickness per side will be used.

The stator will have the usual type of concentric winding of the hemi-tropic type.

L.M.T. of stator coil = 101 cm

Resistance per phase at 15°C

$$= \frac{0{\cdot}017 \times 9 \times 28 \times 1{\cdot}01}{3{\cdot}51} = 1{\cdot}23\ \Omega$$

Weight of copper = $52\frac{1}{2}$ lb

The length of air-gap = 0·5 mm

Number of Rotor Slots

The number of rotor slots will be chosen such that there will be 1 slot more or 1 slot less per pair of poles than in the stator. If we use 57 rotor slots, this condition will be satisfied. Much trouble is frequently experienced due to incorrect choice of slot ratio in squirrel-cage machines. The resultant torque-slip curve is compounded of the torque-slip curves due to the fundamental and those of the higher harmonics. In consequence of this, depressions in the torque-slip curve occur; or, in other words, the torque drops at some speed to a lower value than is necessary to continue the acceleration of rotor under the load or even when running unloaded. The torque may actually become negative. There may be more than one such point of low torque. These points are called saddle points. The trouble from saddle points is apt to be due to those harmonic fields which rotate in the same direction as the fundamental field, viz. the seventh, thirteenth, and higher harmonics. These forward harmonics will increase the torque at standstill, but reduce it seriously beyond the synchronous speed of these harmonics.

Again at standstill, locking points may occur when the numbers of stator and rotor slots have a common factor; and the higher this common factor, the poorer the starting torque. Numbers of slots which are prime to one another are best for this purpose. The trouble is chiefly due to zigzag leakage flux. In cases where motors refuse to start up due to this cause, increasing the air-gap length will sometimes effect a cure, but there is a limit to this set by power-factor considerations.

Smooth starting and accelerating conditions will be secured by adopting one slot less or one slot more per pole pair in the rotor than in the stator.*

---

* See discussion on this point in *The Electrician* by Dr. Chapman

With 57 rotor slots, the current per bar at full load

$$= \frac{15 \times 746}{3 \times 440 \times 0{\cdot}9} \times \frac{54 \times 28}{57 \times 1} = 256\ \text{A}$$

Current per ring

$$= \frac{256}{1{\cdot}11} \times \frac{\text{bars}}{2\ \text{poles}} \times \frac{1}{1{\cdot}414} = \frac{256}{1{\cdot}11} \times \frac{57}{12} \times \frac{1}{1{\cdot}414} = 770\ \text{A}$$

Assuming a rotor current density of 3·5 A/mm² in the bars –

$$\text{area of bar} = \frac{256}{3{\cdot}5} = 72\ \text{mm}^2$$

The rotor slot is shown in Fig. 23.2.

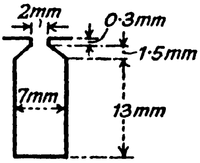


Fig. 23.2

Rotor bar of copper 6 mm wide by 12 mm deep.

Resistance of rotor bars at 15°C

$$= \frac{0{\cdot}017 \times 57 \times 0{\cdot}19}{72} = 0{\cdot}00256\ \Omega$$

Resistance (hot) = 0·00307 Ω

In order to get the desired starting torque, end rings of brass 20 mm × 20 mm will be used.

Resistance of each ring = 0·000163 Ω hot

Magnetizing Current

Part	Area (cm²)	Length (cm)	ϕ	B_{60}	A_{60}
Stator core . . .	67·5	13·30	0·4025	5 950	33
Stator teeth . . .	126·0	7·20	0·8050	8 170 13 150	62
Gap	145·5	0·10	0·8050	7 070	563
Rotor teeth . . .	79·0 62·0	2·60	0·8050	13 050 16 600	78
Rotor core . . .	89·0	5·22	0·4025	4 520	10

Total ampere-turns at 60° = A_{60} = 746

$$\text{Magnetizing current per line wire} = \frac{\sqrt{3} \times 746}{2{\cdot}12 \times 3 \times 28} = 7{\cdot}27$$

Ideal Short-circuit Current

$$\lambda_s = \frac{36}{24} + \frac{4}{11} + \frac{0{\cdot}5}{3{\cdot}0} = 2{\cdot}03$$

$$\lambda_r = \frac{13}{14} + \frac{3{\cdot}0}{9{\cdot}0} + \frac{0{\cdot}3}{2{\cdot}0} = 1{\cdot}413$$

$$\lambda_1 = \left[\frac{2{\cdot}03}{0{\cdot}96 \times 3} + \frac{1{\cdot}413}{3{\cdot}16}\right] \times 14 = 16{\cdot}2$$

$$\lambda_2 = 0{\cdot}73(0{\cdot}36\tau + A) = 0{\cdot}73(0{\cdot}36 \times 13{\cdot}1 + 14) = 13{\cdot}7$$

$$\lambda_3 = \frac{L\varepsilon\tau}{48 \times \delta \times q_1 \times q_2} = \frac{14 \times 13{\cdot}1 \times 1{\cdot}337}{48 \times 0{\cdot}05 \times 3 \times 3{\cdot}16} = 10{\cdot}75$$

Note $\delta = 0{\cdot}5 \text{ mm} = 0{\cdot}05 \text{ cm}$

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 40{\cdot}65$$

$$\underset{\text{(ideal)}}{I_{sc}} \text{ line} = \frac{\sqrt{3} \times 0{\cdot}805 \times 10^6}{3{\cdot}55 \times 3 \times 28 \times 40{\cdot}65} = 115 \text{ A}$$

Our rough method gave 113 A.

Maximum horse-power = 31·8

Actual short-circuit current = 93 A

$$\sigma = \frac{7{\cdot}27}{115} = 0{\cdot}063$$

cos ϕ F.L. = 87·5 per cent

Efficiency

Stator copper loss = $3 \times 11{\cdot}2^2 \times 1{\cdot}45 = 545$

Iron loss $\left\{\begin{array}{l}\text{Teeth} = 204\\ \text{Core} = 178\end{array}\right\} = 382$

Rotor copper loss = 394

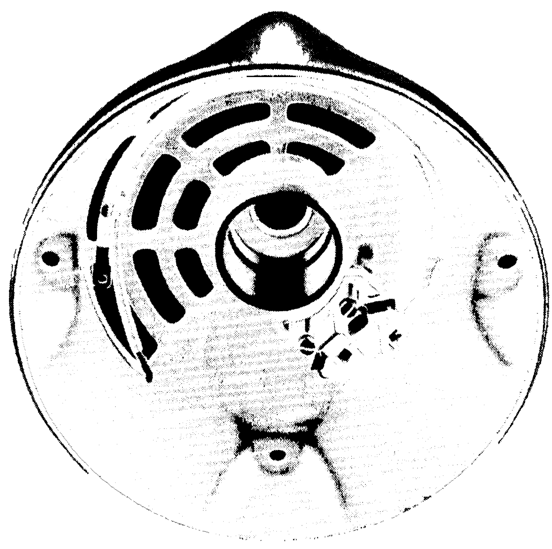
F.L. and windage = 140

Total losses = 1 461

Output = 11 200

Input = 12 661

Efficiency = 88½ per cent



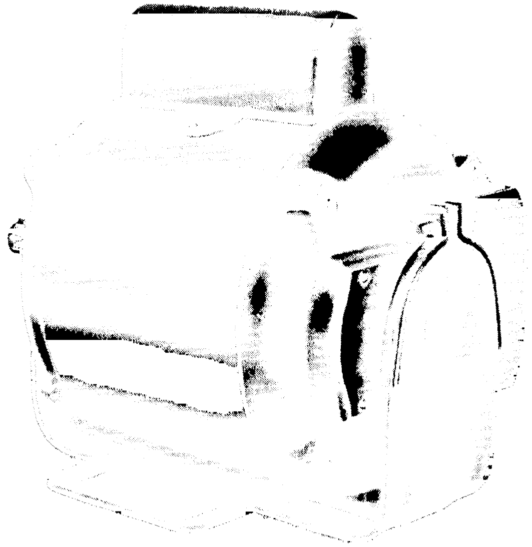


Plate XVII

(*Upper*) End Bracket of Type CAP Motor (Single-phase) Showing Starting Capacitor Switch

(*Courtesy Westinghouse Electric Corp. of America*)

(*Lower*) Capacitor-start Motor Type FJ

(*Courtesy Westinghouse Electric Corp. of America*)

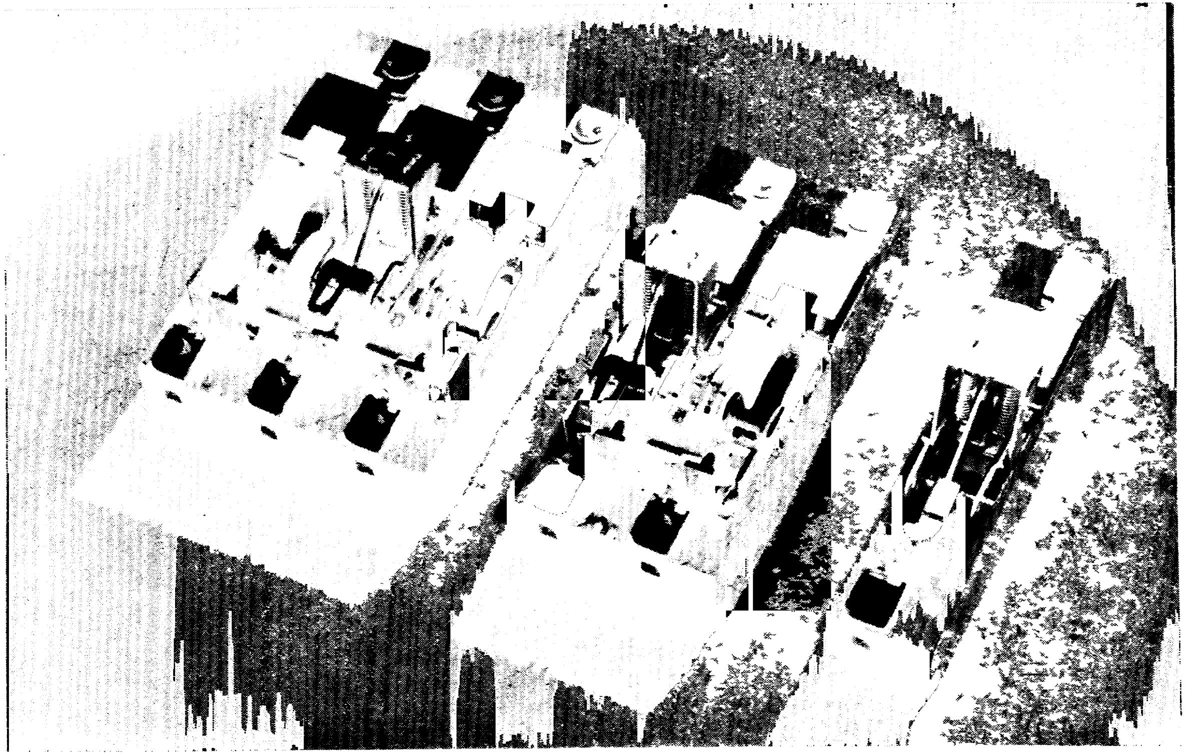


Plate XVIII Control Switches for F H P Motors

(Courtesy Westinghouse Electric Corp of America

Starting torque as percentage of full-load torque

$$= \left(\frac{93}{\sqrt{3}} \times \frac{54 \times 28}{256 \times 57}\right)^2 \times \text{slip percentage}$$

$$= \left(\frac{1420}{256}\right)^2 \times 3{\cdot}4 = 105 \text{ per cent}$$

$$= \left(\frac{\text{rotor current at start}}{\text{rotor current at full load}}\right)^2 \times \text{percentage slip}$$

$$\text{Slip} = \frac{394}{11\,594} = 0{\cdot}034 \text{ and in percentage} = 3{\cdot}4$$

This machine was built and tested. The results of the test are as follows—

Temperatures after 6-hr full-load run—

Part	Temperature (°F)	Atmosphere (°F)	Rise (°F)
Stator core . . .	116	66	50
Stator winding (back) .	117	66	51
Stator winding (front) .	120	66	54
Rotor end ring . .	128	66	62

Magnetizing current line = 6·7 A

(Air-gap a little under 0·5 mm.)

Short-circuit current = 93 A

CHAPTER XXIV

Design for Double Squirrel-cage Motor

THE application of squirrel-cage induction motors has increased greatly, due to their reliability and simplicity of construction, low first cost (both of motor and control gear), and to improved design. The modern squirrel-cage rotor, being practically indestructible, also ensures a high starting torque, which compares favourably with that obtained from a slip-ring type motor. For certain requirements, it is necessary to have a high starting torque, with comparatively low starting current. It is possible to ensure these results by the use of a double squirrel-cage rotor. With full voltage starting, by direct switching, it is possible to obtain 2 to $2\frac{1}{2}$ full-load torque, with a starting current of $4\frac{1}{2}$ to $5\frac{1}{4}$ times full-load current. Even with a star-delta starter, 60 to 75 per cent of full-load torque is obtained at starting, with a starting current of only $1\frac{1}{2}$ to $1\frac{3}{4}$ times full-load current. By using an auto-transformer starter, full-load starting torque can be obtained with slightly increased current. In making use of such motors, where high starting torque with low starting current is desired, due allowance must be made for the possible effects on gearing, belt or rope, of the high starting torque. For centrifugal pumps and fans, unless one desires to limit the starting current, an ordinary single-cage motor is desirable, since the torque, with this class of load, rises rapidly with the speed, and is like the speed-torque curve of the ordinary squirrel-cage.

Speed as a Fraction of Synchronous Speed	Full-load Torque		Current/Full-load Current	
	Single-cage (%)	Double-cage (%)	Single-cage	Double-cage
0	130	220	5·50	4·80
20	145	215	5·30	4·60
40	175	193	5·10	4·45
60	218	185	4·80	4·00
80	255	205	3·75	3·00
100	0	0	0·50	0·50

For driving compressors, fitted with unloading valves, it is unnecessary to use a double cage, for the starting torque required is but 20 to 30 per cent of full-load torque. For compressors, without unloading valves, a double-cage motor is very desirable. We have already seen, in an earlier chapter, that the double-squirrel cage has a higher starting torque than the single-cage machine, but its *maximum* torque is smaller than that of the single-cage machine. This motor should be used, only in those cases where a high starting torque is required, with a relatively low starting current.

Comparative values for single-cage and double-cage motors are appended in the table on page 474.

We will now design a double-squirrel cage for the following output and speed: 12 b.h.p.; 220 V; 50 c/s; 4 poles; 1500/1455 r.p.m.

The machine is required to give about twice full-load starting torque with 4·5 to 5 times full-load current.

The $$D^2L = \frac{4.06 \times 10^{11} \times \text{b.h.p.}}{B \times q \times \text{r.p.m.} \times \cos\theta \times \eta}$$

Here $q = 220$, $B = 4000$, r.p.m. $= 1455$, $\cos\theta = 0{\cdot}85$, $\eta = 0{\cdot}87$.

$$\therefore \quad D^2L = \frac{4{\cdot}06 \times 10^{11} \times 12}{4000 \times 1455 \times 220 \times 0{\cdot}85 \times 0{\cdot}87}$$
$$= 5150 \text{ cm}^3$$

For best power factor

$$D = 1{\cdot}35p\sqrt{L}$$
$$= 1{\cdot}35 \times 4\sqrt{L} = 5{\cdot}40\sqrt{L}$$
$$\therefore \quad 29{\cdot}2L^2 = 5150$$
$$L = 13{\cdot}25 \text{ cm}$$
$$\therefore \quad D = 19{\cdot}6$$

Make $D = 19{\cdot}5$ cm and $L = 14$ cm.
There will be one vent duct of 1 cm wide,

$$\therefore \quad L_{\text{net}} = 13 \text{ cm}$$

Pole pitch $$= \frac{\pi \times 19{\cdot}5}{4} = 15{\cdot}3 \text{ cm}$$

We will use 9 slots per pole; $q_1 = 3$.

Number of stator slots = 36

Line amperes

$$= \frac{12 \times 746}{\sqrt{3} \times 220 \times 0{\cdot}85 \times 0{\cdot}87} = 31{\cdot}7 \text{ A}$$

With delta connection, the current per phase

$$= \frac{31{\cdot}7}{\sqrt{3}} = 18{\cdot}3 \text{ A}$$

Apparent gap area per pole

$$= \frac{\pi \times 19{\cdot}1 \times 14{\cdot}0}{4} = 210 \text{ cm}^2$$

Assume $B_{\text{av}} = 4500$.

Then $\phi = 4500 \times 210 = 0{\cdot}945 \times 10^6$

Conductors in series per phase

$$= Z = \frac{215 \times 10^8}{0{\cdot}945 \times 10^6 \times 50 \times 2{\cdot}12}$$

$$= 214$$

$$\text{Conductors per slot} = \frac{214 \times 3}{36} = 17{\cdot}8$$

We will use 18 conductors per slot, i.e. 18 turns per coil with a mush winding.

Current density in conductors $= 3{\cdot}5 \text{ A/mm}^2$

$$\text{Area of conductor} = \frac{18{\cdot}3}{3{\cdot}5} = 5{\cdot}25 \text{ mm}^2$$

Use 5 wires in parallel per turn, and each wire of 18 s.w.g. wire, diameter base = 1·2192 mm, d.c.c. to 1·4192 mm.

Area occupied by the wires per slot

$$= 90 \times 0{\cdot}7854 \times 1{\cdot}4192 \times 1{\cdot}4192 = 142 \text{ mm}^2$$

$$\text{l.m.t.} = 2\left(14 + \frac{\pi \times 22{\cdot}65}{4} \times 1{\cdot}4\right) + 5 = 83 \text{ cm}$$

Resistance per phase at 15°C

$$= \frac{0{\cdot}017 \times 6 \times 18 \times 0{\cdot}83}{5 \times 1{\cdot}1675} = 0{\cdot}261\ \Omega$$

$$R_{\text{hot}} = 0{\cdot}308\ \Omega$$

I^2R loss in stator

$$= 3 \times 18{\cdot}3 \times 18{\cdot}3 \times 0{\cdot}308 = 310 \text{ W}$$

Copper weight

$$= 0{\cdot}0197 \times 18 \times 18 \times 0{\cdot}83 \times 5 \times 1{\cdot}1675 = 31 \text{ lb}$$

The stator will have a mush winding, connected in delta; 6 coils per phase, 18 turns per coil, with each turn consisting of 5 wires in parallel, and each wire of 18 s.w.g. copper wire.

The stator slot, shown in Fig. 24.1 is suitable.

Stator core area

$$= \frac{0{\cdot}945 \times 10^6}{2 \times 9500} = 49{\cdot}6 \text{ cm}^2$$

Radial depth of core

$$= \frac{49{\cdot}6}{11{\cdot}7} = 4{\cdot}25 \text{ cm}$$

External diameter

$$= 195 + 63 + 85 = 343 \text{ mm} = 34{\cdot}3 \text{ cm}$$

Single air-gap length = 0·4 mm.

Number of rotor slots = 34.

The rotor will have two cage windings, the upper one of high resistance and low leakage reactance, and the lower or inner cage will have high reactance, produced by arranging the depth of the inner-cage below the upper cage and by providing a narrow air-slit between.

The rotor current at full load

$$\hat{=} \frac{12 \times 746}{3 \times 215 \times 0{\cdot}9} \times \frac{36 \times 18}{34 \times 1} = 294 \text{ A}$$

At standstill, the current (total), per bar, of a single-cage rotor would be, say, 1400 A, i.e. 4·75 times full-load current. We may assume that, at standstill, about 0·90 of the total current flows in the outer cage. We shall have about 1265 A in the bars of the outer cage, if our starting conditions are not to be violated.

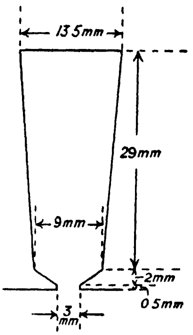


FIG. 24.1

Now we require about twice full-load torque at start with a current not greater than 4·5 to 5 times full-load current. The starting torque in synchronous watts = loss in the rotor circuits. So we must have a loss of 24 b.h.p., i.e. 24 × 746 W loss in the rotor cages.

Since the current, at the start, is about 90 per cent of the total in the outer cage, it follows that the outer cage produces, at the start, nearly the whole torque required. We shall assume it does so.

The resistance of the outer cage

$$= \frac{24 \times 746}{1265 \times 1265} = 0{\cdot}0114 \, \Omega$$

The current per ring, in the outer cage,

$$= \frac{1265}{1{\cdot}11} \times \frac{34}{8} \times \frac{1}{1{\cdot}414} = 3420 \text{ A}$$

The current density, in the ring of the outer cage, may be chosen quite high at the start, for as the speed rises, the current is transferred to the inner cage. We will use aluminium cast rotors, and the area of the outer ring will be about 100 mm.^{2}

Resistance of the aluminium ring at 15°C

$$= \frac{0{\cdot}0282 \times \pi \times 0{\cdot}181}{100} = 0{\cdot}000161\ \Omega$$

Outer Cage

$$R_{\text{hot}} = 0{\cdot}000189\ \Omega$$

Loss in 2 rings at standstill

$$= 2 \times 3420 \times 3420 \times 0{\cdot}000189 = 4420\ \text{W}$$

Therefore, loss in bars

$$= 17\,900 - 4420 = 13\,480\ \text{W}$$

Loss per bar, at start,

$$= \frac{13\,480}{34} = 396\ \text{W}$$

Resistance per bar (hot)

$$= \frac{396}{1265 \times 1265} = 0{\cdot}000248\ \Omega$$

$$= 0{\cdot}00021\ \Omega \text{ at } 15°\text{C}$$

Length of bar $= 16$ cm

Area of bar $= \dfrac{0{\cdot}0282 \times 0{\cdot}16}{0{\cdot}00021} = 21{\cdot}5\ \text{mm}^2$

We will use a circular bar of aluminium of 5·25 mm diameter. The area of the bar

$$= 5{\cdot}25 \times 5{\cdot}25 \times 0{\cdot}7854 = 21{\cdot}7\ \text{mm}^2$$

The inner cage we will make also of cast aluminium and shall make the area of the bar = 49 mm.^{2}

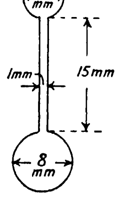


FIG. 24.2

The following arrangement of slots, shown in Fig. 24.2, will be used.

Ring of inner cage = 165 mm.^{2}

Resistance of Outer Cage (Aluminium)

Resistance of bars at 15°C

$$= \frac{0{\cdot}0282 \times 34 \times 0{\cdot}16}{21{\cdot}7} = 0{\cdot}00707$$

$$R_{\text{hot}} = 0{\cdot}00835\ \Omega$$

Resistance per bar (hot)

$$= \frac{0{\cdot}00835}{34} = 0{\cdot}000246\ \Omega$$

Resistance per ring at 15°C

$$= \frac{0{\cdot}0282 \times \pi \times 0{\cdot}181}{100} = 0{\cdot}000161\ \Omega$$

Resistance per ring (hot) $= 0{\cdot}000189\ \Omega$

Resistance of Inner Cage (Aluminium)

Resistance of bars at 15°C

$$= \frac{0{\cdot}0282 \times 34 \times 0{\cdot}16}{49} = 0{\cdot}00314\ \Omega$$

Resistance of bars (hot) $= 0{\cdot}00372\ \Omega$

Resistance *per bar* (hot)

$$= \frac{0{\cdot}00372}{34} = 0{\cdot}000109\ \Omega$$

Resistance of ring at 15°C

$$= \frac{0{\cdot}0282 \times \pi \times 0{\cdot}1374}{165} = 0{\cdot}000074\ \Omega$$

$$R_{\text{hot}} = 0{\cdot}0000875\ \Omega$$

Resistance of the outer cage, referred to the stator

$$r_{20}' = 12(T_1K_1)^2 \left[\frac{r_b}{N_2} + \frac{R_c}{2p^2\pi^2}\right]$$

where T_1 = turns in series per phase of the stator = 108

K_1 = breadth factor

r_b = resistance per bar (hot)

R_c = resistance per ring (hot)

$$\therefore r_{20}' = 12 \times 108 \times 108 \times 0{\cdot}96 \times 0{\cdot}96 \left[\frac{0{\cdot}000246}{34} + \frac{0{\cdot}000189}{2 \times 16 \times \pi^2}\right]$$

$$= 1{\cdot}29 \times 10^5 \left[\frac{7{\cdot}22}{10^6} + \frac{0{\cdot}6}{10^6}\right] = 1{\cdot}010\ \Omega$$

Resistance of the inner cage, referred to the stator,

$$= r_{2i}' = 12 \times 108 \times 108 \times 0{\cdot}96 \times 0{\cdot}96 \left[\frac{0{\cdot}000109}{34} + \frac{0{\cdot}000087}{2 \times 16 \times \pi^2}\right]$$

$$= 1{\cdot}29 \times 10^5 \left|\frac{3{\cdot}2}{10^6} + \frac{0{\cdot}276}{10^6}\right| = 0{\cdot}45\ \Omega$$

The reactance of the outer cage, referred to the stator,

$$x_{20}' = 12 \times (T_1K_1)^2 \times 2\pi f \times 10^{-8}\left[\frac{l\lambda}{N_2} + \frac{l_T\lambda_T}{2\pi^2p^2}\right]$$

where λ = slot and zigzag permeance per centimetre

λ_T = permeance per centimetre of ring leakage path

$$\lambda_n = 0{\cdot}62 + \frac{0{\cdot}5}{2{\cdot}0} = 0{\cdot}87$$

$$\lambda_T = 0{\cdot}5$$

$$l_c = \pi \times 18{\cdot}1$$

The zigzag permeance will be taken into account, when estimating the leakage impedance of the stator winding.

$$\therefore\ x_{20}' = \frac{12 \times 108 \times 108 \times 0{\cdot}96 \times 0{\cdot}96 \times 6{\cdot}28 \times 50}{10^8} \times \left[\frac{14 \times 0{\cdot}87}{34} + \frac{\pi \times 18{\cdot}1 \times 0{\cdot}5}{2 \times 16 \times \pi^2}\right]$$

$$= 0{\cdot}404[0{\cdot}358 + 0{\cdot}0905] = 0{\cdot}182$$

The reactance of the inner cage, referred to the stator,

$$= \frac{12 \times 108 \times 108 \times 0{\cdot}96 \times 0{\cdot}96 \times 6{\cdot}28 \times 50}{10^8} \times \left[\frac{14 \times 15{\cdot}62}{34} + \frac{\pi \times 13{\cdot}74 \times 0{\cdot}5}{2 \times 16 \times \pi^2}\right]$$

$$= 0{\cdot}404[6{\cdot}45 + 0{\cdot}069] = 2{\cdot}64\ \Omega$$

Resistance of stator winding per phase = 0·308

Leakage reactance of stator winding per phase

$$= \frac{0{\cdot}4\pi \times (q_1z_1)^2 \times \lambda \times p \times 2\pi f}{10^8}$$

$$\lambda_1 = \frac{29}{33{\cdot}75} + \frac{4}{12} + \frac{0{\cdot}5}{3{\cdot}0} = 1{\cdot}356$$

$$\lambda_S = \left(\frac{\lambda_1}{0{\cdot}96 \times q_1}\right)L = \frac{1{\cdot}356}{0{\cdot}96 \times 3} \times 14{\cdot}0 = 6{\cdot}56$$

$$\lambda_T = 17{\cdot}0$$

$$\lambda_Z = 19{\cdot}8$$

$$\lambda = 43{\cdot}36$$

$$\therefore\ x_1 = \frac{1{\cdot}257 \times 3 \times 18 \times 3 \times 18 \times 43{\cdot}36 \times 4 \times 2\pi \times 50}{10^8} = 2{\cdot}0\ \Omega$$

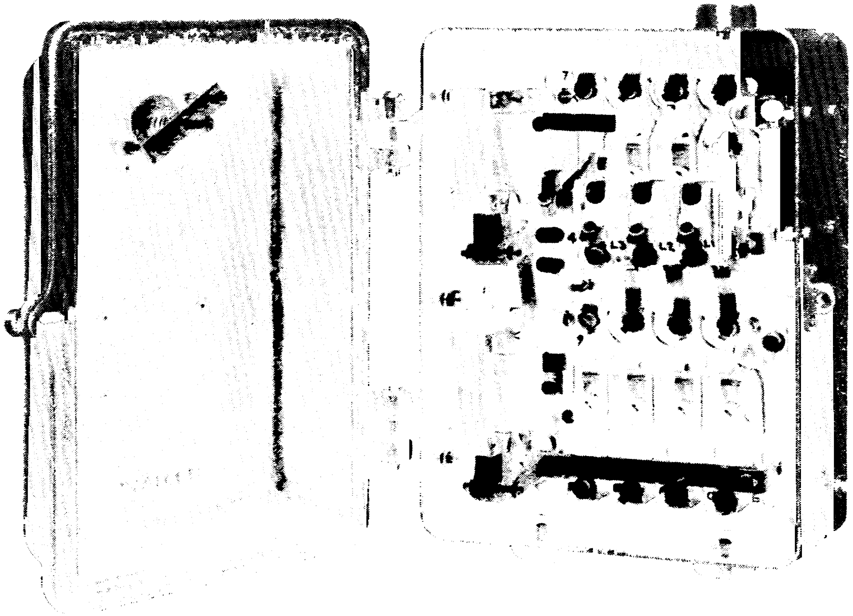


Plate XIX Three-horse-power Reversing Contactor Starter

The two triple-pole contactors mechanically interlocked for forward and reverse running may be seen

Courtesy B T H Co , Rugby)

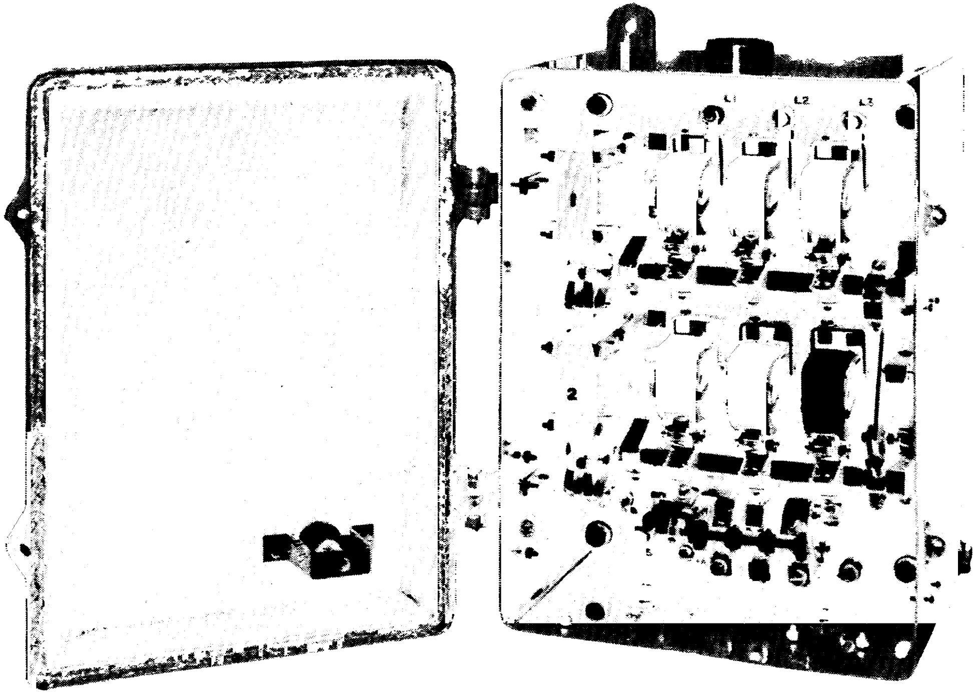


PLATE XX. CONTACTOR STARTER WITH MECHANICALLY-INTERLOCKED TRIPLE-POLE CONTACTOR

Thermal overload relays in the three phases are mounted at the bottom

(Courtesy P T H C P)

The equivalent circuit at standstill is shown in Fig. 24.3.

The magnetizing current, approximately,

$$= 6{\cdot}75 \text{ A}$$

The magnetizing reactance

$$jx_m \simeq \frac{220}{6{\cdot}75} = 32{\cdot}5\ \Omega$$

Actually $$jx = \frac{\mathbf{V} - \mathbf{I}_1\mathbf{Z}}{6{\cdot}75}$$

and is slightly smaller than $32{\cdot}5\ \Omega$.

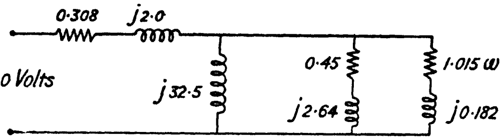


FIG. 24.3

Absolute impedance of inner cage

$$= \sqrt{0{\cdot}45^2 + 2{\cdot}64^2} = \sqrt{0{\cdot}202 + 6{\cdot}96} = 2{\cdot}67\ \Omega$$

$$\tan\phi_1 = \frac{2{\cdot}64}{0{\cdot}45} = 5{\cdot}87$$

and $$\phi_1 = 80^\circ\ 19'$$

Absolute impedance of outer cage

$$= \sqrt{1{\cdot}015^2 + 0{\cdot}182^2} = \sqrt{1{\cdot}02 + 0{\cdot}033} = 1{\cdot}03\ \Omega$$

$$\tan\phi_2 = \frac{0{\cdot}182}{1{\cdot}010} = 0{\cdot}18$$

and $$\phi_2 = 10^\circ\ 12'$$

Combined impedance of two cages, at standstill,

$$= \frac{(0{\cdot}45 + j2{\cdot}64)(1{\cdot}015 + j0{\cdot}182)}{1{\cdot}465 + j2{\cdot}822}$$

$$= \frac{0{\cdot}457 + 2{\cdot}66j + 0{\cdot}082j - 0{\cdot}48}{1{\cdot}465 + j2{\cdot}822}$$

$$= \frac{-0{\cdot}023 + j2{\cdot}742}{1{\cdot}465 + j2{\cdot}822}$$

Rationalizing, we have, combined impedance

$$= \frac{(-0{\cdot}023 + j2{\cdot}742)(1{\cdot}465 - j2{\cdot}822)}{1{\cdot}465^2 + 2{\cdot}822^2}$$
$$= \frac{-0{\cdot}0338 + 4{\cdot}02j + 0{\cdot}065j + 7{\cdot}75}{2{\cdot}15 + 7{\cdot}96}$$
$$= \frac{7{\cdot}7162 + 4{\cdot}085j}{10{\cdot}11}$$
$$= 0{\cdot}763 + 0{\cdot}403j$$

Total series impedance, at standstill, approximately,

$$= 0{\cdot}308 + j2{\cdot}0 + 0{\cdot}763 + 0{\cdot}403j$$
$$= 1{\cdot}071 + 2{\cdot}403j$$
$$|Z_{\text{total}}| = \sqrt{1{\cdot}071^2 + 2{\cdot}403^2}$$
$$= \sqrt{1{\cdot}15 + 5{\cdot}8} = 2{\cdot}64\,\Omega$$

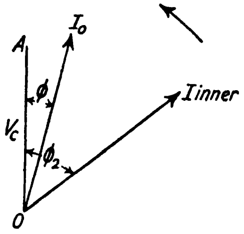


FIG. 24.4

Actual short-circuit current per phase

$$= \frac{220}{2{\cdot}64} = 83{\cdot}5\text{ A}$$

$$\text{Cos }\phi\text{ at short-circuit} = \frac{1{\cdot}071}{2{\cdot}64} = 0{\cdot}405$$

$$\frac{\text{current in outer cage at standstill}}{\text{current in inner cage at standstill}} = \frac{\text{impedance of inner at standstill}}{\text{impedance of outer at standstill}}$$
$$= \frac{2{\cdot}67}{1{\cdot}03} = 2{\cdot}6$$

If *OA* (Fig. 24.4) represents the voltage across each cage at standstill—

$I_{\text{outer cage}}$ makes an angle of 10° 12′ with $OA = \phi_1$

and $I_{\text{inner cage}}$ makes an angle of 80° 19′ with $OA = \phi_2$

Also, I_{outer}, at standstill $= 2{\cdot}6I_{\text{inner}}$

The resultant of I_0 and $I_i = 83{\cdot}5$ A and the angle between the two = 70° 7′.

$$\therefore \quad 83{\cdot}5 = \sqrt{I_i^2 + (2{\cdot}6I_i)^2 + 5{\cdot}2I_i^2 \cos 70^\circ\, 7'}$$
$$= I_i\sqrt{1 + 6{\cdot}75 + 5{\cdot}2 \times 0{\cdot}34}$$
$$= I_i\sqrt{9{\cdot}515}$$
$$= 3{\cdot}08I_i$$

$\therefore$ $I_{\text{inner}} = I_i = 27$ A at standstill, referred to stator

and $I_{\text{outer}} = I_0 = 70{\cdot}5$ A at standstill, referred to stator

Current in bars of outer cage, at standstill

$$70{\cdot}5 \times \frac{36 \times 18}{34} = 1345 \text{ A}$$

Loss in inner cage at start

$$= 3 \times 27 \times 27 \times 0{\cdot}45 = 980 \text{ W}$$

Loss in outer cage at the start

$$= 3 \times 70{\cdot}5 \times 70{\cdot}5 \times 1{\cdot}015 = 15\,150 \text{ W}$$

Therefore, loss in both cages at the start

$$= 16\,130 \text{ W} = 21{\cdot}7 \text{ h.p.}$$

Thus, we have 1·8 full-load torque with 4·55 times full-load current at the start.

For twice full-load torque at start, we might use 17 turns per coil in the stator winding.

It will be seen we have 84 per cent of total current in outer cage instead of 90 per cent as assumed, hence our starting torque is down a little. This could be remedied by increasing the resistance of the outer cage, and by strengthening the field slightly.

At full load, the reactances of both cages are negligible, and the currents are shared between the cages in the inverse ratio of their resistances.

At full load, with a single cage, the current in the rotor would be

$$\frac{12 \times 746}{3 \times 220 \times 0{\cdot}9} \times \frac{36 \times 18}{34 \times 1} = 287 \text{ A per bar}$$

The current carried by the outer cage, at full load,

$$= \frac{0{\cdot}45}{1{\cdot}465} \times 287 = 87{\cdot}75 \text{ A}$$

The current carried by the inner cage, at full load,

$$= \frac{1{\cdot}015}{1{\cdot}465} \times 287 = 199{\cdot}25 \text{ A}$$

Loss in outer cage at full load, in bars

$$= 87{\cdot}75 \times 87{\cdot}75 \times 0{\cdot}00835 = 64{\cdot}5 \text{ W}$$

Current per ring at full load, outer cage,

$$= \frac{87{\cdot}75}{1{\cdot}11} \times \frac{34}{8} \times \frac{1}{1{\cdot}414} = 237 \text{ A}$$

Loss in 2 rings, outer cage, at full load,

$$= 2 \times 237 \times 237 \times 0{\cdot}000189 = 21{\cdot}2 \text{ W}$$

Total loss, outer cage, at full load $= 85{\cdot}7$ W

Loss in inner cage at full load, in bars

$$= 199{\cdot}25 \times 199{\cdot}25 \times 0{\cdot}00372 = 148 \text{ W}$$

Current per ring

$$= \frac{199{\cdot}25}{1{\cdot}11} \times \frac{34}{8} \times \frac{1}{1{\cdot}414} = 540 \text{ A}$$

Loss in 2 rings

$$= 2 \times 540 \times 540 \times 0{\cdot}000087 = 50{\cdot}5 \text{ W}$$

Loss in inner cage, at full load = 198·5 W

I^2R loss in 2 cages, at full load = 284·2 W

Output = 8950 W

Rotor input = 9234 W

Per cent slip at full load

$$= \frac{284}{9234} \times 100 = \text{nearly } 3{\cdot}08 \text{ per cent}$$

R.p.m. at full load = 1450

Current densities, at full load—

Outer cage: bars $= \frac{87{\cdot}75}{21{\cdot}7} = 4{\cdot}03 \text{ A/mm}^2$

Rings $= \frac{237}{100} = 2{\cdot}37 \text{ A/mm}^2$

Inner cage: bars $= \frac{199{\cdot}25}{49} = 4{\cdot}05 \text{ A/mm}^2$

Rings $= \frac{540}{165} = 3{\cdot}27 \text{ A/mm}^2$

Stator: current density $= \frac{18{\cdot}3}{5 \times 1{\cdot}1675} = 3{\cdot}14 \text{ A/mm}^2$

Magnetizing Current

Apparent gap area per pole = 210 cm²

Carter's factor—

Stator slots = 1·15
Rotor slots = 1·08
Vents = 1·07

True gap area per pole $= \frac{210}{1{\cdot}34} = 157 \text{ cm}^2$

Stator teeth: top = 94·7 cm²
bottom = 89 cm²

Stator core: area = 49·6 cm²
Length of path = 16 cm

Rotor teeth: $\dfrac{\pi \times 18{\cdot}07}{34} = 1{\cdot}665$

Width of tooth $= 1{\cdot}135$

Area at A $= 1{\cdot}135 \times 8{\cdot}5 \times 11{\cdot}7 = 113 \text{ cm}^2$

$$\frac{\pi \times 13{\cdot}74}{34} = 1{\cdot}27$$

Width of tooth $= 0{\cdot}47$

Area at B $= 0{\cdot}47 \times 8{\cdot}5 \times 11{\cdot}7 = 56{\cdot}6 \text{ cm}^2$

Shaft diameter $= 70$ mm

Rotor core $= 2{\cdot}97 \times 11{\cdot}7 = 34 \text{ cm}^2$

Length of path $= \dfrac{\pi \times 9{\cdot}97}{4} \times \dfrac{2}{3} = 5{\cdot}2$ cm

$$\phi = \frac{215 \times 10^8}{2{\cdot}12 \times 12 \times 18 \times 50} = 0{\cdot}945 \times 10^6$$

Air-gap

$$B_{60} = 1{\cdot}28 \times \frac{0{\cdot}945 \times 10^6}{157} = 6000 \times 1{\cdot}28$$

$$\bar{A}_{g\,60} = 0{\cdot}796 \times 6000 \times 0{\cdot}08 \times 1{\cdot}28 = 381 \times 1{\cdot}28 = 486$$

$$B_{ST\,60}\,A = \frac{0{\cdot}945 \times 10^6 \times 1{\cdot}28}{94{\cdot}7} = 12\,750$$

$$x = 1{\cdot}06$$

$$B = \frac{0{\cdot}945 \times 10^6 \times 1{\cdot}28}{89} = 13\,650$$

$$\bar{A}_{ST\ \text{teeth}} = 15 \times 6{\cdot}3 = 94{\cdot}5$$

$$B_{ST\ \text{core}} = \frac{0{\cdot}4725 \times 10^6}{49{\cdot}6} = 9500$$

$$\bar{A}_{ST\ \text{core}} = 6 \times 16 = 96$$

Rotor Teeth

$$B_{60}\,A = \frac{0{\cdot}945 \times 10^6 \times 1{\cdot}28}{113} = 10\,700$$

$$B_{60} = \frac{0{\cdot}945 \times 10^6 \times 1{\cdot}28}{56{\cdot}6} = 21\,300$$

$$\bar{A}_{RT} = 60$$

Rotor Core $\quad B_c = \dfrac{0{\cdot}4725 \times 10^6}{34} = 13\,800$

$$\bar{A}_{rc} = 20 \times 5{\cdot}2 = 104$$

$$\bar{A}_{60}\ (\text{gap}) = 486$$

Per pole pair

$$\begin{aligned} ST &= 95 \\ SC &= 96 \\ RT &= 60 \\ RC &= 104 \\ &\overline{\underline{741}} \end{aligned}$$

$$I_\mu = \frac{741}{2{\cdot}12 \times 3 \times 18} = 6{\cdot}5 \text{ A/phase}$$

$$= 6{\cdot}5 \times \sqrt{3} \text{ line} = 11{\cdot}3 \text{ A}$$

Ideal short-circuit current per phase $= \dfrac{220}{2{\cdot}403} = 91$ A

Ideal short-circuit current per line $= 91 \times \sqrt{3} = 158$ A

Dispersion coefficient $= \dfrac{11{\cdot}3}{158} = 0{\cdot}0715$

Maximum horse-power $= \dfrac{\sqrt{3} \times 220 \times 144}{2{\cdot}81 \times 746} = 26{\cdot}2$ h.p.

cos ϕ (F.L.) = 85 per cent

Weight of core = 32·6 kg
Weight of teeth = 8·7 kg

Iron loss: core = 228
teeth = 84
312

Stator copper loss, full load

= 3 × 18·3 × 18·3 × 0·308
= 310 W

Iron loss = 312 W
Rotor I^2R = 276 W
B.F.W. = 200 W

Total loss = 1 098 W

Output = 8 960 W
Input = 10 058 W

Efficiency = 89 per cent

CHAPTER XXV

Design of Fractional Horse-power Motors

THERE is an enormous field for fractional h.p. motors, and it may be desirable to indicate some of the types which are available. They are built for single-phase, two-, and three-phase supply circuits. There are three types of single-phase motor available, namely, the "general purpose" split phase, the "high-torque" split-phase, and the capacitor-start motor.

The general-purpose split-phase motor is built for 110 to 225 per cent starting torque and 200 to 280 per cent breakdown torque in relation to full-load torque. Its applications include oil-burners, fans, blowers, office appliances, and certain types of small tools. It is built in outputs of from $\frac{1}{20}$ to $\frac{1}{3}$ h.p. for 60, 50, and 25 c/s and 115 to 230 V, and speeds of 2850, 1425, 960 r.p.m. are available on 50 c/s.

The split-phase high-torque motor has 200 to 300 per cent starting torque with 260 to 350 per cent breakdown torque in relation to full-load torque. Its applications include washing and ironing machines. It is usually built for $\frac{1}{6}$, $\frac{1}{4}$, and $\frac{1}{3}$ h.p. on 60, 50, and 25 c/s for 115 to 220 V. Approximate speeds are 1725 r.p.m. on 60 c/s, 1425 on 50 c/s, and 1425 on 25 c/s.

Capacitor-start motors are built for 300 to 450 per cent starting torque, with 225 to 300 per cent breakdown torque with low starting current. They are built in sizes varying from $\frac{1}{6}$ to $\frac{3}{4}$ h.p. on 60, 50, and 25 c/s with speeds of 2850, 1425, and 960 r.p.m. on 50 c/s. The size of the capacitor is 70 to 80 μF for $\frac{1}{8}$ h.p., 120 to 150 μF for $\frac{1}{4}$ h.p. and 230 to 285 μF for $\frac{1}{2}$ h.p. The starting winding is opened by a centrifugal switch at approximately 70 per cent of synchronous speed. The starting torque is of the order of 3 to 3·5 oz-ft/A of starting current at 110 V. These machines have high efficiency and power factor, and are ideal for compressors, pumps, stokers, refrigerators, and air-conditioning equipment.

It should be stated that the starting winding, with its capacitance in series, is sometimes left permanently in circuit when running. The motor is then operated as a rather unbalanced two-phase motor. When so used, the capacitance used is much smaller than when used for *starting* purposes only, and is of the order of 3 to 15 μF. The starting torque is much smaller in this case and of the order of

40 to 50 per cent of full-load torque. An electrolytic capacitor is used for starting purposes, and a paper capacitor for permanent running.

Polyphase motors of the squirrel-cage type are built for outputs, in the fractional h.p. range, of $\frac{1}{6}$ to $\frac{3}{4}$ h.p. for 2, 4, and 6 poles.

To illustrate the design of these motors, we will take two examples, namely, a three-phase type, and a single-phase type.

THREE-PHASE SQUIRREL-CAGE MOTOR

Output $= \frac{1}{4}$ b.h.p.; 220 V; 960 r.p.m.

Number of poles = 6

$$= \frac{120 \times 50}{1000} = \frac{120 \times f}{\text{r.p.m.}_{\text{syn.}}}$$

$$D^2L = \frac{4{\cdot}06 \times 10^{11} \times \text{b.h.p.}}{B_g \times q \times \text{r.p.m.} \times \cos\theta \times \eta}$$

$$= \frac{4{\cdot}06 \times 10^{11} \times 0{\cdot}25}{3500 \times 150 \times 960 \times 0{\cdot}7 \times 0{\cdot}6} = 480 \text{ cm}^3$$

Internal diameter of stator = 12 cm.
Core length = 3·33 cm, say, 3·3 cm.
Apparent gap area per pole

$$= \frac{\pi \times 11{\cdot}95 \times 3{\cdot}3}{6} = 20{\cdot}7 \text{ cm}^2$$

$$\phi = 20{\cdot}7 \times 3500 = 72\,400$$

With star-connected stator, volts per phase = 127.

$$\bar{E} \simeq 122$$

Conductors in series per phase

$$= \frac{122 \times 10^8}{2{\cdot}12 \times 0{\cdot}72 \times 10^5 \times 50} = 1600$$

$$3Z = 4800$$

$$\text{Pole pitch} = \frac{\pi \times 12}{6} = 6{\cdot}28 \text{ cm}$$

This suggests about 6 slots per pole.
Number of slots = 36.

At this stage, it would be well to check the approximate overload capacity. Volts per conductor at 50 c/s $= \frac{127}{1560} = 0{\cdot}081$

Short-circuit volts per conductor at 1000 ampere-conductors per slot $= 0{\cdot}05 \times 1{\cdot}3 = 0{\cdot}065$.

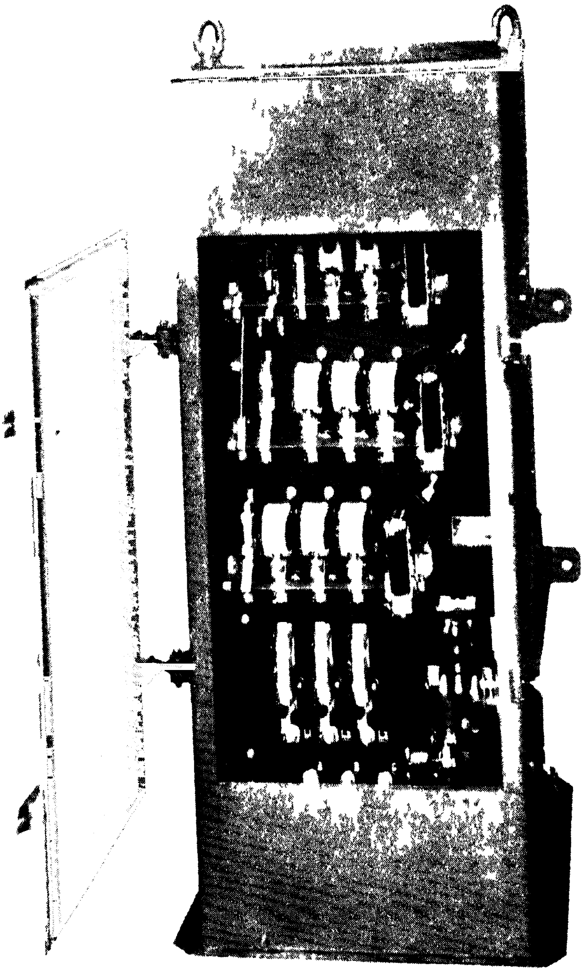


PLATE XXI. THE KMC AUTO-TRANSFORMER STARTER FOR 50 B.H.P. 440 V. SQUIRREL-CAGE MOTOR

([illegible] Ltd.)

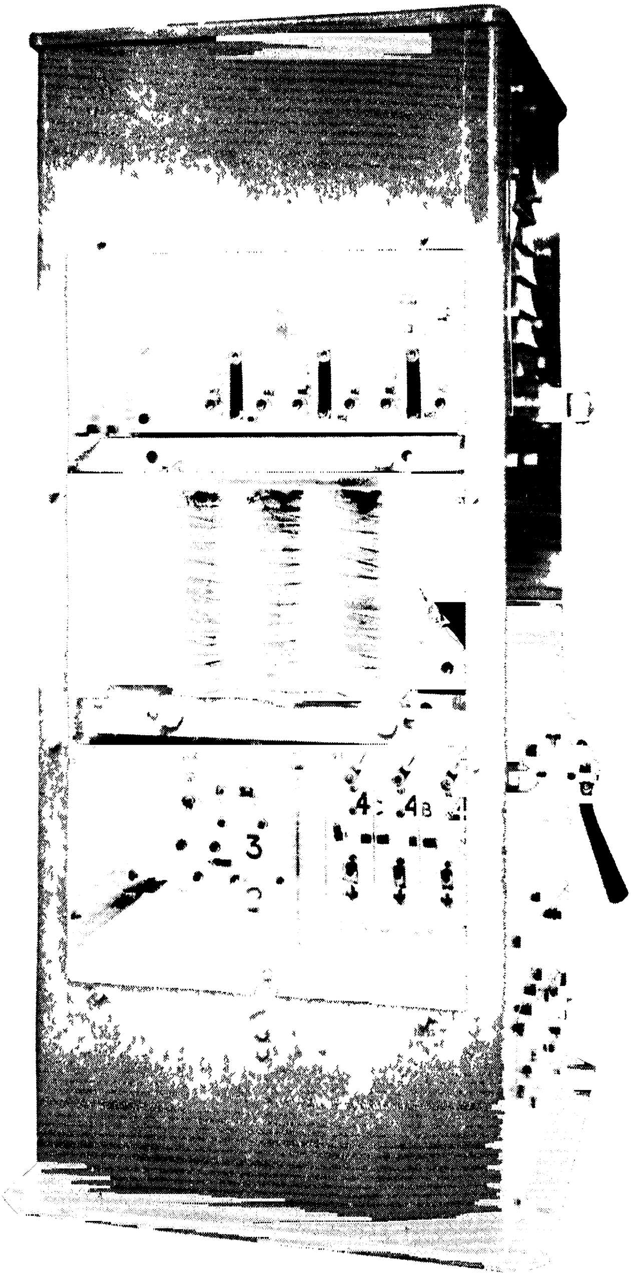


Plate XXII Rear View of Unit Seen in Plate XXI

(Courtesy B T H Co Rugby)

Short-circuit volts per conductor, at 1000 ampere-conductors per slot, for the end connections $= \dfrac{0{\cdot}012 \times (2{\cdot}48)^{1\cdot5}}{0{\cdot}415} = 0{\cdot}112$

Total short-circuit volts per conductor, at 1000 ampere-conductors per slot $= 0{\cdot}177$.

Short circuit amperes $= \dfrac{1000}{130} \times \dfrac{0{\cdot}081}{0{\cdot}177} = 3{\cdot}52$

Maximum horse-power $= 0{\cdot}434$.

This suggests that 130 turns per coil gives too small an overload capacity. We will, therefore, adopt 110 turns per coil. We will use a single-layer mush winding.

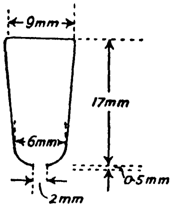


FIG. 25.1

Total number of coils = 18.
Turns per coil = 110.
Number of coils in series per phase = 6.
Turns in series per phase = 660.
Conductors in series per phase = 1320.

Line amperes $= \dfrac{0{\cdot}25 \times 746}{\sqrt{3} \times 220 \times 0{\cdot}7 \times 0{\cdot}7} = 1{\cdot}0$ A

Use 22 s.w.g. wire, diameter bare = 0·028 in.
Lewmex covering, diameter covered = 0·857 mm.
Space required for wires in the slot = 64 mm^2.
The slot shown in Fig. 25.1 is suitable.

Slot lining consists of presspahn and leatheroid 0·7 mm thickness per side.

$$\text{L.M.T.} = 2\left[3{\cdot}3 + \frac{\pi \times 13{\cdot}75}{6} \times 1{\cdot}4\right]$$
$$= 2\,[3{\cdot}3 + 10{\cdot}1] = 26{\cdot}8 \text{ cm}$$

Resistance per phase at 15°C

$$= \frac{0{\cdot}017 \times 6 \times 110 \times 0{\cdot}268}{0{\cdot}3973} = 7{\cdot}56\ \Omega$$

Resistance per phase (hot) $= 8{\cdot}9\ \Omega$
Stator I^2R loss $= 3 \times 1{\cdot}0 \times 1{\cdot}0 \times 8{\cdot}9 = 26{\cdot}7$ W
Copper weight

$$= 0{\cdot}0197 \times 18 \times 110 \times 0{\cdot}268 \times 0{\cdot}3973 = 4{\cdot}15 \text{ lb}$$

Stator Core

Area $= \dfrac{0{\cdot}43 \times 10^5}{8500} = 5{\cdot}06$ cm^2

Radial depth $= 1{\cdot}7$.

External diameter of stator

$$= 12 + 3{\cdot}5 + 3{\cdot}4 = 18{\cdot}9 \text{ cm} = 7{\cdot}45 \text{ in.}$$

Single air-gap = 0·25 mm.
External diameter of rotor = 11·95 cm.
Number of rotor slots = 33.

Rotor current per bar

$$= \frac{0\cdot25 \times 746}{\sqrt{3} \times 220 \times 0\cdot9} \times \frac{36 \times 110}{33 \times 1} = 64\cdot5 \text{ A}$$

$$\text{Amperes per ring} = \frac{64\cdot5}{1\cdot11} \times \frac{33}{12} \times \frac{1}{1\cdot414} = 113$$

$$\text{Area of bar} = \frac{64\cdot5}{3} = 21\cdot5 \text{ mm}^2$$

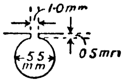


FIG. 25.2

Bar, 5·25 mm diameter, slot = 5·5 mm. (*See* Fig. 25.2.)
Length of bar = 4·5 cm.
Resistance of bars at 15°C

$$= \frac{0\cdot017 \times 33 \times 0\cdot045}{21\cdot6} = 0\cdot001165\ \Omega$$

Resistance (hot) = 0·00138 Ω.
Loss in bars = 64·5 × 64·5 × 0·00138 = 5·8 W
Two end rings of copper, each of 40 mm² area.

$$\text{Resistance of ring at } 15^\circ\text{C} = \frac{0\cdot017 \times \pi \times 0\cdot113}{40} = 0\cdot000151$$

$$R_{\text{hot}} = 0\cdot000178$$

Loss in 2 rings = 2 × 113 × 113 × 0·000178 = 4·56

Total rotor loss = 10·36
Rotor output = 186·50

196·86

$$\text{Slip} = \frac{10\cdot3}{196\cdot86} \times 100 = 5\cdot25 \text{ per cent}$$

Full-load speed = 947·5 r.p.m.

Inductance of Stator Winding

$$L = \frac{0\cdot4\pi(q_1 z_1)^2 \times \lambda \times p}{10^8}$$

$$\lambda_1 = \frac{17\cdot0}{22\cdot5} + \frac{0\cdot5}{2\cdot0} = 1\cdot01$$

$$\lambda_S = \left[\frac{1{\cdot}01}{0{\cdot}96 \times 2}\right] \times 3{\cdot}3 = 1{\cdot}71$$

$$\lambda_Z = 6{\cdot}9$$
$$\lambda_T = 6{\cdot}44$$
$$\lambda_S = 1{\cdot}71$$
$$\lambda = 15{\cdot}05$$

$$L = \frac{0{\cdot}4\pi \times 2 \times 2 \times 110 \times 110 \times 15{\cdot}05 \times 6}{10^8} = 0{\cdot}0546 \text{ H}$$

$$L\omega = 2\pi \times 50 \times 0{\cdot}0546 = 17{\cdot}1 = X_1$$

$$X_1 I_1 = 17{\cdot}1 \times 1{\cdot}0 = 17{\cdot}1 \text{ V}$$

$$R_1 I_1 = 8{\cdot}9 \times 1{\cdot}0 = 8{\cdot}9 \text{ V}$$

Resistance of Rotor Referred to Stator

$$r_2' = \left\{r_b + \frac{r_r}{2\left(\frac{\pi p}{2Q_2}\right)^2}\right\} \times \frac{4m_1 \times (T_1 K)^2}{Q_2}$$

where r_b = resistance of one bar $= \frac{0{\cdot}00138}{33} = 0{\cdot}000042$

$$r_r = \frac{\text{resistance of ring}}{\text{number of rotor bars}} = \frac{0{\cdot}000178}{33} = 0{\cdot}0000054$$

$$\therefore \; r_2' = \left\{0{\cdot}000042 + \frac{0{\cdot}0000054}{2\left(\frac{\pi \times 6}{66}\right)^2}\right\} \frac{12 \times 660 \times 660}{33}$$

$$= \left\{0{\cdot}000042 + \frac{0{\cdot}0000054}{0{\cdot}164}\right\} \times \frac{12 \times 660 \times 660}{33}$$

$$= \frac{0{\cdot}000075 \times 12 \times 660 \times 660}{33} = 11{\cdot}9\ \Omega$$

Reactance of Rotor Referred to Stator

$$x_2' = \frac{x_{be} 4 m_1 (T_1 K)^2}{Q_2}$$

where x_{be} = reactance of one bar and ring segment

$$= \frac{4{\cdot}65}{10^8} \times 2\pi \times 50 = \frac{1{\cdot}46}{10^5}$$

$$\therefore \qquad x_2' = \frac{1\cdot 46}{10^5} \times 12 \times \frac{660 \times 660}{33} = 2\cdot 32$$

The magnetizing reactance, approximately, per phase

$$= \frac{127}{0\cdot 6} = 212\ \Omega$$

Our equivalent circuit is as in Fig. 25.3.

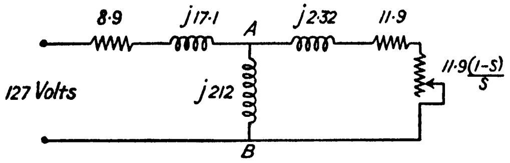


FIG. 25.3

Combined impedance, at $s = 1$

$$= \frac{(11\cdot 9 + j2\cdot 32)(j212)}{11\cdot 9 + j214\cdot 32} = 11\cdot 6 + 2\cdot 92j$$

So short-circuit impedance $= 20\cdot 5 + 20\cdot 02j$

$$Z_{T\text{absol}} = \sqrt{20\cdot 5^2 + 20\cdot 02^2} = 28\cdot 7\ \Omega$$

Short-circuit current $= 4\cdot 43$ A.
Power factor at short-circuit $= 0\cdot 707$.
Impedance of rotor circuit (referred to stator), at standstill

$$= \sqrt{11\cdot 9^2 + 2\cdot 32^2} = \sqrt{141 + 5\cdot 36} = 12\ \Omega$$

Current in rotor circuit $= \dfrac{4\cdot 43 \times j212}{11\cdot 9 + j214\cdot 32}$ A

Now to calculate the magnetizing current, we have

$$V \simeq E + R_1 I \cos\phi + X_1 I \sin\phi \text{ approximately}$$

The full-load current $= 1\cdot 0$ A.

$$\therefore \quad V \simeq E + 1\cdot 0 \times 8\cdot 9 \times 0\cdot 7 + 17\cdot 1 \times 1\cdot 0 \times 0\cdot 707$$
$$= E + 6\cdot 25 + 12\cdot 0 = E + 18\cdot 25$$

Therefore, the counter e.m.f.

$$= 127 - 18\cdot 25 = 108\cdot 75 \text{ V}$$

Note, on these small motors, the drop in stator impedance is large and must be taken into account in calculating the magnetizing current and iron losses.

Magnetizing Current

Apparent gap area per pole = 20·7 cm².

Correction for slot openings—

Stator $\frac{\omega_s}{\delta} = \frac{2 \cdot 0}{0 \cdot 25} = 8; \ \frac{\omega_s}{p_s} = \frac{0 \cdot 2}{1 \cdot 05} = 0 \cdot 191$

Carter's factor for stator slots = 1·175.

Rotor $\frac{\omega_r}{\delta} = \frac{1 \cdot 0}{0 \cdot 25} = 4; \ \frac{\omega_r}{p_r} = \frac{0 \cdot 10}{1 \cdot 14} = 0 \cdot 087$

Carter's factor for rotor slots = 1·05.

True gap area per pole $= \frac{20 \cdot 7}{1 \cdot 23} = 16 \cdot 8$ cm²

Stator

Teeth

Area at top of slot = 0·45 × 6 × 3·3 × 0·9 = 8·05 cm²

Area at bottom of slot = 0·51 × 6 × 3·3 × 0·9 = 9·1 cm²

Core

Area = 5·05 cm²

Length of path $= \frac{\pi \times 17 \cdot 2}{6} \times \frac{2}{3} = 6$ cm

Rotor

Teeth

Area = 0·525 × 5·5 × 3·3 × 0·9 = 8·6 cm².

Shaft

Diameter = 2 cm.

Core

Area = 13 cm². Length of path = 2·23 cm.

$$\phi = \frac{108 \cdot 75 \times 10^8}{2 \cdot 12 \times 1320 \times 50} = 0 \cdot 78 \times 10^5$$

$$B_{g\,\mathrm{av}} = \frac{0 \cdot 78 \times 10^5}{16 \cdot 8} = 4650; \ B_{g\,60} = 5950$$

$$A_{g\,60} = 0 \cdot 796 \times 5950 \times 0 \cdot 05 = 236$$

$$B_{ST}\ \mathrm{top} = \frac{0 \cdot 78 \times 10^5}{8 \cdot 05} = 9700$$

$$B = \frac{0{\cdot}78 \times 10^5}{9{\cdot}1} = 8600; \; B_{60} = 11\,000$$

$$A_{ST} = 10 \times 3{\cdot}5 = 35$$

$$B_{SC} = \frac{0{\cdot}39 \times 10^5}{5{\cdot}05} = 7775$$

$$A_{SC} = 4{\cdot}5 \times 6 = 27$$

$$B_{RT} = \frac{0{\cdot}78 \times 10^5}{8{\cdot}6} = 9050; \; B_{60} = 11\,600$$

$$A_{RT} = 9{\cdot}5 \times 1{\cdot}0 = 9{\cdot}5$$

$$B_{RC} = \frac{0{\cdot}39 \times 10^5}{13} = 3000$$

$$A_{RC} = 2{\cdot}0$$

$$\bar{A}_{60} = 310$$

Magnetizing amperes

$$= \frac{310}{2{\cdot}12 \times 2 \times 110} = 0{\cdot}665 \text{ A}$$

Short-circuit Current

$$\lambda_1 = \frac{17{\cdot}0}{22{\cdot}5} + \frac{0{\cdot}5}{2{\cdot}0} = 0{\cdot}76 + 0{\cdot}25 = 1{\cdot}01$$

$$\lambda_2 = 0{\cdot}62 + \frac{0{\cdot}5}{1{\cdot}0} = 1{\cdot}12$$

$$\lambda_S = \left[\frac{1{\cdot}01}{0{\cdot}96 \times 2} + \frac{1{\cdot}12}{1{\cdot}83}\right] \times 3{\cdot}3$$

$$= [0{\cdot}52 + 0{\cdot}61] \times 3{\cdot}3 = 3{\cdot}72$$

$$\lambda_T = 0{\cdot}635 \times 10{\cdot}1 = 6{\cdot}44$$

$$\lambda_Z = \frac{L\varepsilon\tau}{48 \times \delta \times q_1 \times q_2} = \frac{3{\cdot}3 \times 6{\cdot}28 \times 1{\cdot}475}{48 \times 0{\cdot}025 \times 2 \times 1{\cdot}83} = 6{\cdot}9$$

$$A = \frac{S_2 - a_1}{0{\cdot}96\tau_2} = \frac{1{\cdot}035 - 0{\cdot}2}{0{\cdot}96 \times 1{\cdot}135} = \frac{0{\cdot}835}{0{\cdot}96 \times 1{\cdot}135} = 0{\cdot}76$$

$$B = \frac{S_1 - a_2}{\tau_1} = \frac{0{\cdot}85 - 0{\cdot}1}{1{\cdot}05} = \frac{0{\cdot}75}{1{\cdot}05} = 0{\cdot}715$$

$$\varepsilon = 1{\cdot}475$$

$$\lambda_S + \lambda_Z + \lambda_T = 17{\cdot}06$$

$$I_{sci} = \frac{0{\cdot}78 \times 10^5}{3{\cdot}55 \times 2 \times 110 \times 17{\cdot}06} = 5{\cdot}85 \text{ A}$$

Check by Rough Method

Volts per conductor at 50 c/s

$$= \frac{127}{1320} = 0{\cdot}0965$$

$$I_{sc} = \frac{1000}{110} \times \frac{0{\cdot}0965}{0{\cdot}177} = 5 \text{ A}$$

$$\sigma = \frac{0{\cdot}665}{5{\cdot}0} = 0{\cdot}133$$

$\cos \phi$ full load = 73 per cent approximately

Iron Loss

Volume of core $= 5{\cdot}06 \times \pi \times 17{\cdot}2 \text{ cm}^3$

Weight of core

$$= 5{\cdot}06 \times \pi \times 17{\cdot}2 \times \frac{7{\cdot}9}{1000} \times 2{\cdot}2 = 4{\cdot}75 \text{ lb}$$

Weight of teeth

$$0{\cdot}48 \times 1{\cdot}75 \times 36 \times 3{\cdot}3 \times 0{\cdot}9 \times \frac{7{\cdot}9}{1000} \times 2{\cdot}2 = 1{\cdot}56 \text{ lb}$$

Loss in core $= 4{\cdot}75 \times 2{\cdot}50 = 11{\cdot}90$
Loss in teeth $= 1{\cdot}56 \times 4{\cdot}77 = 7{\cdot}45$
Total iron loss $= 19{\cdot}35$

Stator I^2R loss	=	26·70
Rotor I^2R	=	10·36
Iron	=	19·35
B.F.W.	=	10·00
Total losses	=	66·41
Output	=	186·50
		252·91

Efficiency = 73·5 per cent

Loss in the rotor with ideal short-circuit current

$$= 510 \times 510 \times 0{\cdot}00138 = 360 \text{ W in bars}$$

Loss in rings with ideal short-circuit current

$$= 2 \times 895 \times 895 \times 0{\cdot}000178 = 288 \text{ W}$$

Total loss in rotor = 648 W
Equivalent loss per phase = 216 W

Loss in stator per phase with ideal short-circuit current

$$= 5{\cdot}0 \times 5{\cdot}0 \times 8{\cdot}9 = 223 \text{ W}$$

Loss per phase, due to ideal short-circuit current,

$$= 216 + 223 = 439 \text{ W}$$

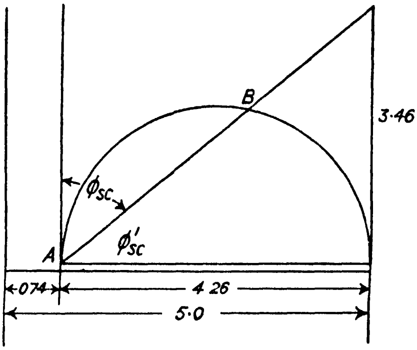


FIG. 25.4

Watt-amperes corresponding

$$= \frac{439}{127} = 3{\cdot}46$$

$$\tan \phi_{sc}' = \frac{3{\cdot}46}{4{\cdot}26} = 0{\cdot}81$$

$$\phi_{sc}' = 39^\circ$$

$$\phi_{sc} = 90^\circ - 39^\circ = 51^\circ$$

$$\cos \phi_{sc} = 0{\cdot}6293$$

In Fig. 25.4, $AB = 4{\cdot}26 \cos 39^\circ = 4{\cdot}26 \times 0{\cdot}7771 = 3{\cdot}31$

Maximum horse-power

$$= \frac{\sqrt{3} \times 220 \times 3{\cdot}31}{2(1 + 0{\cdot}6293) \times 746} = \frac{\sqrt{3} \times 220 \times 3{\cdot}31}{3{\cdot}258 \times 746}$$

$$= 0{\cdot}515 \text{ b.h.p.}$$

$$= 2{\cdot}06 \text{ F.L.}$$

Loss in Rotor at Standstill

Loss in bars $= 398 \times 398 \times 0{\cdot}00138 = 218$ W

Loss in rings $= 2 \times 695 \times 695 \times 0{\cdot}000178 = 174$ W

Total loss in rotor $= \underline{\underline{392}}$ W

Starting torque in synchronous watts

$$= 2{\cdot}08 \times \text{full-load torque}$$

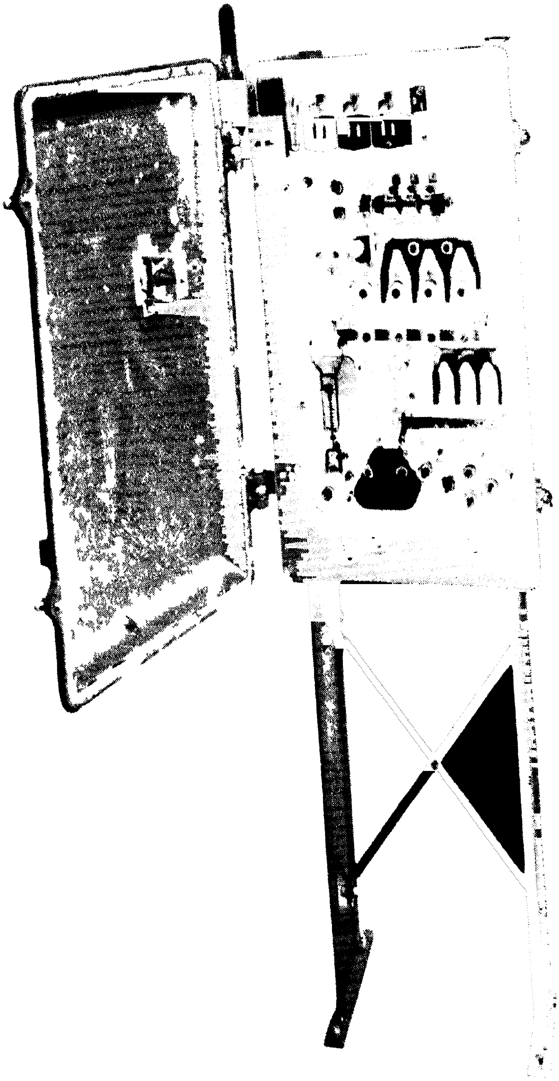


Plate XXIII Star-delta Starter

(Courtesy B. T. H. Co., Rugby)

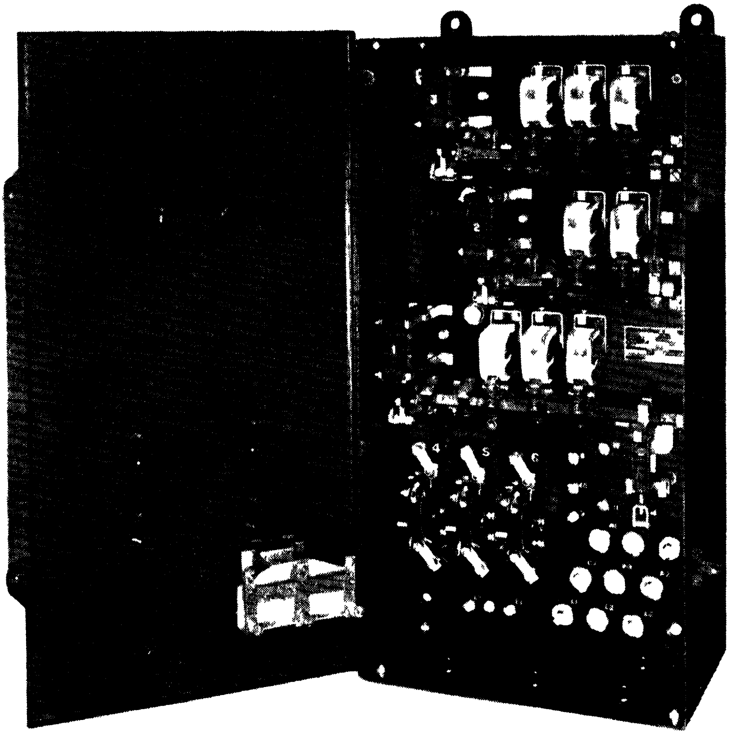


PLATE XXIV. CONTACTOR STARTER FOR SQUIRREL-CAGE MOTORS UP TO 50 H.P.

(Courtesy B.T.H. Co., Ltd.)

DESIGN OF SINGLE-PHASE MOTOR

Output = $\frac{1}{6}$ b.h.p.; 220 V; 50 c/s; 1000/955 r.p.m.

Number of poles = 6; ventilated type with fan.

$$D^2L = \frac{6{\cdot}1 \times 10^{11} \times \text{b.h.p.}}{B_g \times q \times \text{r.p.m.} \times \cos\theta \times \eta}$$

$$= \frac{6{\cdot}1 \times 10^{11} \times 1}{3500 \times 150 \times 6 \times 960 \times 0{\cdot}62 \times 0{\cdot}55} = 590 \text{ cm}^3$$

Internal diameter of stator = 12 cm.
Core length = 4 cm.

$$\text{Apparent gap area per pole} = \frac{\pi \times 11{\cdot}95 \times 4{\cdot}0}{6} = 25 \text{ cm}^2$$

$$\text{The pole-pitch} = \frac{\pi \times 12}{6} = 6{\cdot}28 \text{ cm}$$

Number of slots = 36.

$$\text{Approximate current} = \frac{125}{220 \times 0{\cdot}55 \times 0{\cdot}65} = 1{\cdot}6 \text{ A}$$

Conductor: No. 21 s.w.g. wire.
Diameter bare = 0.032 in.
Lewmex covering = 0·006 in.
Diameter covered = 0·038 in. = 0·965 mm.

Two-thirds of the stator slots per pole will be occupied by the *running winding*, and one-third by the *starting winding*.

$$\phi = 25 \times 3500 = 87\,000$$

$$Z = \frac{164 \times 10^8}{1{\cdot}84 \times 87\,000 \times 50} = 2060$$

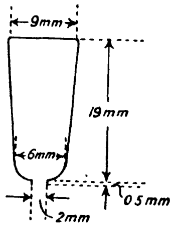


FIG. 25.5

Running Winding

$$\text{Conductors per slot} = \frac{2060}{24} = 86$$

Take 90 conductors per slot.

Space required in slot

$$= 90 \times 0{\cdot}7854 \times 0{\cdot}965 \times 0{\cdot}965 = 66 \text{ mm}^2$$

Area of slot required (*see* Fig. 25.5)

$$= 132 \text{ mm}^2$$

$$\text{L.M.T.} = 2\left[4{\cdot}0 + \frac{\pi \times 13{\cdot}95}{6} \times 1{\cdot}4\right] = 28{\cdot}4 \text{ cm}$$

Resistance of running winding at 15°C

$$= \frac{0{\cdot}017 \times 12 \times 90 \times 0{\cdot}284}{0{\cdot}5189} = 10{\cdot}05\ \Omega$$

$$R_{\text{hot}} = 11{\cdot}9\ \Omega$$

Copper weight

$$= 0{\cdot}0197 \times 12 \times 90 \times 0{\cdot}284 \times 0{\cdot}5189 = 3{\cdot}14 \text{ lb}$$

$$RI \text{ drop} = 1{\cdot}6 \times 11{\cdot}9 = 19{\cdot}0 \text{ V}$$

$$I^2R \text{ in running winding} = 19{\cdot}0 \times 1{\cdot}6 = 30{\cdot}3 \text{ W}$$

$$\text{Current density} = \frac{1{\cdot}6}{1{\cdot}5189} = 3{\cdot}08 \text{ A/mm}^2$$

Running Winding

Single-layer, 24 slots, 12 coils, 90 turns per coil; number of turns in series = 1080.

Conductors in series per phase = 2160.

Starting Winding

12 slots, 6 coils; turns per coil = 180.

$$180 \times 0{\cdot}7854 \times d_c^2 = 66$$

$$d_c^2 = 0{\cdot}466$$

$$d_c = 0{\cdot}68 \text{ mm} = 0{\cdot}0268 \text{ in.}$$

No. 24 s.w.g., diameter bare = 0·022 in.
Lewmex covering = 0·0052 in.
Diameter covered = 0·0272 in. = 0·69 mm.

$$180 \times 0{\cdot}7854 \times 0{\cdot}69 \times 0{\cdot}69 = 67{\cdot}5$$

Resistance at 15°C

$$= \frac{0{\cdot}017 \times 6 \times 180 \times 0{\cdot}284}{0{\cdot}2453} = 21{\cdot}3\ \Omega$$

Resistance (hot) = 25·0 Ω

Copper weight

$$= 0{\cdot}0197 \times 6 \times 180 \times 0{\cdot}284 \times 0{\cdot}2453 = 1{\cdot}48 \text{ lb}$$

Single air-gap length = 0·25 mm.

Rotor

External diameter of rotor = 11·95 cm.
Number of rotor slots = 33
Fig. 25.6 shows a slot, skewed one-slot pitch.
Length of bar = 5·5 cm.
Bar diameter = 5·25 mm.
Resistance of bars at 15°C

$$\frac{0{\cdot}017 \times 33 \times 0{\cdot}055}{21{\cdot}6} = 0{\cdot}00143\ \Omega$$

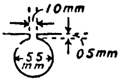


FIG. 25.6

Resistance of bars (hot) $= 0{\cdot}00169\ \Omega$

Resistance per bar (hot) $= \dfrac{0{\cdot}00169}{33} = 0{\cdot}0000512$

Two end rings of copper, each ring of area 40 mm.^{2}

$$\text{Resistance of ring} = \frac{0{\cdot}017 \times \pi \times 0{\cdot}113}{40} = 0{\cdot}000151$$

$$R_{\text{hot}} = 0{\cdot}000178$$

Resistance of sector of ring between 2 bars

$$= \frac{0{\cdot}000178}{33} = 0{\cdot}0000054$$

Secondary Resistance Referred to Primary

I^2R loss in rotor $= I_2{}^2Q_2r_b + 2I_r{}^2R_r$

where I_2 = current per bar (r.m.s.) (p = poles)

Q_2 = number of rotor bars

r_b = resistance *per bar*, hot

r_r = resistance of one ring sector between two bars

R_r = resistance of one ring

α = electrical angle of displacement between 2 bars

I_r = current per ring

$$I_r = \frac{I_2}{2\sin\frac{\alpha}{2}} = \frac{I_2}{2\sin\left(\frac{p\pi}{2Q_2}\right)}$$

$$\text{Loss in rotor} = I_2{}^2Q_2r_b + \frac{2I_2{}^2R_r}{4\sin^2\left(\frac{p\pi}{2Q_2}\right)}$$

$$I_2{}^2r_2 \simeq Q_2I_2{}^2\left[r_b + \frac{r_r}{2\left(\frac{p\pi}{2Q_2}\right)^2}\right]$$

since $Q_2r_r = R_r$ and since α is small, the sine can be replaced by the angle in circular measure.

$$\therefore \qquad r_2 = \left[r_b + \frac{r_r}{2\left(\frac{p\pi}{2Q_2}\right)^2}\right] \times Q_2$$

If I_2' = rotor current, referred to the stator

and T_1 = number of turns in series per phase in stator

K_1 = winding factor for fundamental of stator winding

Then $2I_2'T_1K_1 = I_2Q_2$

$$\therefore \qquad I_2' = \frac{I_2 Q_2}{2T_1K_1}$$

since

$$(I_2')^2 r_2' = I_2^2 r_2$$

$$r_2' = \frac{I_2^2 r_2}{(I_2')^2} = \frac{4T_1^2K_1^2 r_2}{Q_2^2}$$

$$= \frac{4T_1^2K_1^2}{Q_2}\left\{r_b + \frac{r_r}{2\left(\frac{p\pi}{2Q_2}\right)^2}\right\}$$

Therefore, in our case

$$r_2' = \frac{4 \times 1080 \times 1080 \times 0{\cdot}96 \times 0{\cdot}96}{33}\left[0{\cdot}000051 + \frac{0{\cdot}0000054}{2\left(\frac{6 \times \pi}{66}\right)^2}\right]$$

$$= 1{\cdot}295 \times 10^5 \left[0{\cdot}0000512 + \frac{0{\cdot}0000054}{0{\cdot}164}\right]$$

$$= 1{\cdot}295 \times 10^5[0{\cdot}0000512 + 0{\cdot}000033]$$

$$= 1{\cdot}295 \times 10^5 \times \frac{8{\cdot}42}{10^5} = 10{\cdot}9\ \Omega$$

The reactance of the rotor, referred to the stator,

$$x_2' = x_{be}\frac{4T_1^2K_1^2}{Q_2}$$

where x_{be} = equivalent reactance of one bar and one ring sector

$$x_2' = \frac{4 \times 1080 \times 1080 \times 0{\cdot}96 \times 0{\cdot}96}{33} \times x_{be}$$

$$= 1{\cdot}295 \times 10^5 \times \frac{1{\cdot}75}{10^5} = 2{\cdot}27\ \Omega$$

Inductance of Stator Winding

$$L = \frac{0{\cdot}4\pi \times (q_1 z_1)^2}{10^8} \times p \times \lambda$$

$$\lambda_S = \left[\frac{19}{22{\cdot}5} + \frac{0{\cdot}5}{2{\cdot}0}\right] \times \frac{4}{0{\cdot}96 \times 4} = (0{\cdot}845 + 0{\cdot}25) \times \frac{1}{0{\cdot}96}$$

$$= \frac{1{\cdot}095}{0{\cdot}96} = 1{\cdot}14$$

$$\lambda_T = 0{\cdot}64 \times 10{\cdot}2 = 6{\cdot}53$$

$$\lambda_Z = \frac{4{\cdot}0 \times 6{\cdot}28 \times 1{\cdot}475}{16 \times 0{\cdot}025 \times 4 \times 5{\cdot}5} = 4{\cdot}2$$

$$\lambda = 1{\cdot}14 + 6{\cdot}53 + 4{\cdot}2 = 11{\cdot}87$$

$$L = \frac{0{\cdot}4\pi \times 4 \times 4 \times 90 \times 90 \times 11{\cdot}87 \times 6}{10^8} = 0{\cdot}1157$$

$$X_1 = 0{\cdot}1157 \times 2\pi \times 50 = 36{\cdot}4\,\Omega$$

Resistance of stator winding (RW) hot

$$= 11{\cdot}9\,\Omega = R_1$$

We have, approximately

$$\bar{V} = \bar{E} + R_1\bar{I}\cos\phi + X_1\bar{I}\sin\phi$$

$$\therefore \quad V \simeq E + 19{\cdot}0 \times 0{\cdot}5 + 58{\cdot}5 \times 0{\cdot}867$$

$$\simeq E + 9{\cdot}5 + 51$$

$$220 \simeq E + 60{\cdot}5$$

$$\therefore \quad E \simeq 159{\cdot}5$$

Apparent gap area per pole = 25 cm^2.
Carter's factor for slots = 1·23.
True gap area per pole = 20·4 cm^2.

Stator

Teeth

Area at top $0{\cdot}49 \times 6 \times 4{\cdot}0 \times 0{\cdot}9 = 10{\cdot}6$

Area at bottom $0{\cdot}51 \times 6 \times 4{\cdot}0 \times 0{\cdot}9 = 11{\cdot}05$

Core

$1{\cdot}5 \times 4 \times 0{\cdot}9 = 5{\cdot}4$ cm^2

Length of path $= \dfrac{\pi \times 17{\cdot}4}{6} \times \dfrac{2}{3} = 6{\cdot}05$ cm

External diameter of stator = 18·9 cm = 7·5 in.

Rotor

Teeth

$0{\cdot}525 \times 5{\cdot}5 \times 4{\cdot}0 \times 0{\cdot}9 = 10{\cdot}3$

Core

Area = 15·75 cm^2. Length of path = 2·23 cm

$$\hat{\phi} = \frac{155{\cdot}4 \times 10^8}{1{\cdot}84 \times 2160 \times 50} = 0{\cdot}78 \times 10^5$$

$$B_{g\,\mathrm{av}} = \frac{0{\cdot}78 \times 10^5}{20{\cdot}4} = 3820;\ B_M = 6000$$

$$A_M = 0{\cdot}796 \times 6000 \times 0{\cdot}05 = 240$$

$$B_{ST\,\mathrm{av}}\ \mathrm{top} = \frac{0{\cdot}78 \times 10^5}{10{\cdot}6} = 7375$$

$$B = \frac{0\cdot78 \times 10^5}{11\cdot05} = 7070;\ B_M = 11\,100$$

$$A_{ST} = 9 \times 3\cdot9 = 35$$

$$B\ (\text{stator core}) = \frac{39\,000}{5\cdot4} = 7250$$

$$\bar{A}\ (\text{stator core}) = 3\cdot5 \times 6\cdot05 = 21$$

$$B\ (\text{rotor teeth}) = \frac{78\,000}{10\cdot3} = 7570;\ B_M = 11\,900$$

$$\bar{A}_{RT} = 5$$

$$B\ (\text{rotor core}) = \frac{39\,000}{15\cdot75} = 2500$$

$$\bar{A}_{RC} = 2\cdot0$$

Ampere-turns:
gap = 240
stator teeth = 35
stator core = 21
rotor teeth = 5
rotor core = 2

303

And approximate magnetizing current

$$= \frac{303}{0\cdot755 \times 4 \times 90} = 1\cdot11\ \text{A}$$

We need to calculate x_m

$$x_m = 3\cdot2f \times (T_1K_1)^2 \times \frac{\tau l_c}{p\delta C_1\beta} \times 10^{-8}\ \Omega$$

This is derived as follows—

The e.m.f. generated by $\hat{\phi}$

$$= \hat{E} = 4\cdot44 \times K_1T_1 \times \hat{\phi} \times 10^{-8} = I\mu x_m'$$

where x_m' = magnetizing reactance
and I = magnetizing current
if $\hat{B}$ = maximum flux density in the gap

Then $$\hat{\phi} = \frac{2}{\pi} \times \hat{B} \times \tau \times l_c$$

where τ = pole pitch
and l_c = core length in centimetres

The alternating m.m.f. of the single-phase machine has the amplitude

$$= \frac{1\cdot8\ T_1K_1\ I_\mu}{p} \times 0\cdot4\pi$$

and $$\hat{B}_{\max} = \frac{1 \cdot 8 T_1 K_1 I_\mu}{p} \times \frac{1 \times 0 \cdot 4\pi}{\delta \times C_1 \times \beta}$$

where C_1 = Carter's factor for slots

$\beta = \dfrac{\hat{A}_{\text{total}}}{\hat{A}_{\text{gap}}}$ = saturation factor

δ = air-gap length

$$\therefore\ E = 4 \cdot 44 K_1 T_1 \times \frac{2}{\pi} \times \frac{1 \cdot 8 T_1 K_1 I_\mu}{p\delta C_1 \beta} \times \tau \times l_c \times 10^{-8} \times 0 \cdot 4\pi \times f$$

$$= 6 \cdot 4 (K_1 T_1)^2 \times \frac{\tau l_c}{p\delta C_1 \beta} \times I_\mu \times \frac{1}{10^8} \times f$$

$$\therefore\qquad x_m' = \frac{E}{I_\mu} = 6 \cdot 4 (K_1 T_1)^2 \times \frac{\tau l_c}{p\delta C_1 \beta} \times \frac{1}{10^8} \times f$$

x_m' is the reactance of the alternating field.

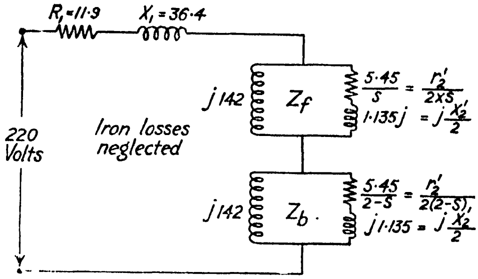

$R_1 = 11 \cdot 9$
$X_1 = 36 \cdot 4$
220 Volts
Iron losses neglected
$j142$
Z_f
$\frac{5 \cdot 45}{S} = \frac{r_2'}{2 \times S},$
$1 \cdot 135 j = j\frac{X_2'}{2}$
$j142$
Z_b
$\frac{5 \cdot 45}{2-S} = \frac{r_2'}{2(2-S)},$
$j1 \cdot 135 = j\frac{X_2'}{2}$

FIG. 25.7

The reactance x_m, which we need

$$= \frac{x_m'}{2} = 3 \cdot 2 (K_1 T_1)^2 \times \frac{\tau l_c}{p\delta C_1 \beta} \times \frac{1}{10^8} \times f$$

In our case

$$x_m = \frac{3 \cdot 2 \times 0 \cdot 84 \times 0 \cdot 84 \times 1080 \times 1080 \times 6 \cdot 28 \times 4 \times 50}{6 \times 0 \cdot 025 \times 1 \cdot 23 \times 1 \cdot 265 \times 10^8}$$
$$= 142\ \Omega$$

This could have been obtained at once from

$$\frac{E}{I_\mu} = \frac{155 \cdot 5}{1 \cdot 11} = 140$$

Our equivalent circuit (*see* Fig. 25.7) from which the performance can be calculated, has the following constants—

$$r_2' = 10 \cdot 9; \quad \frac{r_2'}{2} = 5 \cdot 45; \quad x_2' = 2 \cdot 27; \quad \frac{x_2'}{2} = 1 \cdot 135$$

$$x_m = 142\ \Omega; \quad r_1 = 11 \cdot 9; \quad x_1 = 36 \cdot 4$$

Standstill

$$s = 1$$

$$Z_f = \frac{(5{\cdot}45 + j1{\cdot}135)(j142)}{5{\cdot}45 + j143{\cdot}135} = \frac{774j - 161}{5{\cdot}45 + j143{\cdot}135}$$

Rationalizing $Z_f = \dfrac{(774j - 161)(5{\cdot}45 - j143{\cdot}135)}{5{\cdot}45^2 + 143{\cdot}135^2}$

$$= \frac{4220j - 875 + 111\,000 + 23\,000j}{29{\cdot}8 + 20\,500}$$

$$= \frac{110\,125 + 27\,220j}{20\,529{\cdot}8}$$

$$= 5{\cdot}39 + 1{\cdot}325j$$

$Z_b = Z_f$ when $s = 1$.

Therefore, total series impedance ($s = 1$) $= Z_T$

$$= 11{\cdot}9 + j36{\cdot}4 + 10{\cdot}78 + 2{\cdot}650j$$

$$= 22{\cdot}68 + j39{\cdot}05$$

$$|Z_T| = \sqrt{22{\cdot}68^2 + 39{\cdot}05^2}$$

$$= \sqrt{606 + 1530} = 46{\cdot}25\ \Omega$$

Short-circuit current

$$= \frac{220}{46{\cdot}25} = 4{\cdot}75\ \text{A}$$

$$\tan\phi_{sc} = \frac{39{\cdot}05}{22{\cdot}68} = 1{\cdot}725;\ \phi_{sc} = 59^\circ\ 54'$$

$$\cos\phi_{sc} = 0{\cdot}5015$$

Ideal short-circuit current

$$= \frac{220}{39{\cdot}05} = 5{\cdot}63\ \text{A}$$

Maximum horse-power

$$= \frac{220 \times 3{\cdot}86}{3{\cdot}0 \times 746} = 0{\cdot}378$$

$$\sigma = \frac{1{\cdot}11}{5{\cdot}63} = 0{\cdot}197$$

At a slip of 3 per cent

$$Z_f = \frac{\left(\frac{5{\cdot}45}{0{\cdot}03} + j1{\cdot}135\right)(j142)}{\frac{5{\cdot}45}{0{\cdot}03} + j143{\cdot}135} = \frac{(181{\cdot}6 + j1{\cdot}135)j142}{181{\cdot}6 + j143{\cdot}135}$$

$$= \frac{24\,800j - 161}{181{\cdot}6 + j143} = \frac{248j - 1{\cdot}61}{1{\cdot}816 + j1{\cdot}43}$$

$$s = 0{\cdot}03$$

Rationalizing

$$Z_f = \frac{248j - 1{\cdot}61}{1{\cdot}816 + j1{\cdot}43} \times \frac{(1{\cdot}816 - j1{\cdot}43)}{1{\cdot}816 - j1{\cdot}43}$$

$$= \frac{450j - 2{\cdot}93 + 353 + 2{\cdot}3j}{3{\cdot}3 + 2{\cdot}1}$$

$$= \frac{350{\cdot}07 + 452{\cdot}3j}{5{\cdot}4} = 65 + 84j$$

$$Z_b = \frac{\left(\frac{5{\cdot}45}{1{\cdot}97} + j1{\cdot}135\right)(j142)}{\frac{5{\cdot}45}{1{\cdot}97} + j143} = \frac{(2{\cdot}76 + j1{\cdot}135)(j142)}{2{\cdot}76 + j143}$$

$$= \frac{392j - 161}{2{\cdot}76 + j143}$$

Rationalizing

$$Z_b = \frac{(392j - 161)(2{\cdot}76 - j143)}{2{\cdot}76^2 + 143^2}$$

$$= \frac{1082j - 445 + 56\,000 + 23\,000j}{7{\cdot}6 + 20\,500}$$

$$= \frac{55\,555 + 24\,082j}{20\,507}$$

$$= 2{\cdot}7 + 1{\cdot}175j$$

Total series impedance at $s = 0{\cdot}03$

$$\underset{(s\,=\,0{\cdot}03)}{Z_T} = 11{\cdot}9 + j36{\cdot}4 + 65 + 84j + 2{\cdot}7 + 1{\cdot}175j$$

$$= 79{\cdot}6 + 121{\cdot}575j$$

$$|Z_T| = 10^2\sqrt{0{\cdot}635 + 1{\cdot}475} = 10^2\sqrt{2{\cdot}110} = 145\ \Omega$$

$$\text{Line current} = \frac{220}{145} = 1{\cdot}52\text{ A}$$

$$\tan\phi = \frac{121{\cdot}57}{79{\cdot}6} = 1{\cdot}51$$

$$\phi = 56^\circ\ 30'$$

$$\underset{(s = 0{\cdot}03)}{\cos\phi} = 0{\cdot}5519$$

$$\text{Current in } \left(\frac{r_2'}{2s}\right) \text{ of } Z_f = 1{\cdot}51 \times \frac{142}{230} = 0{\cdot}935\text{ A}$$

$$\text{Loss in } \frac{r_2'}{2s} = 0{\cdot}935 \times 0{\cdot}935 \times 181{\cdot}6 = 159\text{ W}$$

$$\text{Current in } \frac{r_2'}{2(2-s)} \text{ of } Z_b = \frac{142}{143} \times 1{\cdot}51 = 1{\cdot}50$$

$$\text{Loss in } \frac{r_2'}{2(2-s)} = 1{\cdot}5 \times 1{\cdot}5 \times 2{\cdot}76 = 6{\cdot}2\text{ W}$$

Torque in synchronous watts at 3 per cent slip

$$= 159 - 6{\cdot}2 = 153{\cdot}8$$

The forward torque in lb-ft

$$= \frac{7{\cdot}04}{n_s} \times \frac{(I_{2f}')^2 r_2''}{s} = T_1$$

The backward torque in lb-ft

$$= -\frac{7{\cdot}04}{n_s} \times \frac{(I_{2b}')^2 r_2''}{2-s} = T_2$$

Resultant torque $= T_1 - T_2$

Loss

Loss at 3 per cent slip in rotor

$= 0{\cdot}935 \times 0{\cdot}935 \times 5{\cdot}45 = 4{\cdot}8$ W
$+\ 1{\cdot}5 \times 1{\cdot}5 \times 5{\cdot}45 = 12{\cdot}2$ W

Total rotor loss = 17·0 W

Stator copper loss = 30·3 W
Iron loss = 20·0 W
Rotor copper loss = 17·0 W
Friction and windage = 10·0 W

77·3 W

Output = 125·0 W

Input = 202·3 W

Efficiency = 62 per cent

The slip for full load will be less than 4 per cent.

The performance can be readily obtained, for any assumed value of the slip, from the equivalent circuit.

Note that the voltage across Z_b

$$= 1{\cdot}5 \times 2{\cdot}95 = 4{\cdot}45 \text{ V}$$

This shows that the negative-sequence circuit acts simply as an added inductance to the circuit. The field produced, i.e. the negative-sequence field is very small, for practically, at full load, nearly the whole volts are absorbed by the positive-sequence field. It will be noted, however, that the rotor loss is increased greatly by the negative-sequence current.

This motor can be arranged for capacitor start or resistance can be connected in series with one phase. Inductance of starting winding.

$$L_s = \frac{0{\cdot}4\pi(q_s Z_s)^2 \times p \times \lambda_s}{10^8}$$

$$\begin{aligned} \lambda_S &= 2{\cdot}28 \\ \lambda_T &= 6{\cdot}53 \\ \lambda_Z &= \underline{8{\cdot}40} \\ & \underline{\underline{17{\cdot}21}} \end{aligned}$$

$$L_s = \frac{0{\cdot}4\pi \times 2 \times 2 \times 180 \times 180 \times 6 \times 17{\cdot}21}{10^8} = 0{\cdot}167 \text{ H}$$

$$X_s = 0{\cdot}167 \times 2\pi \times 50 = 52{\cdot}5\ \Omega$$

Equivalent Impedances

$$R_s = 25\ \Omega$$

Equivalent impedance of running winding at standstill

$$= 22{\cdot}38 + j39{\cdot}05$$

$$\tan \phi_{sc}\ (\text{RW}) = \frac{39{\cdot}05}{22{\cdot}38} = 1{\cdot}75$$

$$\phi_{sc}\ (\text{RW}) = 60^\circ\ 18'$$

Equivalent impedance of starting winding

$$= 25 + 10{\cdot}48 + j(52{\cdot}5 + 3{\cdot}05)$$

$$= 35{\cdot}48 + j55{\cdot}55$$

$$\tan \phi_{sc}\ (\text{SW}) = \frac{55{\cdot}55}{35{\cdot}48} = 1{\cdot}56$$

Now the current in the running winding at standstill lags the p.d. applied by 60° 18′.

For quadrature relation, we need the current in the starting winding to lead the p.d. by 29° 42′. Therefore, tan ϕ_{sc} (SW) must be 0·5938 *leading*,

$R_{(SW)}$ (including rotor resistance referred to stator)

$$\therefore \qquad X = 0{\cdot}5938 \times 35{\cdot}48 = 21\ \Omega$$

Therefore, the capacitive reactance required

$$= 55{\cdot}5 + 21 = 76{\cdot}5\ \Omega$$

and $$X_c = +\ 76{\cdot}5\ \Omega$$

$$\therefore \qquad C = \frac{10^6}{76{\cdot}5 \times 2\pi \times 50} = 42\ \mu\text{F}$$

The equivalent impedance of the starting winding will then be $(s = 1) = 35{\cdot}48 - j21$

Equivalent impedance of running winding $(s = 1) = 45\ \Omega$

Equivalent impedance of starting winding $(s = 1)$

$$= \sqrt{35{\cdot}48^2 + 21^2} = \sqrt{1257 + 440} = 41\ \Omega$$

With capacitor start, we require about 42 μF in series with the starting winding.

With resistance start, we must be content with much smaller starting torque. We can place resistance in series with the running winding.

The torque

$$T = \frac{450{\cdot}8 \times r_2'}{n_s} \sqrt{\frac{X_s}{X_m}} \frac{1}{\left(1 + \dfrac{x_2'}{x_m}\right)^2} I_M I_S \sin \theta \text{ oz-ft approximately}$$

where n_s = synchronous revolutions per minute

I_M = current in the main winding

I_S = current in the starting winding

X_s and X_m = total *leakage* reactances of starting and main windings

θ = angle between I_M and I_S

The starting torque with resistance start is bound to be low, for I_M is reduced by the addition of external resistance, and sin θ is low and of the order of 0·5.

The performance of this machine is not very good. It is difficult to get good performance on six-pole *single*-phase motors of such small output. It could be improved by using a larger diameter for the machine.

Through the courtesy of the Westinghouse Electric Corporation of America, the several illustrations of fractional horse-power motors are appended. They show, in a remarkably clear manner, how these machines are constructed and built. Plate XVI (upper) shows a totally-enclosed, fan-cooled Life-line Motor, Plate XVI (lower) a drip-proof motor, and Plate XVII (upper) a capacitor start motor. Plate XVII (lower) shows the capacitor starting switch in the end brackets of a single-phase motor, and Plate XVIII shows typical control switches. (Plate XVI faces page 357, plate XVII faces page 472, and plate XVIII faces page 473.)

CHAPTER XXVI

How to Adapt Windings to a Given Frame

It frequently happens in a design office that a winding is required for a given frame for a different horse-power and frequency and speed than the standard.

Let z_1 = turns per coil for standard motor of horse-power $(\text{h.p.})_1$ and frequency f_1 and voltage V_1

z_2 = turns per stator coil for motor of horse-power $(\text{h.p.})_2$ and frequency f_2 and volts V_2

Then for the same number of phases and the same overload capacity

$$z_2 = z_1 \times \frac{V_2}{V_1} \times \sqrt{\frac{(\text{h.p.})_1}{(\text{h.p.})_2}} \times \sqrt{\frac{f_1}{f_2}} \quad . \quad . \quad (26.1)$$

This may be shown as follows—

$$\frac{I_{d1}}{I_1} = \frac{I_{d2}}{I_2} \text{ for same overload capacity} \quad . \quad . \quad (26.2)$$

I_{d1} = ideal short-circuit current per phase of motor 1

I_{d2} = ideal short-circuit current per phase of motor 2

I_1 = full-load current of motor 1

I_2 = full-load current of motor 2

V_1 = applied volts per phase of motor 1

V_2 = applied volts per phase of motor 2

f_1 = frequency of motor 1

f_2 = frequency of motor 2

$$\therefore \qquad \frac{I_{d1}}{I_1} = \frac{V_1}{Kq_1^2 z_1^2 f_1 I_1} \quad . \quad . \quad . \quad (26.3)$$

since

$$I_{d1} = \frac{V_1}{Kq_1^2 z_1^2 f_1}$$

and K = constant

q_1 = slots per pole per phase

z_1 = conductors per slot in stator

$$\frac{I_{d2}}{I_2} = \frac{V_2}{Kq_2^2 z_2^2 f_2 I_2} = \frac{I_{d1}}{I_1} \quad . \quad . \quad . \quad (26.4)$$

$$\therefore \quad q_2^2 z_2^2 = \frac{V_2}{V_1} \times \frac{q_1^2 z_1^2 \times f_1 \times I_1}{f_2 \times I_2} \quad . \quad . \quad (26.5)$$

Now

$$\frac{(\text{h.p.})_1}{(\text{h.p.})_2} = \frac{V_1 I_1}{V_2 I_2} \quad . \quad . \quad . \quad . \quad (26.6)$$

for same power factor and efficiency.

$$\therefore \quad \frac{I_1}{I_2} = \frac{V_2}{V_1} \times \frac{(\text{h.p.})_1}{(\text{h.p.})_2} \quad . \quad . \quad . \quad (26.7)$$

$$\therefore \quad q_2^2 z_2^2 = \frac{V_2^2}{V_1^2} \times \frac{(\text{h.p.})_1}{(\text{h.p.})_2} \times q_1^2 z_1^2 \times \frac{f_1}{f_2} \quad . \quad . \quad (26.8)$$

$$\therefore \quad q_2 z_2 = \frac{V_2}{V_1} \times \sqrt{\frac{(\text{h.p.})_1}{(\text{h.p.})_2}} \times q_1 z_1 \times \sqrt{\frac{f_1}{f_2}} \quad . \quad . \quad (26.9)$$

and if $q_2 = q_1$

$$\therefore \quad z_2 = z_1 \times \frac{V_2}{V_1} \times \frac{(\text{h.p.})_1}{(\text{h.p.})_2} \times \sqrt{\frac{f_1}{f_2}} \quad . \quad . \quad (26.10)$$

To Illustrate This

In the previous example, we had a 15 h.p., three-phase squirrel-cage, 50 c/s, 1000 r.p.m. synchronous induction motor for 440 V.

The turns per stator coil were 28.

It is required to find the turns per stator coil for the same motor running on 40 c/s and giving 12 h.p. at 800 r.p.m. synchronous. The voltage is 500 V. The overload capacity is the same in the two cases.

Turns per coil for 12 h.p., 40 c/s motor

$$= 28 \times \sqrt{\frac{(\text{h.p.})_1}{(\text{h.p.})_2}} \times \frac{V_2}{V_1} \times \sqrt{\frac{f_1}{f_2}}$$

$$= 28 \times \sqrt{\frac{15}{12}} \times \frac{500}{440} \times \sqrt{\frac{50}{40}}$$

= 40 turns per coil

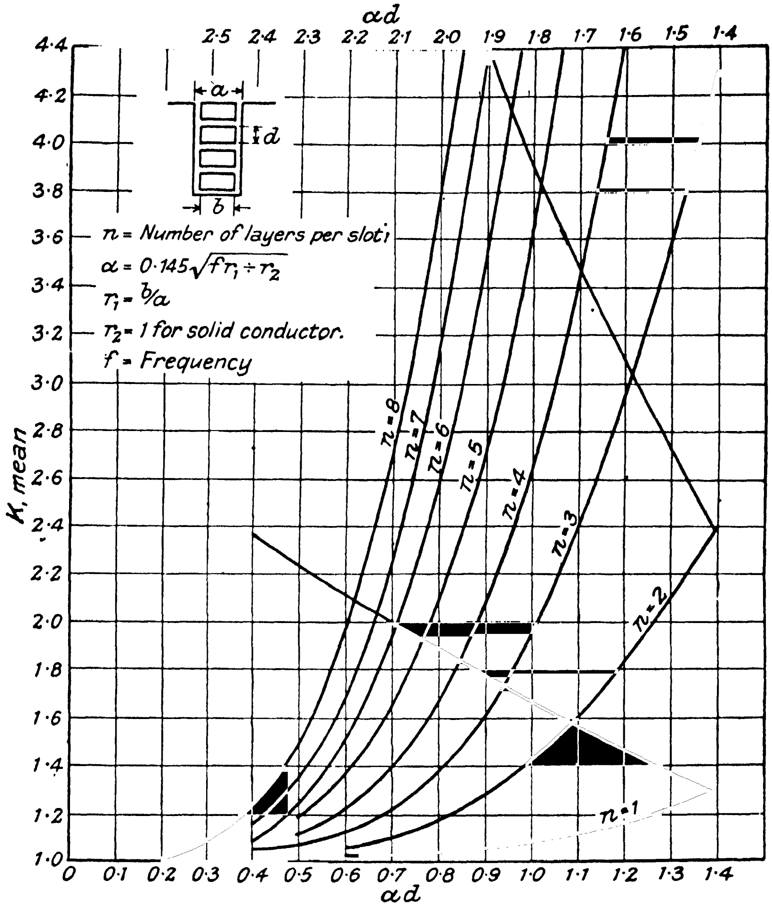


FIG. 26.1. INCREASE IN LOSS DUE TO EDDY CURRENTS IN SLOT COPPER (MEAN VALUE OF FIELD'S CURVES)

Increase in Copper Losses due to Eddy Currents

In estimating the copper losses in the conductors, it is necessary to take into account the increased losses due to eddy currents. This is conveniently done by the aid of Field's curves, which are redrawn to represent the mean increases in loss with different number of layers in the slot.

CHAPTER XXVII

Ratings of Motors

For protected-type motors, the full-load rating is that which will give a temperature rise not exceeding 40°C by thermometer after a continuous run. The time taken to reach a steady temperature will depend on the size of machine. Small machines may take 2 to 3 hr. Large machines seldom longer than 6 hr. Usually a 6-hr run at full load is specified. For machines which have the openings covered with $\frac{1}{16}$ in. mesh gauze covers, the outputs will be approximately 85 per cent of the protected-type outputs for 40°C rise.

For totally-enclosed machines for continuous rating, the

$$\frac{\text{total-enclosed output}}{\text{protected-type output}}$$

will vary with the size of machine. For machines up to about 10 b.h.p. as protected-type machine, the totally-enclosed output will be roughly 50 per cent. For protected-type machines of outputs between 10 and 30 b.h.p., the totally-enclosed output for continuous rating will be about 40 per cent of the protected-type output. Between 30 h.p. and 50 b.h.p., the ratio is about 33 per cent, and for larger machines and ratio decreases to 25 per cent or lower.

Generally it may be said that, for 50°C use by thermometer after 6-hr full-load run as totally-enclosed type, the watts per square centimetre of case area (excluding the bearings) should be roughly about 0·078 or $\frac{1}{2}$ W/in.^{2} of case. One calculates the total losses, excluding the bearing losses, and computes the radiating area of the frame excluding the bearing areas, and the ratio

$$\frac{\text{total losses} - \text{bearing losses}}{\text{radiating area of case}} = \frac{1}{2}\,\text{W/in.}^2$$

for 50°C by thermometer after 6-hr run.

Intermittent Ratings

The following are the relative ratings for protected-type, enclosed ventilated, and totally-enclosed machines for 1 hr and $\frac{1}{2}$ hr respectively—

$$\frac{\text{rating as intermittent type}}{\text{continuous protected-type rating}} = K$$

Protected type	1 hr	40°C	use $K = 1{\cdot}35$
	½ hr	40°C	use $K = 1{\cdot}45$
Enclosed vent	1 hr	50°C	$K = 1{\cdot}13$
	½ hr	50°C	$K = 1{\cdot}25$
Totally enclosed	1 hr	50°C	$K = 100\%$
	½ hr	50°C	$K = 125\%$
Enclosed vent	1 hr	40°C	$K = 1{\cdot}10$
	½ hr	40°C	$K = 1{\cdot}20$
Totally enclosed	1 hr	40°C	$K = 0{\cdot}70$
	½ hr	40°C	$K = 1{\cdot}10$

The author found, after analysis of a large number of totally-enclosed motor tests, that the time constant in minutes of the motor

$$= \frac{1310 \times \text{total weight in pounds}}{\text{area of frame in square centimetres}}$$

Area of frame considered is the sum of the areas of yoke + end shields.

The temperature rise at any time t seconds after the commencement of the run θ

$$\theta = \theta_f(1 - e^{-t/T_c})$$

where θ = temperature use at time t

θ_f = final temperature rise

t = time from commencement of run

T_c = time constant in the same units, either seconds or minutes, whichever is used

Sparking and Leakage Distances

The following table shows the sparking and leakage distances required, and thickness of insulation for various voltages—

Limit Pressure (Volts)	Test Volts		Distance in mm		Thickness of Slot Insulation Tubes	Test Pressure for Slot Insulation
	To Earthed Metal (Volts)	Between Insulated Phases (Volts)	Earthed Metal (mm)	Between Phase Windings (mm)		
650	2 000	1 300	10	5	0·7–1·0 mm pphn.	2 000
1 200	3 000	2 500	15	10	2·0 mm pphn.	4 000
2 200	5 500	4 400	25	20	2·0 mm micanite	16 000
3 300	8 200	6 600	30	25	2·0 ,, ,,	16 000
6 600	14 000	13 000	40	35	2·5 ,, ,,	20 000
8 000	16 000	16 000	50	45	3·5 ,, ,,	30 000
10 000	20 000	20 000	70	60	4·0 ,, ,,	35 000
13 000	26 000	26 000	90	80	5·0 ,, ,,	45 000

Limit Voltage	Test Pressure (Volts)	Sparking Distance to Earthed Metal (mm)	Leakage Distance to Earthed Metal		
			Exposed Surface		Enclosed Surface to Taped Conductor (mm)
			To Bare Conductor (mm)	To Taped Conductor (mm)	
650	2 000	5	10	5	5
1 200	3 000	10	12	10	8
2 200	5 500	15	30	25	20
3 300	8 200	20	35	30	25
6 600	14 000	30	70	60	45
8 000	16 000	40	95	80	60
10 000	20 000	50	130	110	80
13 000	26 000	65	190	160	120

The above are minimum limits and give a safety factor of about 2 on test voltage, assuming the surface damp and dusty.

INSULATION FOR INDUCTION MOTOR WINDINGS

Concentric Coils

The voltage between adjacent conductors should preferably not exceed 40.

Single section coils should be used wherever possible, but the ratio of width to depth of copper should not exceed 5. Two copper strips, side by side in parallel, may be used when the ratio of width to depth of the whole conductor is > 5, and are generally preferable when the ratio exceeds 4. Such coils are more expensive than multiple section coils in series. The latter type should be given preference on voltages greater than 3500.

Insulation between conductors must be either *d.c.c.* or *t.c.c.* *D.c.c.* or *t.c.c.* wire should not be used when either the width or depth of wire exceeds 0·25 in.

Taped conductors of 2 or more wires in parallel may be used when neither the width nor depth of each single wire is less than $\frac{1}{8}$ in. When neither dimension of the whole conductor exceeds $\frac{1}{4}$ in., separate *d.c.c.* or *t.c.c.* wires should be used.

Mica separators between turns are necessary with *d.c.c.* conductors above 3500 V, and with *t.c.c.* or taped conductors above 3500 V.

Mica separators cannot be used if the width of the conductor is less than $\frac{1}{8}$ in.

INSULATION BETWEEN SECTIONS IN SERIES

Insulation between sections in series. For 3500 V and below, alternate sections will be taped with empire cloth. For higher voltages, mica separators will be used.

INTERNAL INSULATION

Around conductors—*d.c.c.* or *t.c.c.*—0·015 in.

	Width (in.)	Depth (in.)
Mica between *d.c.c.* conductors 3500–6600 V	$0·02 \times s$	$0·01 \times n - 1$
Mica between *d.c.c.* conductors above 6600 V	$0·04 \times s$	$0·02 \times (n - 1)$
Mica between *t.c.c.* or tapped conductors above 6600 V	$0·02 \times s$	$0·01 \times n - 1$
Empire tape on alternate sections 3500 V and below	$0·028 \times (s - 1)$	0·028
Mica between sections of coils above 3500 V	$0·03 \times (s - 1)$	0·03

s = number of sections in series
n = number of conductors in depth

EXTERNAL INSULATION

Voltage	Width (in.)	Depth (in.)
0–600	0·10	0·22
601–2,200	0·15	0·27
2,201–3,500	0·17	0·29
350–6,600	0·21	0·33
6,601–9,000	0·26	0·39
9,001–11,000	0·31	0·42
11,001–13,000	0·38	0·47

The overhang will be taped with white tape ½ in. lap.

Insulation for Rotor Windings, with 2 Bars per Slot

A slot lining of 0·15 mm paper is used. On the straight portion of the coil the following thicknesses of paper (ironed on to the coil) are used—

Volts between Rings at Start	*Thickness of Paper*
$\leq$ 300	0·5 mm
301–600	1·0 mm
Above 600	1·5 mm

The insulation projects a distance of 7 to 15 mm, according to the voltage from the core.

Double thickness of 0·2 mm tape on overhang.

CHAPTER XXVIII

Control Devices

THIS chapter is devoted to the consideration of switching and control devices, necessary for placing the motor in service, to protect it from overloads or undervoltage, to take it out of service and to control all abnormal conditions which might affect it. The various types of manual and automatic controls, relays of different types, starters, etc., will be examined.

SMALL MOTORS: MANUAL CONTROLS

ORDINARY knife switches with fuses are used to connect the motor to the supply circuit, and the fuses protect the generating equipment against short-circuit. The motors are further protected against continuous overload by the use of a thermal relay or a solenoid relay.

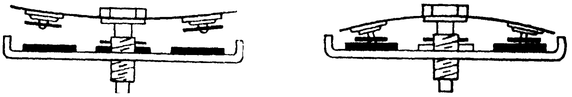


FIG. 28.1

(*Courtesy Westinghouse Electric Corp. of America*)

When thermal overload relays are used, the main line contactors are closed against a spring with a latch support. The thermal strip deflects, when overheated, and releases the latch which supports the spring. Manual starters of this type are generally used in connection with fractional-horse-power motors only.

The motors which are subject to overload or abnormal heating conditions, such as domestic refrigerator motors or stokers, are provided by Westinghouse with "Thermoguard" burn-out protection (Fig. 28.1).

Direct-on-line starters are only used for small motors in this country, because of the large starting current, which, as we have seen, is largely wattless. Such currents of six to eight times full-load value, of a power factor of the order of 0·5, have marked effect on the voltage regulation of the system, and, of course, if many such motors are started together, the effect is very objectionable. In America, where the systems are large, larger size motors are switched direct on.

Magnetic Controls

These use contactors which are opened or closed by means of solenoids or magnets. Thermal or solenoid relays are used in this type of control to break the circuit to the solenoids. Magnetic controls are designed to operate either automatically, or by push-button controls. Overload protection, under-voltage protection, and under-voltage release are usually provided. Overload protection is usually provided by the thermal or solenoid overload relay. The thermal relay can be reset as soon as it is cool enough. This relay is not affected by the peak starting current and remains inoperative on small overloads of short duration. Its operation depends entirely on the heating characteristics of the current flowing and the temperature conditions of the motor are followed closely. Solenoid relays, with two or three single-poles with or without adjustable dashpot time-lags, are often used. Such relays are calibrated at four tripping points over the range of current for which they are designed from full-load to twice full-load current.

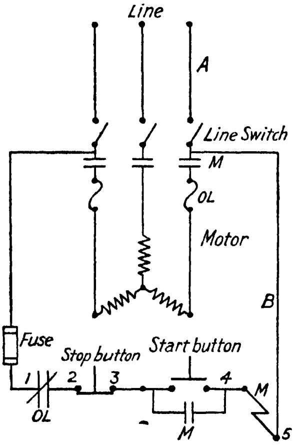


Fig. 28.2

When operated at reduced voltage, motors take excessive current values, and it is essential to provide some control which will take the motor off the line in such cases. The power circuit consists of a line switch, main line contacts, and overload relay coils or heaters. The control circuit consists of a control fuse, contacts of overload relay, stop push-button, start push-button of the momentary contact type, and sealing interlocks, which maintain the circuit after the start button is released.

On conditions of under-voltage occurring, the control circuit is opened by a contactor coil and disconnects the motor from the line. (*See* Fig. 28.2.)

M = main line contacts
OL = overload relay coil or heaters
(1–2) = contacts of overload relay
(3–4) = sealing interlock
(4–5) = contactor coil
A = power circuit
B = control circuit

As soon as the coil (4–5) M is energized, the three main contacts, M, close and connect the motor to the line.

In the case of overload, the *OL* coils in the power circuit will cause the *OL* contacts (1–2) to trip, thereby breaking the circuit to contactor coil *M* (4–5) and allowing the main line contacts to open. The motor is started by pressing the start button in the control circuit. Plate XIX illustrates a 3 h.p. reversing contactor starter, and two triple-pole contactors, mechanically interlocked for forward and reverse running may be seen; while Plate XX shows a contactor starter with mechanically interlocked triple-pole contactors. Thermal overload relays in three phases are mounted at the bottom.

LARGE MOTORS, AUTO-TRANSFORMER STARTERS

For large squirrel-cage motors auto-transformers are generally used for starting purposes. Voltage tappings, giving 45, 65, and 80 per cent of full line voltage, are provided. The starting torque for each voltage is proportional to the square of the starting voltage. Assume a motor takes five times full-load current at full voltage, with a 65 per cent tap the motor will take 3·25 full-load current and exert a torque of 42·3 per cent of the starting torque which it has with full voltage. The current taken from the line, however, will be but 2·11 full-load line current. Thus, the line current at the start varies as the square of the starting voltage.

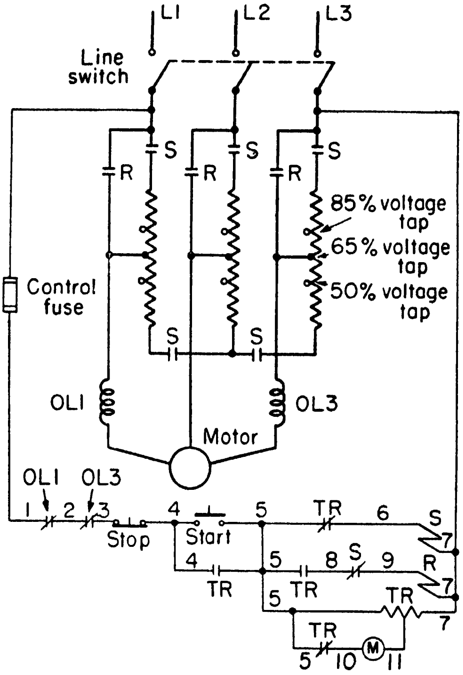


FIG. 28.3. ELEMENTARY WIRING DIAGRAM OF TYPICAL COMPENSATOR REDUCED VOLTAGE STARTER

(Courtesy General Electric Co., U.S.A.)

Fig. 28.3 shows an elementary wiring diagram of a motor started by means of an auto-transformer.

The magnetic contactors are designated *R* for the running position and *S* for the starting position. The starting contactor has five poles. The starting and running contactors have mechanical interlocks to prevent the *R* contactors from picking up during the starting period. The timing relay *TR*, located in the control circuit is used on all auto-transformer starters. Either a pendulum timing device for opening the starter contactor and closing the running

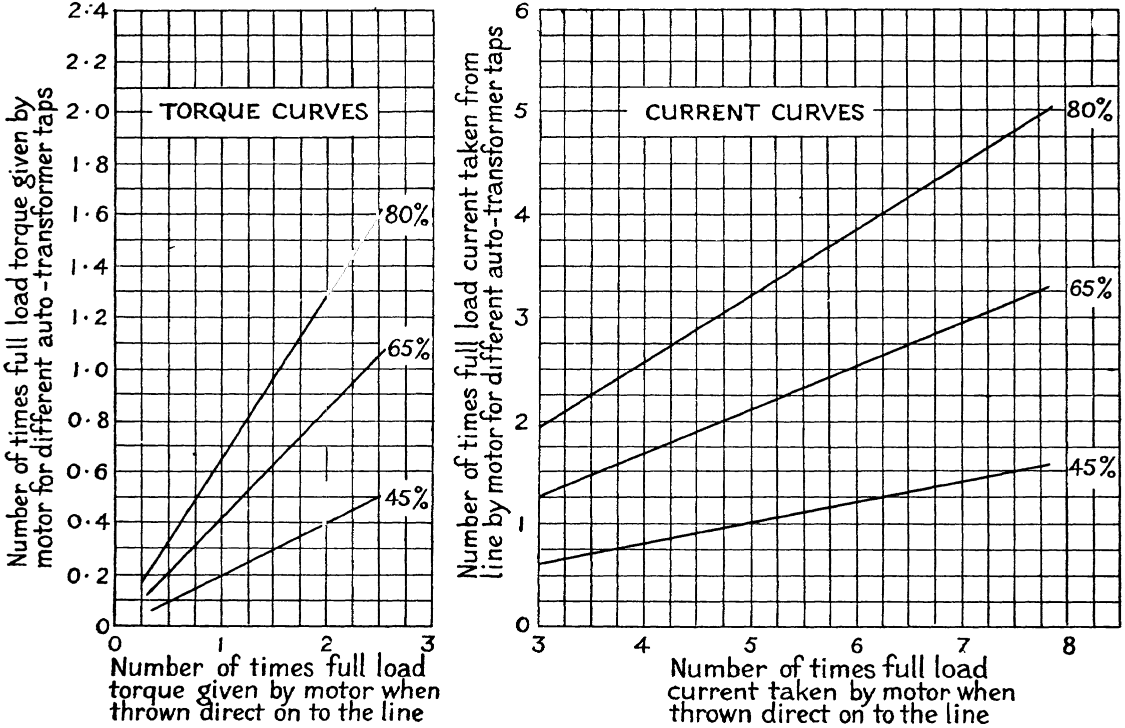

TORQUE CURVES
80%
65%
45%
2·4
2·2
2·0
1·8
1·6
1·4
1·2
1·0
0·8
0·6
0·4
0·2
0
0
1
2
3
Number of times full load torque given by motor for different auto-transformer taps
Number of times full load torque given by motor when thrown direct on to the line
CURRENT CURVES
80%
65%
45%
6
5
4
3
2
1
0
3
4
5
6
7
8
Number of times full load current taken from line by motor for different auto-transformer taps
Number of times full load current taken by motor when thrown direct on to the line

Fig. 28.4

contactor in a predetermined time, or a multi-contact timing relay driven by a synchronous motor is used. A calibration disk is provided on this relay for time adjustment. Referring to Fig. 28.3, with the line switch closed, pressing the starting button energizes the coil (6–7) of the start contactor through *OL*1 contacts (1–2), *OL*3 contacts (2–3) stop, start (4–5), and the normally closed time-opening contacts (5–6) of the timing relay.

This causes the *S* contacts to close (8-9 to open) and energizes the motor at reduced voltage through the auto-transformer. At the same time the timing relay coil (5–7) and the synchronous motor (*M*) are energized, The *TR* coil (5–7) closes the *TR* contacts (4–5) at once, thus maintaining the circuit after the start button is released. The synchronous motor (*M*) starts running, and, after a predetermined time, trips a latch by spring action, thus opening the two normally closed contacts (5–6) and (5–10) of the timing relay, and at the same time closing its normally open contacts (5–8). As soon as the *TR* contacts (5–6) open the circuit, coil (6–7) is de-energized, thereby opening the *S* contactor. When the *S* contactor drops out, it closes its interlock contacts (8–9) to complete the circuit to the coil (9–7) of the run contactor, which connects the motor across the line. The *TR* contacts (5–10) disconnect the synchronous motor from the circuit. When de-energized, the *TR* relay automatically resets.

Plate XXI (facing page 488) shows the KMC.72 auto-transformer starter for a squirrel-cage motor manufactured by B.T.H.

Plate XXII (facing page 489) shows the rear view of this unit.

Fig. 28.4 gives a useful set of curves for determining the torque and current from the line for auto-transformer start. It is self explanatory.

Stator Resistance Method

With resistance in the stator circuit, the starting torque is proportional to the square of the voltage impressed across the motor, but the current from the line depends on the starting voltage to the motor. It is thus inferior to the auto-transformer start.

Star-delta Start

Where starting conditions are easy a favourite method is the star-delta start. The machine is started up with the windings connected in star, and when up to speed, the connection is changed to delta. With the windings in star, the volts per phase $= \frac{\text{line volts}}{\sqrt{3}}$, and the starting torque, being proportional to the square of the applied volts per phase is equal to one-third of that at full voltage. At starting, the phase current is only $\frac{1}{\sqrt{3}}$ of the *phase* current when connected in delta, and since the line current with delta connection $= \sqrt{3}$ times the phase current, it follows that the line-starting current = one-third of the starting current at full voltage.

Plate XXIII shows a star-delta starter with isolator and stand for floor mounting. This starter has a triple-pole contactor, with under-voltage protection, a combined starting (star) and running (delta) contactor, a triple-pole over-current relay, and an adjustable timing device.

Plate XXIV shows a B.T.H. contactor starter for squirrel-cage motors up to 50 h.p., operated by separate push buttons.

Starting by Resistance in the Rotor Circuits

For slip-ring motors, the usual method of start is to connect resistance in the rotor circuits.

The rotor current

$$\bar{I}_r = \frac{\bar{E}_r}{\sqrt{\left(\frac{R_r}{s}\right)^2 + L_r^2\omega^2}}$$

where $\bar{E}_r$ = volts per phase at standstill

R_r = rotor resistance per phase

$L_r\omega$ = rotor reactance at standstill

If the rotor current is to remain constant, and hence the stator current, during the starting period, R_r must vary with the slip. This is possible with a liquid rheostat, but not so with a metal resistance. The current is allowed to vary between a maximum and minimum value.

Let there be n steps in the controller. On the first notch, initially, we have

$$\bar{I}_1 = \frac{\bar{E}_r}{\sqrt{\left(\frac{R_1}{s_1}\right)^2 + L_r^2\omega^2}}$$

R_1 = total resistance of the rotor per phase at slip $s_1 = 1$ at start. As the speed rises, the current will fall; let the slip be s_2, when the current has fallen to $\bar{I}_2$

$$\bar{I}_2 = \frac{\bar{E}_r}{\sqrt{\left(\frac{R_1}{s_2}\right)^2 + L_r^2\omega^2}}$$

The controller is now moved to the next notch, and the current rises to $\bar{I}_1$.

Then $$\bar{I}_1 = \frac{\bar{E}_r}{\sqrt{\left(\frac{R_2}{s_2}\right)^2 + L_r^2\omega^2}}$$

The speed continues to rise and the slip falls, until I_2 is reached, and we have

$$I_2 = \frac{E_r}{\sqrt{\left(\frac{R_2}{s_3}\right)^2 + L_r^2\omega^2}}$$

It is clear that for the $n + 1$ steps, we have

$$\frac{R_1}{s_1} = \frac{R_2}{s_2} = \frac{R_3}{s_3} = \ldots = \frac{R_{n+1}}{s_{n+1}}$$

or $$\frac{s_2}{s_1} = \frac{s_3}{s_2} = \frac{s_4}{s_3} = : = \frac{s_{n+1}}{s_n} = \frac{R_2}{R_1} = \frac{R_3}{R_2} = \frac{R_{n+1}}{R_n} = a$$

Now $s_1 = 1$, $$\begin{aligned} R_2 &= aR_1 \\ R_3 &= aR_2 = a^2R_1 \\ R_n &= aR_{n-1} = a^{n-1}R_1 \\ R_{n+1} &= aR_n = a^nR_1 \end{aligned}$$

where R_{n+1} = rotor resistance with rings short-circuited

$$R_{n+1} = a^nR_1 = a^n\frac{R_{n+1}}{s_{n+1}}$$

s_{n+1} = slip with the rotor short-circuited, i.e. with all external resistance cut out.

Thus $$R_1 = \frac{R_{n+1}}{s_{n+1}}$$

$$a^n = s_{n+1}$$

The steps in the rotor resistance must be

$$\begin{aligned} r_1 &= R_1 - R_2 = R_1(1 - a) \\ r_2 &= R_2 - R_3 = R_1(a - a^2) = ar_1 \\ r_3 &= R_3 - R_4 = ar_2 \end{aligned}$$

and so on

It is also clear that approximately

$$\frac{I_2}{I_1} = a$$

CONTACTORS

These are of two types, known as "vertical-lift" and "clapper" types. They are used for making connection between the motor and supply circuits. The vertical-lift type is used for small motor applications and is provided with two series contacts per pole. This results in a wide gap, which assists in putting out the arc.

The vertical-lift type is shown in Fig. 28.5.

The clapper-type contactors are used for large currents, and need special provision to extinguish the arc. It is necessary to force the arc against a cold insulating material or use a magnetic blow-out coil.

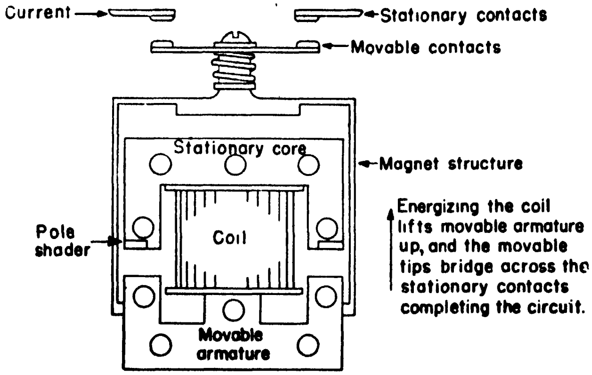
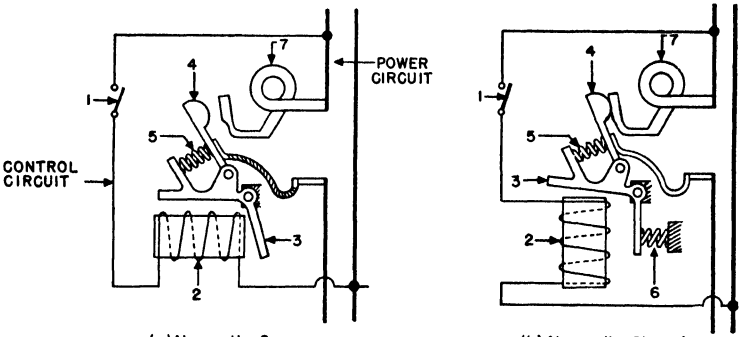

Current
Stationary contacts
Movable contacts
Stationary core
Magnet structure
Energizing the coil lifts movable armature up, and the movable tips bridge across the stationary contacts completing the circuit.
Pole shader
Coil
Movable armature

FIG. 28.5. PRINCIPLE OF OPERATION OF VERTICAL-LIFT CONTACTOR

(*Courtesy General Electric Co., U.S.A.*)

Fig. 28.6 shows the principal components of a clapper-type contactor, arranged for (*a*) normally open and (*b*) normally closed contacts.



(a)Normally Open

(b) Normally Closed

FIG. 28.6. PRINCIPAL COMPONENTS OF CLAPPER-TYPE CONTACTOR

1. Contact which completes control circuit; push button, pressure switch, float switch, etc.
2. Shunt coil which energizes solenoid.
3. Armature connected to movable portion of contactor.
4. Movable contact.
5. Spring for maintaining proper tip pressure in closed position.
6. Spring for closing contacts when coil is de-energized on normally closed contactor.
7. Blow-out coil.

(*Courtesy General Electric Co., U.S.A.*)

The solenoids used on alternating current must have laminated cores and a shader or short-circuited coil to prevent the armature from dropping out each time the magnetic field passes through zero value.

Fig. 28.7 shows the function of the blow-out coil and arc chute.

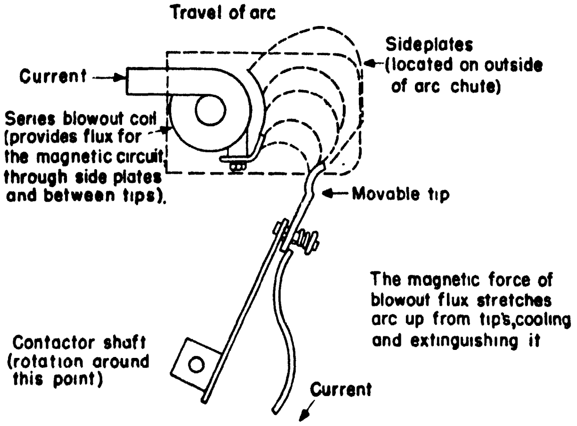

Travel of arc
Sideplates (located on outside of arc chute)
Current
Series blowout coil (provides flux for the magnetic circuit through side plates and between tips).
Movable tip
The magnetic force of blowout flux stretches arc up from tip's, cooling and extinguishing it
Contactor shaft (rotation around this point)
Current

FIG. 28.7. FUNCTION OF BLOW-OUT COIL AND ARC CHUTE
(Courtesy General Electric Co., U.S.A.)

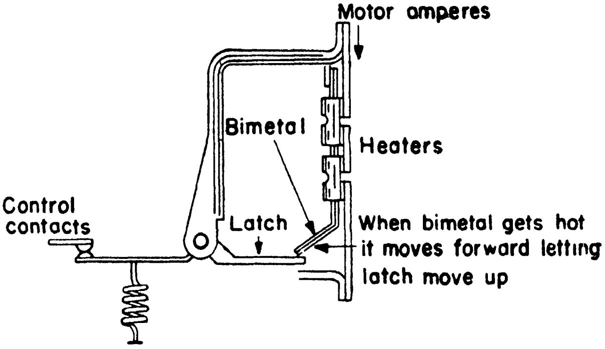

Motor amperes
Bimetal
Heaters
Control contacts
Latch
When bimetal gets hot it moves forward letting latch move up

FIG. 28.8. NON-COMPENSATED TYPE OF THERMAL OVERLOAD RELAY
(Courtesy General Electric Co., U.S.A.)

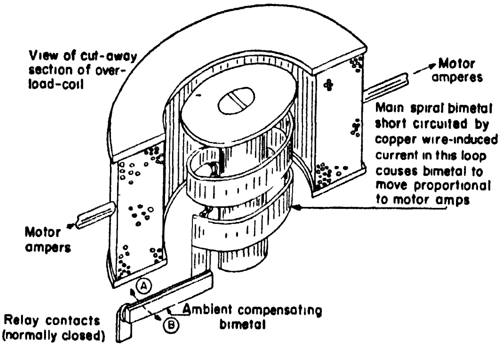

View of cut-away section of over-load-coil
Motor amperes
Main spiral bimetal short circuited by copper wire-induced current in this loop causes bimetal to move proportional to motor amps
Motor ampers
A
B
Relay contacts (normally closed)
Ambient compensating bimetal

FIG. 28.9. COMPENSATED TYPE OF THERMAL OVERLOAD RELAY
(Courtesy General Electric Co., U.S.A.)

RELAYS

These consist of (*a*) voltage relays of the normally closed type; (*b*) thermal overload relays; (*c*) instantaneous overload relays; and (*d*) timing relays.

The simplest form of relay is one which employs a bimetal, or heat-sensitive, strip made of two metals, which expand at different rates, when subjected to heat. These two metals are fused together and bend, due to unequal expansion. This bending is sufficient to permit a latch to trip and open the contacts, if the motor is overloaded. Since the expansion of the strip is determined by the temperature to which it is subjected, it is clear that some form of compensation must be used for the temperature of the room in which the motor works. Such relays are either compensated or uncompensated.

Fig. 28.8 shows an uncompensated thermal overload relay.

The compensated type of relay operates on the induction principle. The motor current flows in a series coil, which acts as the primary of a transformer, and a spiral bimetal strip, short-circuited by a copper wire, acts as secondary. The lower end of the spiral rotates and opens a set of contacts, when it has deflected a given amount. The compensating device consists of a bimetal strip, which deflects in the opposite direction to the spiral and thereby corrects for the effect of different ambient temperatures. The principle of this device is shown in Fig. 28.9.

Index

ADVANCER, phase—
  Kapp vibrator, 246
  Leblanc, 240
  Miles-Walker, 244
  Scherbius, 241
Advancing phase, 240
Air-gap length, various formulae, 386
Alternating-current commutator motor—
  series, 211–27
  shunt, 227–33
Ampere speed-torque curve—
  for a.c. excitation, 137
  for d.c. excitation, 144
Amplitude—
  of ampere-turn wave, 271
  of m.m.f. wave, 266
Analysis—
  of m.m.f. diagrams, 266
  of operation of capacitor motor under unbalanced conditions, 120
  of performance curves of squirrel-cage machines, 40
Arrangement of phases for d.c. excitation of three-phase machine when braking, 140
Asynchronous torque due to higher harmonies, 284
Automatic control of induction regulator, 352
Automatic slip regulator, 157
Auto-starter, 519
Autosyn, 320

BARREL winding, 302
Behrend, B. A., 80
  circle diagram, 13
Belt leakage, 440
Best power factor, 383
Blow-out coil, 524
Brake test, 287
Braking of induction motor—
  by a.c. excitation, 134
  by d.c. excitation, 138–44
Breadth factor, 31
Brush—
  friction loss, 402
  pressure, 402

CAGE motor—
  double, 60–7
  single, 55–60
Capacitor motor, 117, 120
Capacity overload, 33, 52, 391
Carter's factors—
  open slot, 454
  semi-enclosed slot, 453
Cascade connection, 179–86
Characteristics of motors—
  a.c. commutator, 217, 218, 236, 237, 238, 239
  polyphase, 40–9
  single-phase, 101–4
Circle diagram—
  approximate, 13–20
  exact, 68–79
Coefficient of self-inductance, 373, 415
Cogging, 66
Cogging mill, 153
Coil-span factor, 31
Commutation in polyphase commutator motor, 218
Commutator motor—
  three-phase series, 211–27
  three-phase shunt, 227
Compensator, line drop, 352
Concentric winding, 295–9
Condensers, 117
Condenser motor, 117
Connection for d.c. braking, 140
Contactor, 523–4
  magnetic, 518
Control—
  circuits, push-button, 518
  devices, 517
  speed, 146
Controller—
  single-stage rotary, 228
  two-stage rotary, 230
Converter phase, 352, 354
Cooling curve, 376
Co-ordinates of centre of circle diagram, 74, 79
Core-loss curve for iron, 376
Crane-motor rating, 513
Crawling in induction motor, 280, 285
Cross-flux theory of single-phase motor, 80–92
Current—
  magnetizing, 396
  short-circuit, 373
Current in bar and ring—
  of double-cage motor, 63
  of single-cage motor, 55, 56
Curves—
  for slip and output, 28
  for slip and torque, 24
  of performance for permanent rotor resistance, 152

Curves (*contd.*)—
  of power factor for various values of the dispersion co-efficient and various values of maximum load, 395

Design—
  double-cage motor, 474
  fractional horse-power motor, 487, 497
  large slip-ring motor, 461
  medium-power slip-ring type motor, 447
  principles, 367
  squirrel-cage motor, 468
Determination of D^2L—
  for double-phase motors, 370
  for single-phase motors, 371
  for three-phase motors, 367, 368
Devices, control—
  auto-starter, 519
  magnetic contactor, 518, 523, 524
  manual starter, 517
  on the line starter, 517
  overload relay, 526
  push-button circuit, 518
  star-delta starter, 521
  thermal and solenoid contactor, 523–5
Diagram, circle, 13–20, 68–79
Diameter of circle, 17
Differential leakage, 440
Direct current—
  excitation for electric braking, 138
  windings connected for three-phase rotor, 309
Dispersion co-efficient, 37, 381, 389, 395
Distributed windings, 293
Distribution factor, 31
Double-cage motor, 60–7, 474
Drysdale slip indicator, 288

Eddy currents—
  in conductor (Field's curves), 512
  in lamination, 377
Efficiency, 21, [illegible]
  curve of motor, 388–91
  tests, 289
Electric braking, 134–45
Electromotive force equation, 13
Equivalent—
  circuit—
    double-cage motor, 60, 481
    polyphase motor, 16–60, 68
    single-phase motor, 86, 107, 503
    unbalanced conditions, 332, 335–8, 341–7
  impedance, 507
  reactance, referred to stator, 59, 66
  resistance, referred to stator, 54, 58
  stator and rotor quantities, 31
Error of Selsyn, 329

Factor—
  coil-span, 31
  power (maximum), 39
Field—
  cross in single-phase motors, 80–92
  revolving (analysis), 266
Field (*contd.*)—
  two revolving, theory of, in single-phase motor, 105–13
Field's curves, 512
Flux—
  density calculations, 380, 400
  wave form, 397
Form factor, 266
Fractional horse-power motor, 497

Generation induction, 359
Geometrical relations from circle diagram, 32–3

Harmonics, 8
  analysis, of m.m.f. diagrams, 266
  crawling due to, 280–5
  influence on performance, 280–5
  stator and rotor, 272, 278
Heating of motor, 376, 513
High-torque motor, 60–7
Hunt cascade induction motor, 188–95

Impedance, equivalent, referred to stator, 32, 507
Inductance—
  of windings, 404, 436
Induction—
  generator, 359
  motor—
    polyphase, 7–79
    single-phase, 80–133
    with d.c. secondary excitation, 250
  regulators—
    single, 350
    twin, 352
Insulation for windings, 515, 516
Intermittent-contact method of measuring slip, 288
Inversion principle, 73, 74
Iron loss, 402, 114, 377
  curves, 376

Kapp vibrator, 246
Kloss, Dr. Max, overhang leakage, 402, 416–32
Kramer system, 196

Leakage—
  factor, 38
  flux—
    belt, 440
    end connection, 416, 439
    slot leakage, 406, 436
    zig-zag, 412–14, 439
  reactance—
    referred to stator, 54
Leblanc, phase advancer, 240
Load vector diagram of polyphase induction motor, 12
Loading and overloading, 374
Losses—
  eddy current, 377–8
  hysteresis, 377–8
  iron, in single-phase motors,

Losses (*contd.*)—
  pulsation, in teeth, 402
  squirrel-cage rotor, 57
Lyon Walds V., 124, 354

MAGNETIC—
  conductivity of leakage path, 420
  flux distribution, 397
  humming, 280
Magnetization curves for tapered teeth, 455
Magnetizing current—
  calculation, 381, 396
  from circle diagram, 381, 396
Manner of working of induction motor, 1–13
Manual starter, 517
Maximum—
  horse-power, 33, 52, 391
  output, 26
  power factor, 39, 52
  torque, 23, 24, 29
Mechanical output, 14
Miles-Walker phase advancer, 244
Moment of inertia, 147
Motors—
  capacitor, 117
  cascade connection, 179
  commutator, a.c., 211–39
  fractional horse-power, 497
  pole-changing, 173
  Schrage, 234
  squirrel-cage—
    double, 60–7
    single, 55–60
Mutual inductive drop, note on calculation (phase converter), 358

NEGATIVE sequence diagram, 131
No-load—
  loss, 401
  vector diagram, 11
No-voltage release, 518
Noise in induction motors, 280

ON-LOAD vector diagram, 12
Operation of three-phase motor with single-phase rotor, 344
Output—
  co-efficient, 369
  equation, 367
  maximum, 52
  slip curves, 28
Overhang leakage, 416–32
Overload capacity, 33

PERFORMANCE of motor with flywheel and automatic slip regulator, 164
Performance curve—
  a.c. commutator motor, 236, 237
  single-phase motor, 102
  slip-ring motor, 47
  squirrel-cage motor, 47
Permeance—
  of various types of slots, 406–14
  overhang, 416
Phase—
  advancer—
    Kapp vibrator, 246
    Leblanc, 240
    Miles-Walker, 244
    Scherbiüs, 241
    theory, 240–9
  converter theory, 352–9
Pitch, short, winding, 433
Pole changing, 173
  windings, 176, 178
Polyphase-motor theory, 7
Polyphase windings, 293–319
Positive sequence network, 131
Power factor—
  correction, 264
  curves, 395
  maximum, 39, 52
  relation between *D* and *L* for best, 379
Pulling into step with synchronous induction motor, 253, 264
Pull-out torque, 53
Pulsation loss in teeth, 402
Punga and Raydt construction, 62
Push-button control circuits, 518

RATING of motors—
  intermittent, 513
  protected, 513
  totally enclosed, 513
  ventilated, 513
Ratio—
  currents, 15
  fluxes and magnetizing currents in pole changing, 175
  voltages, 15
Reactance—
  equivalent, referred to stator, 59, 66
  leakage, 54
    referred to stator, 59, 66
  synchronous, 251
Referred quantities, 115
Regulation and stability, [illegible], 364–66
Regulator—
  induction, 349, 350
  slip—
    automatic, 157
    permanent, 146
Relation between leakage factors and dispersion coefficient, 38
Relay, 526
  thermal, overload, 525
Resistance, referred to stator, 54, 58
Revolving field analysis, 266
Root mean square, h.p., 166
Rotor—
  double squirrel-cage, 60–7
  reactance referred to stator, 59, 66
  resistance—
    referred to stator, 54, 58
    starting, 522
  slip-ring winding—
    d.c. windings opened to form six phases, 314, 315
    mush, 309

Rotor (*contd.*)—
    openings, 311
    slotted for wave windings, 316
    wave-wound bar, 310
  squirrel-cage winding, 55–60

SATURATION curve for a.c. excitation, 143
Scherbius system, 199
Schrage motor, 234
Schwarz patent, 220
Selsyn and autosyn, 320–9
    differential, 328
    errors, 329
  generator and control-transformer system (voltage indication), 328
  instrument, 323
    angular error, 326, 327
    diagram of connections, 324
  power selsyns, 320
  power vector diagram, 322
Series commutator motor, 211–20
Short-circuit current, 18, 373, 403, 405, 441
Short-pitch windings, 433
Shunt commutator motor, 227
Shuttleworth, 222
Single-phase—
  and fractional horse-power motors, 80, 117, 120, 487, 497
  induction motor—
    cross-field theory, 80–91
    new theory, 92–100
    two revolving field theory, 105–13, 133
  regulators, 350
Skewed rotor slots, 59
Slip, 10, 21
  construction, 25
  definition, 10, 2
  for maximum output, 26, 27
  for maximum torque, 23
  output—
    curves, 2[illegible]
    equation, [illegible]
  ring motor design, 447, 461
  torque relation, 21, 34
Slot—
  leakage flux, 403
  permeance, 406–10
  ratio, see noise in induction motor, 280
  star, 306
Solenoid contactor, 524
Sparking and leakage distances, 514
Specific electric loading and peripheral speed for different slot depths, 379
Speed control—
  automatic slip regulator, 157
  below synchronism, 200
  cascade connection, 179
  change of speed by varying the supply frequency, 195
  continuous slip regulator, 147
  Hunt motor, 188
  Kramer system, 196–8
  pole changing, 173, 174
Speed control (*contd.*)—
  regulation of speed and power factor by the use of a three-phase shunt motor, 206
  resistance in rotor circuit, 146
  variation of speed and power factor by the Scherbius system, 199
  Ward-Leonard system, 169
Speed-torque curves for different values of rotor resistance, 148
Squirrel-cage motor, 55–67
Stability and regulation, 363
Stalloy and super stalloy, 375
Starter—
  auto-transformer, 519
  magnetic control, 518
  manual, 517
  on the line, 517
  star delta, 521
Starting—
  by resistance in rotor circuits, 522
  torque, 24
Stator and rotor slot permeances, 406
Steel tube mill, 162
Stroboscopic method of measuring slip, 288
Synchronizing torque, 256
Synchronous—
  and asynchronous torque due to harmonics, 283, 284
  induction motor—
    performance, 251
    theory, 250–65
  limiting value of slip for pulling into step, 262
  pulling into step, 258, 261

TEMPERATURE rise of motor, 376, 513
Testing, 286
Thermal contactor, 523, 525
Three-phase—
  motor—
    operated from single-phase supply, 125
    theory, 7–79
  regulators, 350
  series motor, 211–27
  shunt motor, 227–33
  supply from a single-phase circuit, 128
Torque—
  Asynchronous and synchronous due to harmonics, 283, 284
  full load and slip relation, 35
  line, 77
  maximum, 23, 29
  output curves, 28
  pound-foot, polyphase motors, 30
  revolutions per minute—
    and amperes in electric braking with a.c. excitation, 137
    and secondary amperes with d.c. excitation, 144
  single-phase machines, 124
  slip curves, 24, 113
    in squirrel-cage motors, 67
  slip relations, 21, 34
  starting, 24

Torque (*contd.*)—
  synchronous watts, in, 22, 29, 64
    in single-phase motors, 112
Totally enclosed ratings, 513
Transformation ratios, 15, 31
Treatment of noise, 280
Two-phase induction motors, 370

Unbalanced conditions—
  in induction motors, 330
  machine with stator phase opened, 337
    equivalent circuit, 338
    gross torque in lb-ft, 339
  polyphase motor with single-phase rotor, 344
    gross mechanical power, copper loss and torque, 348
    positive and negative sequence diagrams, 347
    torque curves, 345
  rotor circuits—
    negative-sequence diagram, 343
    negative-sequence equivalent circuit, 343
    positive-sequence diagram, 341
  stator or rotor circuits, 340
  voltage—
    gross mechanical output, 334, 336
    negative-sequence circuit, 335
    positive or negative sequence circuits, 332
Use of twin regulator, 352

Vector diagram—
  cascade, for three-phase motors, 182
  concatenated commutator motor and induction motor, 201, 204, 205, 207
  induction regulator, 351
  phase advancing—
    Kapp vibrator, 247
    Miles-Walker, 245, 246
    Scherbius, 242

Vector diagram (*contd.*)—
  phase converter, 353, 354
  power selsyns, 322
  selsyns power, 322
  single-phase motor, 83, 88, 93, 100, 116, 117, 118, 130, 131
  slot star for windings, 306
  synchronous induction motor, 256
  three-phase—
    no-load, motor, 11
    on-load, motor, 12
    regulator, 351
    series commutator, motor, 212, 213, 215, 223, 224, 225
    shunt commutator motor, 229, 231, 232
Ventilated motors, 513
Vertical-lift contactors, 524
Vibrator, Kapp, 246
Voltage between rings at start, 310

Walker-Miles phase advancer, 244
Ward-Leonard system, 169
Windage loss, 402
Winding—
  adapted to given frame, 510
  concentric—
    divided coil, 299
    undivided coil, 295
  double layer, 301
  fractional slot, 305
  insulation—
    rotor, 516
    stator, 515
  mould, 319
  mush, 293
  rotor, 309–18
  short pitch, 433
  slot star, 306
  small motor, 318

Zigzag leakage, 412–14, 439